Preface

This course is a continuation of the entry level graduate courses in algebraic topology given during the past two years (Mathematics 205C in Spring 2011 and Mathematics 205B in Winter 2012). In these courses we discussed an algebraic construction on spaces known as **singular homology theory**, which gives algebraic "pictures" of topological spaces in terms of certain abelian groups. We did not actually construct the theory, but we did the following:

(1) For a certain class of spaces known as *polyhedra*, we defined simplicial homology groups which turn out to be isomorphic to singular homology groups.

(2) We gave a somewhat lengthy list of properties or **axioms** for singular homology theory which turn out to characterize the theory uniquely up to natural isomorphism. The equivalence of simplicial homology groups with singular homology groups was included in this list of axioms.

This approach allowed us to use work with simplicial homology and use it to answer some easily stated topological and geometric problems, illustrating that homology theory is an effective tool for analyzing some fundamental types of questions in these subjects. However, *the answers derived in the earlier course(s) are contingent upon knowing that there actually is a singular homology theory satisfying the given axioms*. Thus the first goal of this course is to construct such a theory. In order to motivate the construction further, we shall also give a few applications beyond those in the previous entry level course; one possible example is a topological proof of the Fundamental Theorem of Algebra.

The approach described above can be compared to the way that one often studies the real number system, which is completely characterized by the algebraic and order-theoretic axioms for a *complete ordered field*. These axioms suffice to prove everything that one might want to prove in the theory of functions of real variables, but at some point it is necessary to show that there actually is a system which satisfies the axioms. This is generally done either by means of Dedekind cuts or equivalence classes of rational Cauchy sequences. In either case, once the constructions have yielded a complete ordered field, they have basically served their purpose and one does not need to remember the details of the construction.

The situation for singular homology theory is somewhat different, for one needs the details of the formal construction in order to refine the theory even further, and the next phase of the course will involve such refinements. In somewhat oversimplified terms, we can describe the situation as follows: When we think of algebra, we think of a system which has both addition and multiplication. Homology groups have an obvious additive structure, but in the previous course we did not really say anything about a multiplicative structure. It turns out that a very substantial multiplicative structure exists, and an understanding of the standard construction for singular homology is almost indispensable for motivating and working with this additional structure. We shall try to give applications of this extra structure to a few clearly basic mathematical problems whose statements do not involve homology.

The methods of algebraic topology turn out to be extremely effective for studying many sorts of questions involving topological or smooth manifolds, even in simple cases like open subsets of $\mathbb{R}^n$ (*i.e.*, questions in *geometric topology*), and the final portion of the course will be devoted to establishing a rew fundamental algebraic tools for studying such manifolds (the specifics depend upon time constraints). For example, one topic might be a unified approach to certain fundamental results in multivariable calculus involving the ∇ operator, Green's Theorem, Stokes' Theorem and the Divergence Theorem(s) in 2 and 3 dimensions, and to formulate analogs of these results for higher dimensions. A related topic could be the relationships among various approaches to defining an orientation for a manifold.

Course references

Mathematics 205A and 205B are prerequisites for this course. Lecture notes for these courses are available at the sites given below; the directories containing these files also contain exercises and other related documents (remove the `pdf` file names to get the links for the directories).

`http://math.ucr.edu/~res/math205A-2014/gentopnotes2014.pdf`

`http://math.ucr.edu/~res/math205A-2014/fundgp-notes.pdf`

`http://math.ucr.edu/~res/math205B-2012/algtop-notes.pdf`

Some topics near the end of the second document will be covered at the start of this course.

More formally, throughout the course we shall use the following texts for the basic graduate topology courses as references for many topics and definitions (the first and third are the current texts, and the second might be a helpful bridge between them):

J. R. Munkres. *Topology* (Second Edition), *Prentice-Hall, Saddle River NJ*, 2000. ISBN: 0–13–181629–2.

J. M. Lee. *Introduction to Topological Manifolds* (Second Edition), *Springer-Verlag, New York*, 2010. ISBN: 1–441–97939–5.

J. M. Lee. *Introduction to Smooth Manifolds*, *Springer-Verlag, New York*, 2002. ISBN: 0–387–95448–6.

The official text for this course is the following book:

A. Hatcher. *Algebraic Topology* (Third Paperback Printing), *Cambridge University Press, New York NY*, 2002. ISBN: 0–521–79540–0.

This book can be legally downloaded from the Internet at no cost for personal use, and here is the link to the online version:

`www.math.cornell.edu/~hatcher/AT/ATpage.html`

This web page also contains links to numerous updates, including corrections (one might add that solutions to many exercises are posted online and fairly easy to find using Google or something similar).

Comments on Hatcher's book. This text covers far more material than can be covered in two quarters, and in fact one could easily spend four quarters or three semesters covering the topics in that book by inserting a few extra topics. The challenges faced in covering so much ground are formidable. In particular, the gap between abstract formalism and geometrical intuition is significant, and it is not clear how well any single book can reconcile these complementary factors. More often than not, algebraic topology books stress the former at the expense of the latter, and one important strength of Hatcher's book is that its emphasis tilts very much in the opposite direction. The book makes a sustained effort to include examples that will provide insight and motivation, using pictures as well as words, and it also attempts to explain how working mathematicians view the subject. Because of these objectives, the exposition in Hatcher is significantly more casual than in most if not all other books on the subject. Online reviews suggest that many readers find these features very appealing.

Unfortunately, the book's informality is arguably taken too far in numerous places, leading to significant problems in several directions; as noted in several online reviews of the book, these include assumptions about prerequisites, clarity, wordiness, thoroughness and some sketchy motivations that are difficult for many readers to grasp (these points are raised in some online reviews of the book, and in my opinion these criticisms are legitimate and constructive; of course, it is also necessary to give appropriate weight to the many positive comments about the book and to remember that, despite the drawbacks, it was chosen as the text for this course). Regarding the overall organization, the numbers of sections in both Chapters 2 and 3 are misleadingly small — each section tends to contain three to six significant topics which arguably deserve to be separate units on their own — and perhaps the supplementary topics could have been integrated into the basic structure of the text more systematically; other choices may have made the book easier to read and understand, but it not at all certain that any alternatives would not have given rise to new problems. In any case, one goal of the course and these notes is to deal with some of the issues mentioned in this paragraph.

Selected additional references. Here are four other references; many others could have been listed, but one has to draw the line somewhere. The first is a book that has been used as a text at UCR and other places in the past, the second is a fairly detailed history of the subject during its formative years from the early 1890s to the early 1950s, and the last two are classic (but not outdated) books; the first book also has detailed historical notes.

J. W. Vick. *Homology Theory*. (Second Edition). *Springer-Verlag, New York etc.*, 1994. ISBN: 3–540–94126–6.

J. Dieudonné. *A History of Algebraic and Differential Topology* (1900 − 1960). *Birkhäuser Verlag, Zurich etc.*, 1989. ISBN: 0–817–63388–X.

S. Eilenberg and N. Steenrod. *Foundations of Algebraic Topology*. (Second Edition). *Princeton University Press, Princeton NJ*, 1952. ISBN: 0–691–07965–X.

E. H. Spanier. *Algebraic Topology*, *Springer-Verlag, New York etc.*, 1994.

The `amazon.com` sites for Hatcher's and Spanier's books also give numerous other texts in algebraic topology that may be useful.

Finally, there are two other books by Munkres that we shall quote repeatedly throughout these notes. The first will be denoted by [MunkresEDT] and the second by [MunkresAT]; if we simply refer to "Munkres," it will be understood that we mean the previously cited book, *Topology* (Second Edition).

J. R. Munkres. *Elementary differential topology*. (Lectures given at Massachusetts Institute of Technology, Fall, 1961. Revised edition. Annals of Mathematics Studies, No. 54.) *Princeton University Press, Princeton, NJ*, 1966. ISBN: 0–691–09093–9.

J. R. Munkres. *Elements of Algebraic Topology*. *Addison-Wesley, Reading, MA*, 1984. (Reprinted by Westview Press, Boulder, CO, 1993.) ISBN: 0–201–62728–0.

Overview of the course

The course directory file `outline2012.pdf` lists the main topics in the course with references to Hatcher when such references exist. As noted above, the course will begin by building upon the coverage of simplicial complexes and related structures in 205B; this is basically limited to definitions and results that will be needed later in the course. These properties will then be used in the construction of singular homology theory and the proof that it satisfies the axioms presented in 205B; we shall also prove uniqueness results for systems satisfying the axioms and describe additional applications of the theory beyond those of 205B.

At first glance, the next step in the course may seem like formalism gone crazy. Although homology is initially defined to take values in the category of abelian groups, which can be viewed as modules over the integers $\mathbb{Z}$, one can easily modify the definitions to obtain homology theories with coefficients in some field $\mathbb{F}$, which take values in the category of vector spaces over $\mathbb{F}$. For such theories, one can define *cohomology groups* $H^q(X, A; \mathbb{F})$ to be the dual vector spaces to the corresponding homology groups $H_q(X, A; \mathbb{F})$. Since the dual space construction is a contravariant functor, this definition extends to a contravariant functor on pairs of spaces and continuous mappings of pairs.

Why in the world might one want to do this? The following analogies may provide some insight:

(1) When one studies smooth manifolds, the spaces of tangent vectors to points of a manifold are of course central to the subject, but there are also many situations in which it is preferable to work with the dual spaces of *cotangent vectors* or *covectors* at points of the manifold. One key reason for this is that smooth fields of covectors — usually called *differential* 1*-forms* — have many useful formal properties which are at best very awkward to describe in terms of tangent vector fields. Similarly, if we define homology groups with coefficients in a field then their dual spaces turn out to have some nice formal properties which the spaces themselves do not.

(2) A loosely related analogy involves spaces of continuous real valued functions. Given two spaces X and Y with a continuous mapping $f : X \to Y$, the spaces of bounded continuous real valued functions $\mathbf{BC}(X)$ and $\mathbf{BC}(Y)$ can be made into a contravariant functor if we define $f^* : \mathbf{BC}(Y) \to \mathbf{BC}(X)$ so that $f^*(h) = h \circ f$, but usually there is no useful way to make the function spaces into a covariant functor.

It turns out that there is also an extra structure on cohomology groups which has no comparably simple counterpart in homology; namely, we have a functorial multiplicative structure on cohomology groups which is called the **cup product**. As noted on page 185 of Hatcher, these products "are considerably more subtle than the additive structure of cohomology." After defining these products and giving examples which show that they can be highly nontrivial, we shall also give a few applications to homotopy-theoretic questions; we have chosen some applications whose conclusions can be easily stated using concepts from 205A and 205C without mentioning homology or cohomology groups (or fundamental groups).

The last two units deal with the homological and cohomological properties of topological and smooth manifolds. It is unlikely that both can be covered completely in the present course, but each unit deals with fundamentally important results. Unit V proves **de Rham's Theorem**, which states that the cohomology of a smooth manifold can be computed using differential forms. Among other things, this theorem provides a comprehensive setting for answering certain sorts of results which are often stated without proof in multivariable calculus courses like the following:

Theorem. *Let $A \subset \mathbb{R}^3$ be finite, let U be the complement of A, and let $\mathbf{F}$ be a smooth vector field defined on U. Then $\mathbf{F} = \nabla g$ for some smooth function g if and only if its curl satisfies $\nabla \times \mathbf{F} = \mathbf{0}$ (in other words, $\mathbf{F}$ has a potential function if and only if it is irrotational).*

Note that the conclusion fails if, say, we take A to be the z-axis and let $U = \mathbb{R}^3 - A$. In this case the familiar vector field

$$\frac{x\mathbf{j} \; - \; y\mathbf{i}}{x^2 + y^2}$$

is irrotational but is not the gradient of a smooth function defined over all of U (line integrals of this vector field over closed paths are dependent upon the choice of path; if the vector field were a gradient the line integrals would be independent of the choice of path).

Finally, if time permits there will be a Unit VI, which will cover a class of results known as *duality theorems*. One example of such a result is the following:

Simply connected Poincaré duality theorem. *If M^n is a compact simply connected n-manifold and $0 \leq k \leq n$, then the groups $H^k(M^n;\mathbb{F})$ and $H^{n-k}(M^n;\mathbb{F})$ are isomorphic for every field $\mathbb{F}$.*

Note. It is not difficult to check that this result holds in many special cases like products of spheres.

Results like this suggest that homology and cohomology can be applied effectively to study geometrical and topological questions involving manifolds.

Footnote conventions

At a some points of these notes, certain assertions are made without detailed proofs because the details of verifying them are fairly straightforward. In many cases the details are written out in separate files `footnotes`n`.pdf`, where n refers to the unit in question, and a supserscript $^{(\star)}$ denotes a reference to the appropriate file for these details.

I. Further Properties of Simplicial Complexes

Most homology theories for topological spaces can be described using some method of approximating a space X by maps from compact polyhedra into X or maps from X into compact polyhedra. In order to develop such theories, it is necessary to know more about polyhedra and simplicial complexes than we presented in 205B, and accordingly the first unit is devoted to establishing various additional and important facts about simplicial complexes and their (simplicial) homology groups. The first section describes a way of constructing simplicial chains homology that does not require some auxiliary linear ordering of the vertices, and the second shows that every polyhedron in $\mathbb{R}^n$ admits a simplicial decomposition for which the diameters of the simplices are arbitrarily small. In the third section we consider an extremely useful generalization of simplicial complexes called a **finite cell complex** or a **finite CW-complex**, and in Section 4 we prove a fundamentally important result about such complexes known as the *homotopy extension property*, which states that if X is a finite cell complex and $A \subset X$ is a suitably defined subcomplex, then a continuous map f from A to some space Y extends to X if and only if there is a mapping $g : A \to Y$ such that g is homotopic to f and g extends. Finally, in Section 5 we summarize the basic facts about chain homotopies of chain complexes; these objects were defined and studied in the exercises for 205B, but their role in this course is so important that we are restating the main points here.

I.0 : Review

(Hatcher, various sections)

This is a summary of results from Units IV.2–3 from `algtopnotes2012.tex`. At the end of the first part of that course it was clear that algebraic techniques worked very well for spaces called *graphs*. The effectiveness with which such spaces can be studied can be viewed as an example of the following principle:

> Although topological spaces exist in great variety and can exhibit strikingly original properties, the main concern of topology has generally been the study of spaces which are relatively well-behaved.
>
> RS, *Some recent results on topological manifolds*, Amer. Math. Monthly **78** (1971), 941–952.

One goal of `algtopnotes2012.tex` was to define higher dimensional analogs of graphs which can also be studied effectively using algebraic techniques. It turns out that the appropriate generalization involves spaces which, up to homeomorphism, can be built from a class of building blocks called q-dimensional simplices (*sing.* = simplex), where q runs through all nonnegative integers. Spaces which have geometric decompositions of this form were called *polyhedra*, the building blocks were called a *simplicial decomposition*, and the pair of space with decomposition was called a (finite) *simplicial complex*.

The general versions of several key results from vector analysis — namely, Green's Theorem, Stokes' Theorem and the Divergence Theorem — rely heavily on the fact that certain subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$ are nicely homeomorphic to polyhedra; for Green's Theorem, the subsets are regions in the plane with piecewise smooth boundaries, for Stokes' Theorem, the subsets are oriented piecewise smooth surfaces bounded by piecewise smooth curves, and for the Divergence Theorem, the subsets are regions in space whose boundaries are piecewise smooth surfaces (which have outward pointing orientations). It turns out that many important types of topological spaces are homeomorphic to polyhedra; disks and spheres were particularly important examples in 205B. One large and important class of examples is given by the smooth manifolds which are defined and studied in 205C. A proof of this result is given in the second half of [MunkresEDT]. Furthermore, although it is far beyond the scope of the present course to do so, one can also prove that every closed bounded subsets of some $\mathbb{R}^n$ which is *real semialgebraic set* — namely, definable by finitely many real polynomial equations and inequalities — is homeomorphic to a polyhedron. These results combine to show that the class of spaces homeomorphic to polyhedra is broad enough to include many spaces of interest in topology, other branches of mathematics, and even other branches of the sciences. Here is an online reference for the proof of the result on semialgebraic sets and additional background information:

`http://perso.univ-rennes1.fr/michel.coste/polyens/SAG.pdf`

If a space X is homeomorphic to a polyhedron we often say that a *triangulation* of the space consists of a simplicial complex $(P, \mathbf{K})$ and a homeomorphism from P to X.

In Section IV.1 of `algtopnotes2012.tex` we saw that we could recover the isomorphism type of a connected graphs's fundamental group from a purely algebraic construction given by *chain groups*, which are defined in terms of the edges and vertices of the graph. There are analogous algebraic chain groups for simplicial complexes, and one construction for them was given in 205B. There are several motivations for the algebraic definition of boundary homomorphisms which send chains of a given dimension into their boundaries in lower dimensions. For example, in the previously mentioned results from vector analysis the algebraic boundary behaves as follows:

> In Green's Theorem, the boundary takes a suitably oriented sum of all the 2-simplices in the decomposition into a suitably oriented sum of the 1-simplices in the corresponding decomposition of the boundary.
>
> In Stokes's Theorem, the boundary takes a suitably oriented sum of all the 2-simplices in the decomposition into a suitably oriented sum of the 1-simplices in the corresponding decomposition of the boundary.

In the Divergence Theorem, the boundary takes a suitably oriented sum of all the 3-simplices in the decomposition into a suitably oriented sum of the 2-simplices in the corresponding decomposition of the boundary.

In each of the preceding types of examples, it turns out that the algebraic boundaries of the boundary chains are always zero. More generally, this is always the case for the algebraic chains that were defined in 205B for a simplicial complex with respect to a fixed linear ordering of its (finitely many) vertices. Motivated by the 1-dimensional case, one defines a *cycle* to be a chain whose boundary is zero. Since the boundary of a boundary is zero, every boundary chain is automatically a cycle, and one defines **homology groups** to be the quotients of the subgroups of cycles modulo the subgroups of boundaries.

One obvious question with this definition is the reason(s) for wanting to set boundaries equal to zero. Once again vector analysis provides some insight; in some sense the following discussion is not mathematically rigorous because we have not developed all the tools needed to make it complete, but if one does so then all the assertions can be justified. Suppose we have a connected open subset $U \subset \mathbb{R}^3$, and let $\mathbf{F}$ be a smooth vector field defined on U such that its divergence $\nabla \cdot \mathbf{F}$ is zero; this can be viewed as a model for a moving fluid in U which is *incompressible* — the volume around a point neither increases or decreases with the motion — but we do not need this interpretation. Suppose now that we are given two closed surfaces Σ_i in U for $i = 0$ or 1, oriented with suitably defined outward pointing normals. Then we can form the surface integrals of $\mathbf{F} \cdot d\,\Sigma_i$ over the surfaces Σ_i (we shall call these the flux integrals below). Experience suggests that there is a bounded region between these two surfaces if they are disjoint, and in fact one can prove this is always the case. Suppose now that this region is entirely contained in U, so that we can view $\Sigma_0 \cup \Sigma_1$ as the boundary of something in U; if we do this, then for the inner surface the outward pointing normal for the region is the opposite of the usual orientation (think about two concentric spheres). Under these conditions the Divergence Theorem and $\nabla \cdot \mathbf{F} = 0$ imply that the flux integrals of $\mathbf{F}$ over Σ_0 and Σ_1 are the same, for their difference bounds some subregion E of U, and by the divergense theorem the difference of flux integrals is the integral of $\nabla \cdot \mathbf{F} = 0$ over E. So we have the principle that *the flux integrals of two surfaces agree if their difference bounds a region in U.*

The basic identity $d \circ d = 0$ in a simplicial chain complex arises in several contexts, and it is useful to formulate this abstractly as the definition of a *chain complex*. Homology groups given by $H_k :=$ Kernel d_k/Image d_{k+1} can be defined in this generality, and one can prove many useful formal properties. For example, if one defines morphisms of chain complexes in the obvious fashion, then a morphism of chain complexes induces a morphism of homology, and this construction is functorial.

The usefulness of simplicial chain complexes depends upon our ability to compute their homology groups, so the next step is to develop tools for doing so. The boundary homomorphisms in a simplicial chain complex are defined fairly explicitly, and it is not particularly difficult to write a computer program for carrying out the algebraic computations needed to describe simplicial homology groups up to algebraic isomorphism. However, these calculations do not necessarily provide much geometrical insight into the topological structure of a polyhedron, so one also needs further methods which shed more light on such matters.

For example, given a simplicial complex $(P, \mathbf{K})$ and a homology class $u \in H_r(P, \mathbf{K}^\omega)$, one often wants to know if this class is the image of a homology class $u' \in H_r(Q, \mathbf{L}^\omega)$ of some subcomplex $(Q, \mathbf{L}) \subset (P, \mathbf{K})$. For example, if P is a polyhedral region in $\mathbb{R}^3$ and $r = 2$, then one might want to find a 2-dimensional subcomplex with this property; such subcomplexes always exist, but it is often useful to have more specific information. Questions of this sort can often be answered very effectively using *exact sequences of homology groups*. Two types of such sequences were described in 205B, one of which is the *long exact sequence of a pair* consisting of a complex and a subcomplex, and the other of which is the *Mayer-Vietoris exact sequence* which relates the homology of a union of two subcomplexes

$$(P, \mathbf{K}) \quad = \quad (P_1, \mathbf{K}_1) \ \cup \ (P_2, \mathbf{K}_2)$$

to the homology of the subcomplexes $(P_i, \mathbf{K}_i)$ and the homology of the intersection subcomplex $(P_1, \mathbf{K}_1) \cap (P_2, \mathbf{K}_2)$ in much the same way that the Seifert-van Kampen Theorem relates the fundamental group of a union of two open subsets $X = U_1 \cup U_2$ to the fundamental groups of the subspaces U_i and the intersection $U_1 \cap U_2$ provided that all spaces are arcwise connected.

The material discussed thus far can be used very effectively to analyze homology groups of simplicial complexes. However, there is one fundamental point which was not established in 205B:

> TOPOLOGICAL INVARIANCE QUESTION. If P and P' are homeomorphic polyhedra with corresponding simplicial decompositions, are the associated simplicial homology groups isomorphic?

This turns out to be true for graphs because the homology groups are determined by the fundamental groups of the components of the graph, and these fundamental groups of components are isomorphic if the underlying spaces are homeomorphic. For complexes of higher dimension, the problem was avoided by postulating the existence of some construction for homology groups (which we called a *singular homology theory*) which satisfies the topological invariance condition and also has many other important and useful properties. We made this choice for two reasons:

(i) The construction requires a substantial amount of time and effort, and the motivation for many of the steps involves properties of simplicial complexes beyond those introduced in 205B. Historically, it took about 50 years for mathematicians to perfect the now definitive approach to constructing the singular homology groups in Hatcher's book (Poincaré's first papers on the subject appeared in the 1890s, and the Eilenberg-Steenrod approach was completed in the 1940s).

(ii) One of the strongest motivations for such a construction is an understanding of its usefulness, and the last part of 205B was devoted to using homology groups to prove a few topological results — for example, the fact that open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $m \neq n$, the Jordan Curve Theorem which states that a simple closed curve in S^2 separates its complement into two connected components, the Brouwer Fixed Point Theorem, and the fact that certain graphs are not homeomorphic to subsets of $\mathbb{R}^2$. It is often easier to work slowly through some complicated mathematical constructions if their ultimate benefits are understood.

As noted at the beginning of this unit, the first step in constructing a singular homology theory satisfying the axioms in 205B is to formulate and prove results about simplicial complexes that are needed in the construction or are useful in some other respect, and the present unit is devoted to this process. The construction of singular homology will be given in the next unit.

I.1 : Ordered simplicial chains

(Hatcher, § 2.1)

We have already mentioned the topological invariance question, and in fact there is another issue along these lines which is even more basic. The definition of simplicial chains in 205B required the choice of a linear ordering for the vertices, so the first step is to prove that different orderings yield isomorphic homology groups. In order to show this, we have to go back and give alternate definitions of simplicial homology groups which by construction do not involve any choices of vertex orderings. As noted in the 205B notes, this need to redo fundamental definitions frequently is typical of the subject, and it sometimes makes algebraic topology seem like a real-life parody of the film *Groundhog Day* (see `http://www.imdb.com/title/t0107048`).

> Well, it's Groundhog Day ... **again**. ... I was in the Virgin Islands once ... **That** was a pretty good day. Why couldn't I get **that** day over and over and over?
>
> Phil Connors, in the film *Groundhog Day*

Definition. Suppose that $(P, \mathbf{K})$ is a simplicial complex The *unordered simplicial chain group* $C_k(P, \mathbf{K})$ is the free abelian group on all symbols $\mathbf{u}_0 \ \cdots \ \mathbf{u}_k$, where the $\mathbf{u}_j$ are all vertices of some simplex in $\mathbf{K}$ and repetitions of vertices are allowed. A family of differential or boundary homomorphisms d_k is defined as before, and the k-dimensional simplicial homology $H_k(P, \mathbf{K})$ is defined to be the k-dimensional homology of this chain complex.