of $E_{xe} - \{x\}$ must be contained in $\overline{U}$. However, we have already observed that d does not lie in this subset, and therefore we have a contradiction. The problem arises from our assumption that $Y \subset S^2$ is homeomorphic to a complete graph on 5 vertices, and consequently no such subset can exist.■

Kuratowski's Theorem

The results of this section lead to the more general question of determining which connected graphs are not topologically embeddable in $\mathbb{R}^2$. Clearly a graph which contains a subgraph isomorphic to the utilities network or the complete graph on 5 vertices cannot be homeomorphic to a subset of $\mathbb{R}^2$. The end of Section 64 in Munkres mentions a remarkable converse to this result attributed to C. Kuratowski (1896–1980): *Every graph which is not homeomorphic to subset of* $\mathbb{R}^2$ *must contain a subgraph homeomorphic to* either *the utilities network* or *the complete graph on five vertices*. Here is an online reference for the proof:

`http://cs.princeton.edu/~ymakaryc/papers/kuratowski.pdf`

The file `kuratowski.pdf` contains clickable links to other proofs and further information, including independent discoveries of this result by others.

VII.5 : Rationalizations of abelian groups

(**H**, §2.2)

Frequently it is useful to begin the analysis of fundamental groups by considering their abelianizations, which are the corresponding 1-dimensional homology groups; one reason for this is that the structure theory of finitely generated abelian groups is completely understood while the theory of finitely presented — and not necessarily abelian — groups is not (in fact, their are theorems stating that certain basic problems about groups cannot be solved by systemaic recursive processes). Similarly, since the structure and morphism theory of finite-dimensional vector spaces over a field is much simpler than the structure and morphism theory of finitely generated abelian groups, and there are many situations in which it is useful to work with versions of homology theory that take values in some category of vector spaces over some field **k** and **k**-linear transformations. The purpose of this section is to prove the existence of an axiomatic homology theory valued in the category of rational vector spaces. It turns such a theory can be constructed out of a theory valued in the category of abelian groups by purely algeraic means. Accordingly, we being with a method for converting abelian groups into rational vector spaces; the construction is a straightforward generalization of the standard way to construct the rationals out of the integers using formal fractions.

Modules of quotients

Despite the similarity of names, modules of quotients are quite different from quotient modules. In a very precise sense, modules of quotients resemble the field of quotients associated to an integral domain, while quotient modules correspond to quotient rings associated to an integral domain.

The constructions described in these notes can be carried out in far greater generality than the situations we consider, but we specialize here in order to simplify the discussion.

Definition. Let G be an abelian group. The *rationalization* or G, or the *localization of* G *over the rationals* is formed by a construction very similar to the construction of the rationals from the integers. One starts with ordered pairs (g, r) where $g \in G$ and r is a nonzero integer, and one identifies (g, r) with (h, s) if there is a nonzero integer t such that $t(sg - rh) = 0$ (this is slightly stronger than the condition in the construction of $\mathbb{Q}$ from $\mathbb{Z}$, in which t is always 1). This condition defines an equivalence relation on the set of all ordered pairs, and we let $G_{(0)}$ denote the set of equivalence classes. Formally, the class of (g, r) is supposed to represent an object of the form $r^{-1} \cdot g$, and motivated by this we define addition and multiplication by a rational number as follows:

$$[g, r] + [h, s] = [sg + rh, rs] , \qquad pq^{-1}[g, r] = [pg, qr]$$

At this point it is necessary to verify that our definitions of sums and scalar products do not depend upon the choices of representatives for equivalence classes; this is elementary and entirely similar to the corresponding proof for the formal definition of rational numbers in terms of integers. The following result is also elementary:

THEOREM 1. *The object $G_{(0)}$ constructed above is a rational vector space, and the construction also has the following properties:*

(i) *If $g_1, \cdots, g_m$ generate G, then their images under j_G span the rational vector space $G_{(0)}$.*

(ii) *For each abelian group G there is a group homomorphism $j_G : G \to G_{(0)}$ sending $g \in G$ to the equivalence class $[g, 1]$. This map is an isomorphism if G is a rational vector space.*

(iii) *If $f : G \to H$ is a homomorphism then there is an associated linear transformation of rational vector spaces $f_{(0)} : G_{(0)} \to H_{(0)}$ such that the constructions sending an object or morphism Γ to $\Gamma_{(0)}$ define an* ADDITIVE *covariant functor (call it $\mathcal{L}$ for the sake of definiteness) from the category of abelian groups and homomorphisms to the category of rational vector spaces and linear transformations. Furthermore, the maps j_G define a natural transformation from the identity to this functor $\mathcal{L}$.*

(iv) *The construction sends the infinite cyclic group $\mathbb{Z}$ to $\mathbb{Q}$ and it sends every finite cyclic group to* **0**. *Furthermore, for all abelian groups G and H we have $[G \oplus H]_{(0)} \cong G_{(0)} \oplus H_{(0)}$, and likewise for (weak)* infinite *direct sums.*

In particular, if G is a finitely generated abelian group which is the direct sum of β infinite cyclic groups and several finite cyclic groups, then $G_{(0)}$ is a rational vector space whose dimension is equal to β.

Comments on the proof. Most of the verifications are extremely straightforward and left to the reader, so we shall simply note a few key features. First of all, scalar multiplication by a rational number n/m (where $m \neq 0$) is given by

$$(n/m) \cdot [g, r] = [ng, mr]$$

and similarly the mapping $g_{(0)}$ is defined by the formula

$$f_{(0)}[g, r] = [\, f(g),\, r\,] \,.$$

We shall need the second formula for our next result.∎

The following property of the rationalization construction is somewhat less trivial, and it has far-reaching consequences.

THEOREM 2. *The functor $\Gamma \to \Gamma_{(0)}$ sends exact sequences to exact sequences.*

Proof. Every exact sequence is essentially built from short exact sequences; for example, if $A \to B \to C$ is an exact sequence involving $f : A \to B$ and $g : B \to C$, then the sequence is given by fitting together the following sequences:

$$0 \to \mathrm{Ker}(f) \to A \to \mathrm{Image}(f) = \mathrm{Kernel}(g) \to 0$$

$$0 \to \mathrm{Image}(f) = \mathrm{Kernel}(g) \to B \to \mathrm{Image}(g) \to 0$$

$$0 \to \mathrm{Image}(g) \to C \to \mathrm{Cokernel}(g) \to 0$$

Therefore it will be enough to prove the result for short exact sequences. In other words, if $0 \to A \to B \to C \to 0$ is exact, we need to prove the same holds for $0 \to A_{(0)} \to B_{(0)} \to C_{(0)} \to 0$.

We shall only prove that the sequence is exact at the middle object; the proofs at the other two objects are similar and left to the reader. Suppose that $f : A \to B$ is 1–1 and $g : B \to C$ is onto such that the image of f is the kernel of g. Then $g \circ f = 0$ and additivity imply that $g_{(0)} \circ n_{(0)} = 0$, and therefore it follows immediately that the image of $n_{(0)}$ is contained in the kernel of $g_{(0)}$. Suppose now that $[b, t]$ lies in the kernel of $g_{(0)}$. By definitions it follows that there is a nonzero integer s such that $s \cdot g(b) = 0$. By exactness of the original sequence, there is some $a \in A$ such that $f(a) = sb$, and we claim that $n_{(0)}$ maps $[a, st]$ to $[b, t]$. To see this, note that $n_{(0)}[a, st] = [sb, st]$ and the right hand side is equal to to $[b, t]$ because $stb - tsb = 0$.■

The preceding results have the following implication for chain complexes.

COROLLARY 3. *Let (C, d) be a chain complex of abelian groups. Then rationalization defines a chain complex $(\,C_{(0)}, d_{(0)}\,)$ of rational vector spaces, and the homology of this chain complex is isomorphic to the rationalized homology groups $H_*(C)_{(0)}$.*■

Application to homology theories

Since the functor $\Gamma \to \Gamma_{(0)}$ sends exact sequences to exact sequences and sends $\mathbb{Z}$ to $\mathbb{Q}$, we have the following:

Construction of rational simplicial homology. *The rationalized simplicial chain groups $C_*(\mathbf{K}, \mathbf{L})_{(0)}$ form chain complexes whose homology groups are the rationalized simplicial homology groups $H_*(\mathbf{K}, \mathbf{L})_{(0)}$. These rationalized groups have the same exactness and excision properties as ordinary homology groups, and they have the analogous properties except for the results on H_0 of an arcwise connected space except that if $\mathbf{K}$ is starshaped with respect to a vertex then $H_0(\mathbf{K})_{(0)} \cong \mathbb{Q}$.*■

Construction of rational singular homology. *Suppose that we are given the data for an axiomatic singular homology as in Unit* VI. *If we apply the functor $\Gamma \to \Gamma_{(0)}$ to these data, the result is data which satisfy all the axioms of Unit* VI *with the following modifications:*

(i) *In axiom* (D.3), *the statement should be changed as follows:* If X is arcwise connected then $H_0(X)_{(0)} \cong \mathbb{Q}$.

(ii) *Axiom* (D.4) *is excluded.*■

Frequently it is much simpler to work with the rationalized theory because (i) the rationalized homology groups are rational vector spaces, and the isomorphism type of a vector space is given by its dimension, (ii) linear transformations of rational vector spaces are completely determined by their ranks. In particular, this eliminates issues involving nonzero elements of finite order in abelian groups and proper subgroups of finite index. We shall study one application for which this simplification is particularly useful. It turns out that there are many other places in algebraic topology where it is much easier to "work over the rational numbers," and there is an extensive body of work on this topic. Here are two general references:

`http://en.wikipedia.org/wiki/Rational_homotopy_theory`

P. J. Hilton, Serre's contribution to the development of algebraic topology. Expositiones Mathematicæ **22** (2004), 375–383.

VII.6 : Cell decompositions and Euler's Formula

(**H**, Ch. 0, §2.2, Appendix)

In this final section we shall use ideas from homology theory to derive a well-known formula relating the number of vertices, edges and faces of a convex 2-dimensional polyhedron in the sense of elementary solid geometry; in our terminology, this is the boundary of a convex linear cell with a nonempty interior. By the results of `convexbodies.pdf` and `convexbodies2.pdf`, each of these objects is homeomorphic to S^2.

EULER'S POLYHEDRON FORMULA. *Suppose that $P \subset \mathbb{R}^3$ is the boundary of a convex linear cell with a nonempty interior, so that P has a decomposition into closed regions congruent to convex polygonal regions (faces), every pair of which meets in either one common edge or one common vertex. Then the numbers V, E and F of vertices, edges and faces satisfy the equation*

$$V - E + F = 2 .$$

The online site

`http://www.ics.uci.edu/∼eppstein/junkyard/euler/`

contains further information about this result and its history.

Regular cell complexes

We shall need a generalization of the notion of simplicial complex called a *regular cell complex*. Such objects can be described recursively as follows:

Definition. Let k be a nonnegative integer. If (X, A) is a pair of spaces, we shall say that X *is obtained from A by regularly attaching a k-dimensional cell* if there is a 1–1 continuous mapping $f : (D^k, S^{k-1}) \to (X, A)$ such that $X = f[D^k] \cup A$ and $f[S^{k-1}] = f[D^k] \cap A$; this differs from the usual definition of cell attachment because it assumes that the restriction of f to S^{k-1} is 1–1 (see Hatcher for further information and comments).

A *regular cell decomposition* of a compact (Hausdorff) space X is a finite family $\mathcal{E} = \{\mathcal{E}_\alpha$ of closed subsets E_α (called cells) such that

(*i*) each E_α is homeomorphic to a closed disk $D^{k(\alpha)}$ for some nonnegative integer $k(\alpha)$ called the *dimension* of E_α.

(*ii*) for each cell E_α, the boundary set $\partial E_\alpha \cong S^{k(\alpha)-1}$ is a union of cells whose dimensions are less than $k(\alpha)$ called faces of E_α,

(*iii*) for each pair of distinct cells E_α and E_β, the intersection $E_\alpha \cap E_\beta$ is a common face.

The simplicial decomposition of a simplicial complex is a special type of regular cell complex, but there are many other examples, the most obvious of which are convex polygonal regions in the plane and 3-dimensional objects like a solid cube or pyramid, for which there is one 3-dimensional cell and whose boundary cells are the usual concept of face (for example, in the cube these are the six squares on the boundary). More generally, results in [MunkresEDT] and the book by Hudson yield a similar result for convex linear cells.

STANDARD DECOMPOSITION OF CONVEX LINEAR CELLS. *Let E be a convex linear cell in $\mathbb{R}^n$ with nonempty interior. Then E has a regular cell decomposition with exactly one n-dimensional cell such that each cell in the boundary is also a convex linear cell.*

This provides the geometric input that we need; the next step will involve constructions from algebraic topology.

Homology and cell attachment

The next step is to examine the significance of cell attachment in algebraic topology. Here is the main result:

THEOREM 1. *Suppose that the pair (X, A) is obtained by regularly attaching a k-cell to A, and let $D \subset X$ denote the image $f[D^k]$, and let $S \subset X$ denote the image $f[S^{k-1}]$. Then the inclusion of (D, S) in (X, A) induces isomorphisms of homology groups from $H_*(D, S)$ to $H_*(X, A)$.*

Proof. As suggested in the statement of the theorem, let $f : (D^k, S^{k-1}) \to (D, S)$ be the homeomorphism describing the cell attachment. Define subsets F_0 and G_0 of D^k by the inequalities $|v| > \frac{3}{4}$ and $|v| \geq \frac{1}{2}$ respectively (see the drawing in `cell-add.pdf`), let $F = f[F_0]$ and $G = f[G_0]$, and define new subsets $B = A \cup F$, $C = A \cup G$. Observe that C is a closed subset of X and B is an open subset (its complement is the image of the closed disk of radius $\frac{1}{2}$). Furthermore, the closure $\overline{B}$ is contained in the interior of C.

Consider the following commutative diagram:

$$\begin{array}{ccc} H_*(D-F, G-F) & \xrightarrow{p_*} & H_*(D, G) \\ \big\downarrow g_* & & \big\downarrow f_* \\ H_*(X-B, C-B) & \xrightarrow{q_*} & H_*(X, C) \end{array}$$

The mappings p_* and q_* are induced by inclusions, and g is a homeomorphism of pairs given by f. It follows that g_* is an isomorphism, and the excision axiom implies that p_* and q_* are also isomorphisms. Therefore the commutativity of the diagram implies that the map $H_*(X - B, C - B) \to H_*(X, C)$ is also an isomorphism.

Now S is a strong deformation retract of G, and this implies that A is a strong deformation retract of C. We CLAIM that the homology mappings

$$H_*(D, S) \to H_*(D, G) \ , \qquad H_*(X, A) \to H_*(X, C)$$

are also isomorphisms. prove.

Consider the commutative diagrams of long exact homology sequences given by axiom (B.2) and the pair inclusions $(D, S) \to (D, G)$, $(X, A) \to (X, C)$. In the first diagram the inclusion mappings induce homology isomorphisms $H_*(X) \to H_*(X)$ (which are identity maps) and $H_*(S) \to H_*(G)$ (which are isomorphisms since S is a strong deformation retract of G). Since all these maps are isomorphisms, the Five Lemma (Proposition V.3.4) implies the maps $H_*(D, S) \to H_*(D, G)$ are also isomorphisms. Similar considerations imply that the maps $H_*(X, A) \to H_*(X, C)$ are also isomorphisms.

To conclude the proof, consider the following commutative diagram:

$$\begin{array}{ccc} H_*(D,S) & \longrightarrow & H_*(D,G) \\ \big\downarrow & & \big\downarrow \\ H_*(X,A) & \longrightarrow & H_*(X,C) \end{array}$$

The immediately preceding discussion implies that the horizontal arrows are isomorphisms, and we had previously shown that the left hand vertical arrows are isomorphisms; combining these with a diagram chase, we see that the right hand vertical arrows are also isomorphisms.■

This result has a simple but important application to (finite) regular cell complexes.

PROPOSITION 2. *If X has a finite regular cell complex structure with cells of dimension $\leq n$, then $H_q(X)$ is finitely generated for all q and $H_q(X) = 0$ for $q > n$.*

Proof. The definitions imply that X has an increasing finite sequence of subspaces

$$\emptyset \quad = \quad X_{-1} \quad \subset \quad \cdots \quad X_N \quad = \quad X$$

such that each is obtained from the preceding one by attaching a cell, and if $p < q$ all the p-cells are attached before any q-cells are attached. The conclusion of the proposition is trivial for X_{-1}; assume by induction that the conclusion is true for X_{k-1}. We want to prove that the result is also true for X_k; this will suffice to prove the result for $X = X_N$.

The key step is the following observation: *If $A \to B \to C$ is exact and both A and C are finitely generated abelian groups, then B is also a finitely generated abelian group.* — To see this, note that B lies inside a short exact sequence

$$0 \to A' \to B \to C' \to 0$$

where A' is a quotient of A and C' is a subgroup of C. Since subgroups and quotient groups of finitely generated abelian groups are also finitely generated, it follows that A' and C' are finitely generated, and this implies that B must also be finitely generated.

We now apply this to the exact sequences $H_q(X_{k-1}) \to H_q(X_k) \to H_q(X_k, X_{k-1})$. The first group is finitely generated by the induction hypothesis, and the second is finitely generated by Theorem 1, so the preceding paragraph implies that $H_q(X_k)$ is also finitely generated. Furthermore, if $q > n$ then $H_q(X_{k-1}) = 0$ by induction and $H_q(X_k, X_{k-1}) = 0$ by Theorem 1, and these plus exactness imply that $H_q(X_k) = 0$.■

Euler characteristics

At this point we generalize the Euler characteristic of a graph to finite regular cell complexes in two ways; these invariants are integer valued, and we shall show that these are equal.

Definition. Let X be a topological space such that $H_q(X)$ is a finitely generated abelian group for all $q > 0$. The q^{th} *Betti number* $\beta_q(X)$ is equal to the rank of $H_q(X)$; by definition the rank of a finitely generated free abelian group A is equal to the number of infinite cyclic generators in a standard decomposition (this number depends only upon the group)and is equal to the dimension of the rational vector space $A_{(0)}$.

Definition. Suppose that X has a finite regular cell complex structure, so that Proposition 2 applies. The *homological Euler characteristic* $\chi^H(X)$ is defined to be the alternating sum

$$\sum_q (-1)^q \, \beta_q(X) \ .$$

This sum is actually finite because $H_q(X)$ is trivial for all sufficiently large values of q (and is zero if $q < 0$).

Definition. Suppose that X has a finite regular cell complex structure $\mathcal{E}$, and for each nonnegative integer q let $c_q(X, \mathcal{E})$ denote the number of q-cells in $\mathcal{E}$. The *cellular Euler characteristic* $\chi^C(X, \mathcal{E})$ is defined to be the alternating sum

$$\sum_q (-1)^q \, c_q(X, \mathcal{E}) \ .$$

Euler's Polyhedron Formula — and its analogs in higher dimensions — will follow quickly from the next result.

THEOREM 3. *Suppose that X has a finite regular cell complex structure $\mathcal{E}$. Then the two Euler characteristics $\chi^H(X)$ and $\chi^C(X, \mathcal{E})$ are equal.*

Proof. We shall use the same increasing chain of subspaces

$$\emptyset \ = \ X_{-1} \ \subset \ \cdots \ X_N \ = \ X$$

described in the proof of Proposition 2.

Once again, the result is is trivial for X_{-1}, once again we assume by induction that the conclusion is true for X_{k-1}, and once again it will suffice to prove the conclusion of the theorem for X_k.

The crucial steps are to study what happens to both Euler characteristics when one adds a single cell of dimension r to form X_k from X_{k-1}. It is easy to see what happens to the cellular Euler characteristic; since we are adding a single cell in dimension r we have

$$\chi^C(X_{k-1}, \mathcal{E}_{k-1}) \ + \ (-1)^r \ \ = \ \ \chi^C(X_k, \mathcal{E}_k) \ + \ (-1)^r$$

where $\mathcal{E}_j$ denotes the induced cell structure on X_j.

The analysis of the homological Euler characteristic requires a closer examination of the exact homology sequence of the pair (X_k, X_{k-1}) and its rationalization. Since there is only one nonzero homology group of the latter pair and it is in dimension r, it follows that the inclusion map $H_q(X_{k-1}) \to H_q(X_k)$ is an isomorphism if $q \neq r, r-1$, so that $\beta_q(X_{k-1}) = \beta_q(X_k)$ for $q \neq r, r-1$. To compare the remaining two Betti numbers, we need to look at the nontrivial part of the rationalized exact homology sequence:

$$0 \to H_r(X_{k-1})_{(0)} \to H_r(X_k)_{(0)} \to H_r(X_k, X_{k-1})_{(0)} \cong \mathbb{Q} \to H_{r-1}(X_{k-1})_{(0)} \to H_{r-1}(X_k)_{(0)} \to 0$$

There are two cases depending upon whether or not the map ∂ from $\mathbb{Q} = H_r(X_k, X_{k-1})_{(0)}$ to $H_{r-1}(X_{k-1})_{(0)}$ is trivial or nontrivial. If ∂ is trivial then by exactness we have

$$\beta_{r-1}(X_k) \;=\; \beta_{r-1}(X_{k-1})\,, \qquad \beta_r(X_k) \;=\; \beta_r(X_{k-1}) \;+\; 1$$

and thus we also have

$$\chi^H(X_k) \;-\; \chi^H(X_{k-1}) \;=\; (-1)^r \;=\; \chi^C(X_k, \mathcal{E}_k) \;-\; \chi^C(X_{k-1}, \mathcal{E}_{k-1})$$

which combines with $\chi^H(X_{k-1}) = \chi^C(X_{k-1}, \mathcal{E}_{k-1})$ to imply that $\chi^H(X_k) = \chi^C(X_k, \mathcal{E}_k)$. On the other hand, if ∂ is nontrivial then by exactness we have

$$\beta_{r-1}(X_k) \;+\; 1 \;=\; \beta_{r-1}(X_{k-1})\,, \qquad \beta_r(X_k) \;=\; \beta_r(X_{k-1})$$

and thus we also have

$$\chi^H(X_{k-1}) \;-\; \chi^H(X_k) \;=\; (-1)^{r-1} \;=\; \chi^C(X_{k-1}, \mathcal{E}_{k-1}) \;-\; \chi^C(X_k, \mathcal{E}_k)$$

which combines with $\chi_H(X_{k-1}) = \chi^C(X_{k-1}, \mathcal{E}_{k-1})$ to imply that $\chi^H(X_k) = \chi^C(X_k, \mathcal{E}_k)$.■

Euler's Polyhedral Formula now follows as the specialization of the final result to $n = 2$.

COROLLARY 4. *Let P be the boundary of a convex linear cell in $\mathbb{R}^{n+1}$ with nonempty interior, and for each integer k between 0 and n let V_k denote the number of k-dimensional faces in the standard regular cell decomposition. Then the alternating sum*

$$\sum_k \, (-1)^k \, V_k$$

is 0 if k is odd and 2 if k is even.

Proof. The homological Euler characteristic of P is $1 + (-1)^n$ which is 0 if k is odd and 2 if k is even. Therefore the result follows from Theorem 3.■