Planar and nonplanar graphs

Basic fact: Every graph is homeomorphic to a subset of $\mathbb{R}^3$.

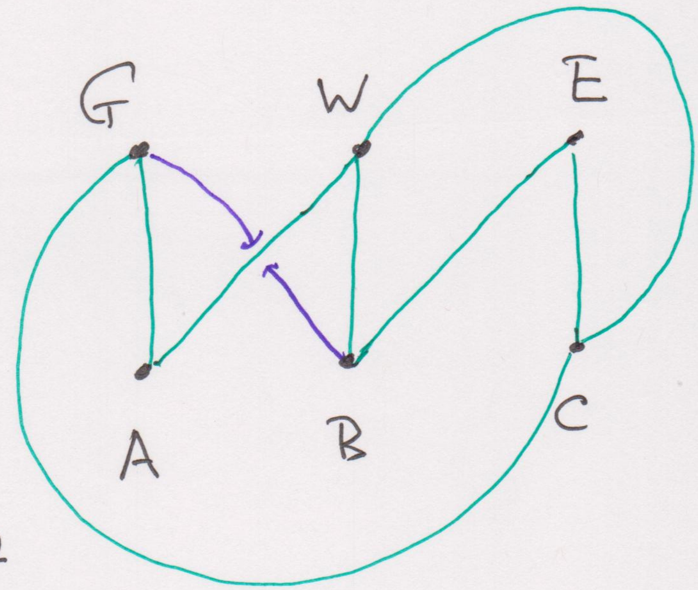
[More generally, every n-dim. simplicial complex $\cong$ polyhedron in $\mathbb{R}^{2n+1}$.]

SEE graphs4.pdf for a proof (of the 1D case).

GOAL: Prove that the gas-water-electricity network* is NOT homeo to a subset of $\mathbb{R}^2$.

* In graph theory, this is usually called $K_{3,3}$.

Recall:



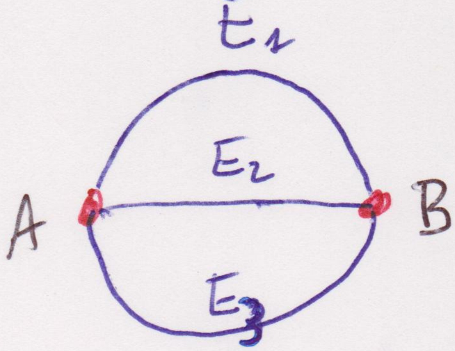
No problem embedding all but one edge in $\mathbb{R}^2$.

Want to show that no clever person will ever find a way of embedding the whole graph.

The proof involves more sophisticated separation theorems like the 2-dim. Jordan-Brouwer Thm. and the result on $S^2 - A$, where $A \cong D^1$ or D^2.

Figure Theta Subspaces

Homeomorphic to



We want to prove that any such subset in $\mathbb{R}^2$ has the separation properties suggested by the picture; i.e., want the following:

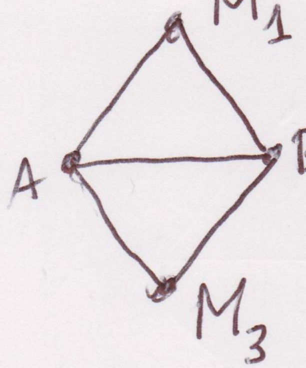
THEOREM. $X \subseteq \mathbb{R}^2 \subseteq S^2$ as above. Then $S^2 - X$ has 3 connected components U, V, W such that

$E_1 \cup E_2$ = Boundary of U

$E_2 \cup E_3$ = Boundary of V

$E_1 \cup E_3$ = Boundary of W.

Note We can triangulate X as



so that

$H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$

$H_0(X) = \mathbb{Z}$

$H_q(X) = 0$ otherwise

So if $X \subseteq S^2$, then $X \neq S^2$ (different homology groups), so $X \subseteq S^2 - \{pt\} \cong \mathbb{R}^2$.

Proof Two parts (actually 3)

1. $S^2 - X$ has three components.

2. There is one of these components whose boundary is $E_1 \cup E_3$.

3. Same for $E_1 \cup E_2$, $E_2 \cup E_3$ (and for the 3 choices, get different components)

since bdies diff.

① Notation

$$U_i = S^2 - E_i, \quad U_{ij} = S^2 - (E_i \cup E_j)$$

$$U_{123} = S^2 - X.$$

Look at the MV sequence involving

$$\begin{tikzcd}
 & U_{13} \arrow[dr] & \\
U_{123} \arrow[ur] \arrow[dr] & & U_3 \\
 & U_{23} \arrow[ur] &
\end{tikzcd}$$

no reduced homology.

So $\widetilde{H}_i(U_{123}) \cong \widetilde{H}_i(U_{13}) \oplus \widetilde{H}_i(U_{23})$

$i=0$: $\widetilde{H}_0(U_{123}) = \mathbb{Z} \oplus \mathbb{Z}$, so

U_{123} has 3 components.

Now U_{13} has 2 components, each of which has $E_1 \cup E_3$ as its boundary.

(2) Let V and W be the components of U_{13}.

Notice that one of these contains the connected set E_2 - endpoints. Say it's W. Then $U_{123} = V \cup W - (E_2 - \text{endpoints})$. Both pieces are nonempty open + closed. So a component of U_{123} is contained in V or W. But $V \subseteq U_{123} \Rightarrow V$ must be a component of U_{123}, and its boundary is known to be $E_1 \cup E_3$.

(3) Interchange the roles of E_1, E_2, E_3 in the preceding to get

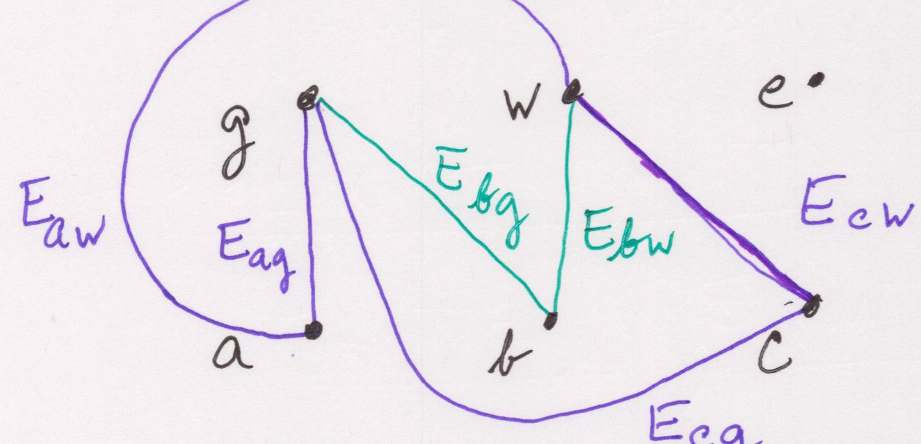
similar conclusions for $E_1 \cup E_2$ and $E_2 \cup E_3$. Since the sets $E_i \cup E_j$ are distinct for different (i,j), this means one gets 3 different components. But there are only 3 components in U_{123}. $\blacksquare$

APPLICATION TO THE UTILITIES NETWORK

Recall. This is a graph with 6 vertices a, b, c, g, w, e and edges ag, aw, ae, bg, bw, be, cg, cw, ce.

CLAIM This graph is not homeomorphic to a subset of S^2 (equivalently, $\mathbb{R}^2$).

A ssume it is, and derive a contradiction. [Source: Munkres, Topology, 394–399]



Let X = utilities graph, X_0 = graph of all edges not containing e as a vertex.

Then X_0 is a figure theta

$$L_1 = E_{ag} \cup E_{aw}$$
$$L_2 = E_{bg} \cup E_{bw}$$
$$L_3 = E_{cg} \cup E_{cw}$$

$S^2 - X_0$ has 3 components, and $e \in S^2 - X_0$ lies in one of them, say U.

It follows that each half-open arc

$$E_{ae} - \{a\}$$
$$E_{be} - \{b\}$$
$$E_{ce} - \{c\}$$

must be contained in U. (each is connected)

and hence $a, b, c \in \overline{U}$.

Why is this impossible?
(Trial and error suggests this!)

The three pieces of X meet in $\{g, w\}$.

$a \in L_1 \quad b \in L_2 \quad c \in L_3$

By the result on theta subspaces the boundary of $\overline{U}$ is the union of exactly two L_i, L_j. If $k \neq i, j$ then one of a, b, c lies on (L_k − its endpoints), and hence cannot be a limit point of U. ∎