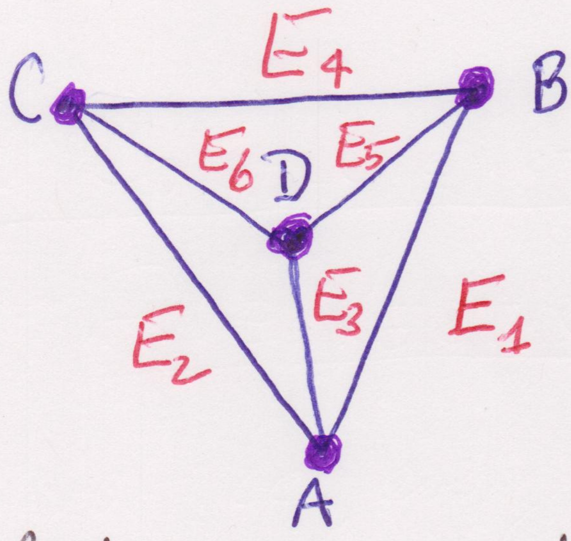
The Complete Graph on Five Vertices

Five vertices, every pair of vertices joined by an edge.

CLAIM This graph is also not homeomorphic to a subset of S^2 or $\mathbb{R}^2$.

Starting point: The complete graph on *four* vertices* is homeomorphic to a subset of S^2 or $\mathbb{R}^2$.

* Four vertices, every pair joined by an edge.



Define edges E_i as at left.

As before, we want to prove that every topological embedding of this graph in S^2 has the separation properties suggested by the picture.

THEOREM. Let $X \subseteq \mathbb{R}^2 \subseteq S^2$ be homeomorphic to the given graph. Then $S^2 - X$ has four connected components U, V, W, O such that

Boundary of $U = E_1 \cup E_3 \cup E_5$
Boundary of $V = E_2 \cup E_3 \cup E_6$
Boundary of $W = E_4 \cup E_5 \cup E_6$
Boundary of $O = E_1 \cup E_2 \cup E_4$.

Same type of strategy as before.

(1) $S^2 - X$ has four components

(2) $E_1 \cup E_2 \cup E_4$ bounds one component of $S^2 - X$.

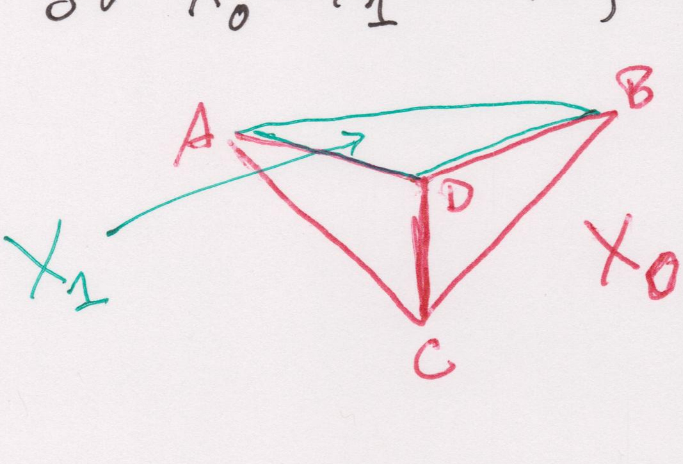
(3) Apply (1) + (2) to the other "triangles" listed above.
(Each triangle then bounds a component, and there are 4 triangles. Hence each of the four components must figure in the conclusion.)

<u>Preliminary note</u> The proofs of the earlier separation theorems imply that $\tilde{H}_i(S^2 - X) = 0$ if $i \geq 1$ and X is homeomorphic to S^1 or a theta space (look at the MV sequences in higher dims.)

① Let X_0 = subgraph of X with edge E_4 removed

$$X_1 = E_4 \cup E_5 \cup E_6$$

So $X_0 \cup X_1 = X$, $X_0 \cap X_1 = E_5 \cup E_6$ $(\cong [0,1)$



Consider the reduced MV sequence for

$$\begin{tikzcd}[column sep=small]
 & S^2-X_0 \arrow[dr, dash] & \\
S^2-X \arrow[ur, dash] \arrow[dr, dash] & & S^2-(E_5\cup E_6) \\
 & S^2-X_1 \arrow[ur, dash] &
\end{tikzcd}$$

$\left\{ \begin{array}{l} E_5 \cup E_6 \cong \\ [0,1] \end{array} \right\}$

$\tilde{H}_*$ is trivial

It follows that

$$\tilde{H}_*(S^2-X) \cong \tilde{H}_*(S^2-X_1) \oplus \tilde{H}_*(S^2-X_0)$$

0	$i \geq 0$	$0 \quad i \geq 0$
$\mathbb{Z}$	$i=0$	$\mathbb{Z} \oplus \mathbb{Z}$
Jordan Brouwer		previous result

Hence S^2-X has $\boxed{4}$ connected components.

(2) By Jordan-Brouwer,
$S^2 - (E_1 \cup E_2 \cup E_4)$ has two
"outside triangle" ABC — call it Γ.
components, say G and H. One
contains the remaining vertex D.
Without loss of generality, suppose
$D \in H$ (reverse the roles of G
and H in the argument below to
treat the other case!).

The half-open edges

$E_3 - \{A\}$ are connected
$E_5 - \{B\}$ and disjoint from
$E_6 - \{C\}$ Γ, so they
must all be contained in H since
D lies on each of them and
$G \subseteq S^2 - X$.

Let Q_1, Q_2, Q_3, Q_4 be the components of $S^2 - X$, and number them so that $G \subseteq Q_1$.

But now

$$S^2 - X = G \cup \left(H - (E_1 \cup E_2 \cup E_3)\right)$$

nonempty, open, disjoint

Hence G is also closed in $S^2 - X$. If $y \in G \subseteq Q_1$, this means that $Q_1 \subseteq G$. Since the boundary of G is $E_1 \cup E_2 \cup E_4$, we have proved the statement we wanted.

③ See previous remarks. ∎

Proof that the complete graph on five vertices is not homeo. to a subset of $\mathbb{R}^2$ or S^2.

Assume that it is homeo. to some subset Y, and let a, b, c, d, e be its vertices. Let $X \subseteq Y$ be the subgraph of all edges which do not have e as an endpoint, so that X is homeo. to ~~the~~ complete graph on four vertices. Use the preceding result to study $S^2 - X$.

Let E_{uv} = edge joining u to v.

Without loss of generality, we can assume e lies in the component U of $S^2 - X$ whose boundary is $E_{ab} \cup E_{bc} \cup E_{ac}$. In any case, e lies in one of these components — treat the other cases by permuting the roles of a, b, c, d.

Note that d is not in $\overline{U}$ by the preceding proof.

Each of the sets $E_{xe} - \{x\}$, where $x = a, b, c$ or d, is connected and contains e, so each lies in U. Hence each end point x lies in $\overline{U}$. Since $d \notin \overline{U}$, this is a contradiction. $\blacksquare$

A result of K. Kuratowski gives a strong converse:

If $(X, \mathcal{E})$ is a graph that is not homeomorphic to a subset of $\mathbb{R}^3$, then X contains a subgraph X_0 which is homeomorphic to either the utilities graph or the complete graph on 5 vertices.

See <u>math205Ctopics16b.pdf</u> for clickable links to proofs and other information (such as independent discoveries of this result by others).