I.3 : Abstract cell complexes

(Hatcher, Ch. 0)

One possible way to view a polyhedron is to think of it as an object that is constructible in a finite number of steps as follows:

(0) Start with the finite set P_0 of vertices,

(n) If P_{n-1} is the partial polyhedron constructed at Step $(n-1)$, at Step (n) one adds finitely many simplices S_j, identifying each face of each simplex S_j with a simplex in P_{n-1}.

In fact, one can do this in order of increasing dimension, attaching all 1-simplices to the vertices at Step 1, then attaching 2-simplices along the boundary faces at Step 2, and so on. It is often useful in topology to consider objects that are generalizations of this procedure that are more flexible in certain key respects. The objects used these days in algebraic topology are known as **cell complexes**.

One immediate difference between cell complexes and simplicial complexes is that the former use the closed unit disk $D^n \subset \mathbb{R}^n$ and its boundary S^{n-1} in place of an n-simplex Δ and its boundary $\partial\Delta_n$. Since the results of pages 84–85 in `algtop-notes.pdf` (in particular, Theorem VII.1.1) imply that D^n is homeomorphic to Δ_n such that S^{n-1} corresponds to $\partial\Delta_n$, it follows that one can view simplicial complexes as special cases of cell complexes.

Adjoining cells to a space

We shall now give the basic step in the construction of cell complexes. The discussion below relies heavily on the material in Unit V of the online Mathematics 205A notes that were previously cited.

Definition. Let X be a compact Hausdorff space and let A be a closed subset of X. If k is a nonnegative integer, we shall say that *the space X is obtained from A by adjoining finitely many k-cells* if there are continuous mappings $f_i : S^{k-1} \to A$ for $i = 1, \cdots, n$ such that X is homeomorphic to the quotient space of the topological disjoint union

$$A \coprod \left(\{1, \cdots, N\} \times D^k \right)$$

modulo the equivalence relation generated by identifying $(j, \mathbf{x}) \in \{j\} \times S^{k-1}$ with $f_j(\mathbf{x}) \in A$, where the homeomorphism maps $A \subset X$ to the image of A in the quotient by the canonical mapping.

By construction, there is a 1–1 correspondence of sets between X and

$$A \coprod \left(\{1, \cdots, N\} \times \mathbf{open}(D^k) \right)$$

where $\mathbf{open}(D^k) \subset D^k$ is the complement of the boundary sphere. The set $E_j \subset X$ corresponding to the image of $\{j\} \times D^k$ in the quotient is called a *(closed) k-cell*, and the subset $E_j^{\mathbf{O}}$ corresponding to the image of $\{j\} \times \mathbf{open}(D^k)$ in the quotient is called an *open k-cell*. One can then restate the observation in the first sentence of the paragraph to say that X is a union of A and the open k-cells, and these subsets are pairwise disjoint.

Before discussing some topological properties of a space obtained by adjoining k-cells, we shall consider some special cases.

Example 1. Let $(P, \mathbf{K})$ be a simplicial complex,let P_k be the union of all k-simplices in $\mathbf{K}$, and let P_{k-1} be defined similarly. Then the whole point of stating and proving Theorem 1 was to justify an assertion that P_k is obtained from P_{k-1} by attaching k-cells, one for each k-simplex in $\mathbf{K}$. Specifically, for each k-simplex A the map f_A is given by the composite of the homeomorphism $S^{k-1} \to \partial A$ with the inclusion $\partial A \subset P_{k-1}$. The homeomorphism from the quotient of the disjoint union to P_k is given by starting with the composite

$$P_{k-1} \coprod \left(\{1, \cdots, N\} \times D^k \right) \longrightarrow P_{k-1} \amalg_{\partial A} A \longrightarrow P_k$$

where $\amalg_A$ runs over all the k-simplices of $\mathbf{K}$, the first map is a disjoint union of homeomorphisms on the pieces where the maps of Theorem 1 are used to define the homeomorphisms $\{j\} \times D^k \cong A$, and the second map is inclusion on each disjoint summand. This composite passes to a map of the quotient of the space on the left modulo the equivalence relation described above, and it is straightforward to show this map is 1–1 onto and hence a homeomorphism (all relevant spaces are compact Hausdorff).

Example 2. (GRAPHS) As in Section 64 of Munkres, one may define a finite (vertex-edge) graph to be a space obtained from a finite discrete space by adjoining 1-cells. Frequently there is an added condition that the attaching maps for the boundaries should be 1–1 (so that each 1-cell has two endpoints), and the weaker notion introduced in `algtop-notes.pdf` (and Hatcher) is then called a *pseudograph*. The graph corresponds to a simplicial decomposition of a simplicial complex if and only if different 1-cells have different endpoints, and the simplest example of a graph structure that does not come from a simplicial complex is given by taking $X = S^1$ and $A = S^0$ with two 1-cells corresponding to the upper and lower semicircles $E^1_\pm$ in the complex plane. The attaching maps are defined to map the endpoints of $D^1 = [-1, 1]$ bijectively to $-1, 1$. — Another example that is historically noteworthy is the Königsberg Bridge Graph, in which the vertices correspond to four land masses in the city of Königsberg (now Kaliningrad, Russia) and the 1-cells (or *edges*) correspond to the bridges which joined pairs of land masses in the 18th century (see Figure ??? in `advnotesfigures.pdf` for a drawing). This is another example of a graph that does not come from a simplicial complex but is not a pseudograph; if there are two bridges joining the same pairs of land masses, then the graph has two edges with the same boundary points.

Example 3. Yet another example is given by S^n, which is homeomorphic to the quotient D^n / S^{n-1} obtained by identifying all points in the boundary to a single point. An explicit attachment map is given by the continuous onto mapping sending $x \in D^n$ to

$$\left(\frac{x}{2\sqrt{|x| - |x|^2]}}, \; 2|x| - 1 \right) \; ;$$

checking that the first coordinate function is continuous at $x = \mathbf{0}$ and $|x| = 1$ with limits equal to $\mathbf{0}$ is a straightforward exercise (look at the limits as $t \to 0$ and $t \to 1^-$, where t replaces $|x|$ and $\pm t$ replaces x). In these examples the attaching maps are constant, which is the complete opposite of being 1–1 for spaces containing more than a single point.

We shall encounter further examples of adjoining cells after we define the main concept of this section. For the time being, we mention a few simple properties of spaces obtained by attaching k-cells for some k

PROPOSITION 2. *If X is obtained from A by attaching 0-cells, then X is homeomorphic to the disjoint union of A with a finite discrete space.*

This is true because the 0-disk D^0 has an empty unit sphere, so there are no attaching maps and the equivalence relation on the space $A \amalg \{1, \; \cdot \;, N\}$ is the equality relation.■

PROPOSITION 3. *If X is obtained from A by attaching k-cells, then each open cell $E_j^{\mathbf{O}}$ is an open subset of X, and each such open cell is homeomorphic to* $\mathbf{open}(D^k)$.

Proof. Each closed cell is compact because it is a continuous image of D^k, and hence each such subset is closed in X. By the set-theoretic description given above, the open cell $E_j^{\mathbf{O}}$ is just the complement of the closed set

$$A \; \cup \; \bigcup_{i \neq j} \; E_i$$

and hence it is open in X. Since the quotient space map from the disjoint union to X defines a 1–1 onto continuous mapping from $\mathbf{open}(D^k)$ to $E_j^{\mathbf{O}}$, it suffices to show that an open subset of $\mathbf{open}(D^k)$ is sent to an open subset of $E_j^{\mathbf{O}}$. Let

$$\varphi : A \coprod \left(\{1, \; \cdots \;, N\} \times D^k \right) \; \longrightarrow \; X$$

be the continuous onto quotient map corresponding to the cell attachments, and suppose that U is open in $\{j\} \times \mathbf{open}(D^k)$. By construction we then have

$$U \quad = \quad \varphi^{-1}\left[\, \varphi[U] \,\right]$$

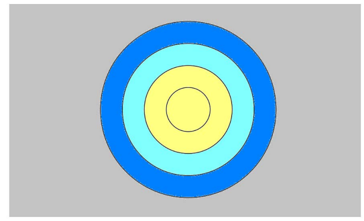
and thus $\varphi[U]$ is open in X by the definition of the quotient topology.■

The last result in this subsection implies that the inclusion of A in X is homotopically well-behaved if X is obtained from A by adjoining k-cells.

PROPOSITION 4. *If X is obtained from A by attaching k-cells and U is an open subset of X containing A, then there is an open subset V such that*

$$A \;\; \subset \;\; V \subset \;\; \overline{V} \;\; \subset U$$

and A is a strong deformation retract of both V and $\overline{V}$.



The case $N = 1$ is illustrated at the right.

Proof. As in the preceding argument, take

$$\varphi : A \coprod \left(\{1, \cdots, N\} \times D^k \right) \longrightarrow X$$

to be the continuous onto map corresponding to the k-cell attachments.

Let $F = X - U$, and let $F_0 = \varphi^{-1}[F]$, so that F_0 corresponds to a disjoint union $\amalg_j F_j$, where each F_j is a compact subset of $\mathbf{open}(D^k)$; compactness follows because the image of each F_j in X is a closed subset of the compact k-cell E_j. Therefore we can find constants c_j such that $0 < c_j < 1$ and F_j is contained in the open disk of radius c_j about the origin in $\{j\} \times D^k$; let c be the maximum of the numbers c_j, and let $V \subset X$ be the image under φ of the set

$$W \quad = \quad A \coprod \left(\bigcup_j \{j\} \times \{ \ \mathbf{x} \in D^k \ | \ c < |\mathbf{x}| \leq 1 \ \} \right) \ .$$

Then V is open because it is the complement of a compact set, and it follows that $\overline{V}$ is the image of

$$Y \quad = \quad A \coprod \left(\bigcup_j \{j\} \times \{ \ \mathbf{x} \in D^k \ | \ c \leq |\mathbf{x}| \leq 1 \ \} \right) \ .$$

Each of the sets W and Y is a strong deformation retract of

$$B \quad = \quad A \coprod \left(\bigcup_j \{j\} \times S^{k-1} \right) \ .$$

Specifically, the homotopies deforming W and Y into B are the identity on A and map each of the sets $\{ \ c < |\mathbf{x}| \leq 1 \ \}$, $\{ \ c \leq |\mathbf{x}| \leq 1 \ \}$ to S^{k-1} by sending a (necessarily nonzero) vector $\mathbf{y}$ to $|\mathbf{y}|^{-1}\mathbf{y}$ and taking a staight line homotopy to join these two points. A direct check of the equivalence relation defining φ shows that the associated maps and homotopies $W \to B \to W$ and $Y \to B \to Y$ pass to the quotients $V \to A \to V$ and $\overline{V} \to A \to \overline{V}$, and these quotient maps display A as a strong deformation retract of both V and $\overline{V}$.■

Cell complex structures

By the preceding discussion, a simplicial complex $(P, \mathbf{K})$ has a finite, linearly ordered chain of closed subspaces

$$\emptyset \quad = \quad P_{-1} \quad \subset \quad P_0 \quad \subset \quad \cdots \quad \subset \quad P_m \quad = \quad P$$

such that for each k satisfying $0 \leq k \leq m$, the subspace P_k is obtained from P_{k-1} by attaching finitely many k-cells. We shall generalize this property into a definition for arbitrary cell complex structures.

Definition. Let X be a topological space. A *finite cell complex structure* (or finite *CW* structure) on X is a chain $\mathcal{E}$ of closed subspaces

$$\emptyset \quad = \quad X_{-1} \quad \subset \quad X_0 \quad \subset \quad \cdots \quad \subset \quad X_m \quad = \quad X$$

such that for each k satisfying $0 \leq k \leq m$, the subspace X_k is obtained from X_{k-1} by attaching finitely many k-cells. The subspace X_k is called the *k-skeleton* of X, or more correctly the k-skeleton of $(X, \mathcal{E})$

At this level of abstraction, the notion of cell complex structure is due to J. H. C. Whitehead (1904–1960); his definition extended to infinite cell complex structures and the letters *CW* were described as abbreviations for two properties of the infinite complexes that are explained in the Appendix of Hatcher's book, but one should also note that the letters also represent Whitehead's last two initials.

It follows immediately that simplicial complexes are examples of cell complexes. Numerous further examples appear on pages 5–8 of Hatcher. Furthermore, the Δ-complexes discussed on pages 102–104 are also examples of cell complexes. In analogy with (edge-vertex) graphs, the main difference between Δ-complexes and simplicial complexes is that two k-simplices in a Δ-complex may have the same faces, but two k-simplices in a simplicial complex have at most a single $(k-1)$-face in common.

Because of the following result, one often describes a cell complex structure as a *cellular decomposition* of X.

PROPOSITION 5. *If X is a space and $\mathcal{E}$ is a cell decomposition of X, then every point of X lies on exactly one open cell of X.*

Proof. Since $X = \cup_k \, (X_k - X_{k-1})$, it follows that every point $y \in X$ lies in a exactly subset of the form $X_k - X_{k-1}$. Therefore there is at most one value of k such that x can lie on an open k-cell. Furthermore, since $X_k - X_{k-1}$ is a union of the open k-cells and the latter are pairwise disjoint, it follows that x lies on exactly one of these open k-cells.■

NOTE. If a cell complex has an n-cell for some $n > 0$ and $0 < m < n$, the cell complex might not have any m-cells (in contrast to the situation for, say, simplicial complexes); see Example 0.3 on page 6 of Hatcher.

Finally, we shall give a slightly different definition of subcomplex than the one in Hatcher.

Definition. If $(X, \mathcal{E})$ is a cell complex, we say that a closed subspace $A \subset X$ determines a *cell subcomplex* if for each $k \geq 0$ the set $A_k = X_k \cap A$ is obtained from A_{k-1} by attaching k-cells such that the every k-cell for A is also a k-cell for X.

There is an simple relationship between this notion of cell subcomplex and the previous definition of subcomplex for a simplicial complex; the proof is straightforward.

PROPOSITION 6. *If $(P, \mathbf{K})$ is a simplicial complex and $(P_1, \mathbf{K}_1)$ is a simplicial subcomplex, then P_1 also determines a cell subcomplex.*■

Finally, here are two further observations regarding subcomplexes. Again, the proofs are straightforward.

PROPOSITION 7. *If X is a cell complex such that $A \subset X$ determines a subcomplex of X and $B \subset A$ determines a subcomplex of A, then B also determines a subcomplex of X. Likewise, if B determines a subcomplex of X then B determines a subcomplex of A.*■

PROPOSITION 8. *If X is a cell complex such that $A \subset X$ determines a subcomplex of X, then for each $k \geq 0$ the set $X_k \cup A$ determines a subcomplex of X.*■

Cellular homology

If P is a polyhedron of positive dimension, the preceding discussion implies that the singular homology groups of P are finitely generated abelian groups. In fact, the conclusion holds more generally if X has the structure of a finite cell complex by the following result:

THEOREM 9. *Let $(X, \mathcal{E})$ be a finite cell complex of dimension n. Then there is a chain complex $\big(C_*(X, \mathcal{E}), d\big)$ such that the chain groups are finitely generated free abelian in every dimension with $C_q(X, \mathcal{E}) = 0$ if $q < 0$ or $q > n$, and the q-dimensional homology of this chain complex is isomorphic to the singular homology group $H_q(X)$.*

The chain complex will be defined explicitly in terms of singular homology and the cell structure for $(X, \mathcal{E})$, and it will be called the *cellular chain complex*. For each k such that $-1 \leq k \leq n$, let X_k denote the k-skeleton of X, where $X_{-1} = \emptyset$. Specifically, we set $C_q(X, \mathcal{E}) = H_q(X_q, X_{q-1})$ and define the differential d_q to be the following composite:

$$H_q(X_q, X_{q-1}) \xrightarrow{\partial[q]} H_{q-1}(X_{q-1}) \xrightarrow{j[q-1]_*} H_{q-1}(X_{q-1}, X_{q-2})$$

These maps define a chain complex since

$$d_{q-1} \circ d_q \quad = \quad j[q-2]_* \circ \partial[q-1] \circ j[q-1]_* \circ \partial[q]$$

and $\partial[q-1] \circ j[q-1]_* = 0$ because the factors are consecutive morphisms in the long exact homology sequence for (X_{q-1}, X_{q-2}). By the results of the preceding section, *the q-dimensional cellular chain group is isomorphic to a free abelian group on the set of q-cells in $\mathcal{E}$.*

Proof of Theorem 9. The result is immediate if $\dim X = 0$ or -1, in which cases X is a nonempty finite set or the empty set. In this case the cellular chain groups are either concentrated in degree zero (the 0-dimensional case) or are all equal to zero (the (-1)-dimensional case).

We shall prove the result for the explicit cellular chain complex described above by induction on $\dim X$, and for this purpose we assume that the result is true when $\dim X \leq n-1$. The inductive hypothesis then implies that the theorem is true for the $(n-1)$-skeleton X_{n-1}. Now the only difference between the cellular chain complex for X and the corresponding complex for X_{n-1} is that the n-dimensional chain group for the latter is zero while the n-dimensional chain group for the latter is nonzero, and likewise the differentials in both complexes are equal except for the ones going from n-chains to $(n-1)$-chains (in the second case the differential must be zero). It follows that the homology groups of these cell complexes are isomorphic except perhaps in dimensions n and $n-1$.

Similarly, since $H_q(X_n, X_{n-1}) = 0$ if $q \neq n$ or $n-1$, it follows that $H_q(X) \cong H_q(X_{n-1})$ except perhaps in these dimensions. Therefore, we have shown the inductive step except when $q = n$ or $n-1$. It will be necessary to examine these cases more closely.

We shall describe the n-dimensional homology of $C_*(X, \mathcal{E})$ first. By definition the map d_n is a composite $j[q-1]_* \circ \partial[q]_*$, and the factors fit into the following long exact sequences:

$$0 \;=\; H_n(X_{n-1}) \;\longrightarrow\; H_n(X) \;\longrightarrow\; H_n(X, X_{n-1}) \;\longrightarrow\; H_{n-1}(X_{n-1}) \;\cdots$$

$$0 \;=\; H_{n-1}(X_{n-2}) \;\longrightarrow\; H_{n-1}(X_{n-1}) \;\longrightarrow\; H_{n-1}(X_{n-1}X_{n-2})$$

It follows that $H_n(X)$ is isomorphic to the kernel of $\partial[q]_*$ and the map $j[q-1]_*$ is injective. Similarly, it also follows that $H_{n-1}(X)$ is isomorphic to the kernel of $\partial[q-1]_*$ and the map $j[q-2]_*$ is injective. Since $d_q = j[q-1]_* \circ \partial[q]$, it follows that $H_n(X)$ is also isomorphic to the kernel of d_n, and since $C_{n+1}(X, \mathcal{E}) = 0$ it follows that the kernel of d_n is also isomorphic to the n-dimensional homology of $C_*(X, \mathcal{E})$. Thus we now know the theorem is true for all dimensions except possibly $(n-1)$.

In order to describe the $(n-1)$-dimensional homology of $C_*(X, \mathcal{E})$ we shall consider the following diagram, in which both the row and the column are exact:

$$\begin{array}{ccccccc} & & H_{n-1}(X_{n-2}) = 0 & & & & \\ & & \big\downarrow & & & & \\ \cdots\; H_n(X, X_{n-1}) & \xrightarrow{\partial[n]} & H_{n-1}(X_{n-1}) & \longrightarrow & H_{n-1}(X) & \longrightarrow & H_{n-1}(X, X_{n-1}) = 0 \\ & & \big\downarrow j[n-1]_* & & & & \\ & & H_{n-1}(X_{n-1}, X_{n-2}) & & & & \end{array}$$

By the exactness of the row we know that $H_{n-1}(X)$ is isomorphic to the quotient group

$$H_{n-1}(X_{n-1}) \,/\, \text{Image } \partial[n]$$

and since $j[n-1]_*$ is injective we know from the previous discussion that $j[n-1]_*$ sends $H_{n-1}(X_{n-1})$ onto the kernel of d_{n-1} (note this map is the same for both X and X_{n-1}). Furthermore, by construction we also know that $j[n-1]_*$ maps the image of $\partial[n]$ onto the image of d_n. If we make these substitutions into the displayed expression above, we see that $H_{n-1}(X)$ is isomorphic to the kernel of d_{n-1} modulo the image of d_n, which proves that the conclusion of the theorem also holds in dimension $n-1$.■

If we let $\mathcal{C}(q) = \{ E^q_\alpha \}$ denote the (finite) set of q-cells for $\mathcal{E}$ and view the cellular chain groups $C_q(X, \mathcal{E})$ as free abelian groups on the sets $\mathcal{C}(q)$ by the preceding construction and result, it follows that for each E^q_α we have

$$d_q \,(E^q_\alpha) \quad = \quad \sum_{\mathcal{C}(q-1)} [\alpha : \beta] \, E^{q-1}_\beta$$

for suitable integers $[\alpha : \beta]$; classically, these coefficients were called *incidence numbers*. Unlike the situation for simplicial chain complexes, there are no general formulas for finding these numbers. If we already know the homology of X from some other result, then it is often possible to recover them by working backwards (*i.e.*, if we know the homology then often there are not many possibilities for the incidence numbers which will yield the correct homology groups).

One condition under which the incidence numbers are recursively computable is if the cell complex is a **regular cell complex**; in other words, each closed n-cell is in fact homeomorphic to to D^n via the attaching map and is a subcomplex in the evident sense of the word (the boundary is a union of cells in the big complex). These will be true for the cell complexes considered in the next subheading.

Here is a very brief summary of the recursive process: Suppose we have worked out the differentials for the chain complex through dimension $n - 1$, and we want to find the differentials in dimension n. Let E be an n-cell; by definition, E determines a cell complex which has the homology of a disk. Let ∂E be the subcomplex given by the boundary, so that we have the incidence numbers on ∂E already. It is only necessary to figure out the map from $\mathbb{Z} = C_n(E)$ to $C_{n-1}(E)$. Now the homology of ∂E is just the homology of S^{n-1}, and since $C_n(\partial E) = 0$ it follows that there are no nontrivial boundaries in $C_{n-1}(\partial E)$, so that $H_{n-1}(\partial E) \cong Z$ may be viewed as a subgroup A of $C_{n-1}(\partial E) = C_{n-1}(E)$. Now the image of this copy of $\mathbb{Z}$ in $C_{n-1}(E)$ represents zero in homology since $H_{n-1}(E) = 0$, and therefore there must be some element in $C_n(E)$ which maps to a generator of A. Since $C_n(E)$ is infinite cyclic, it follows that some multiple of the generator $[E]$ for $C_n(E)$ must map to the generator of A. Let $a \in A$ be the generator such that $d(k[E]) = a$; then it follows that $a = k\, d([E])$. But since $d([E])$ is also a cycle, it follows that $d([E]) = m\, a$ for some integer m. Combining these, we see that $a = km\, a$, and since A is torsion free this implies that $km = 1$, so that $k = m = \pm 1$. Thus we must have $d([E]) = \pm\, a$. the generator of $C_n(E)$. In fact, the exact choice for the sign is unimportant because one obtains the same homology in all cases; we can always choose the generator for $C_n(E)$ so that the incidence number is $+1$. More detailed information is given in the following reference:

> **G. E. Cooke and R. L. Finney.** *Homology of cell complexes (Based on lectures by N. E. Steenrod), Princeton Mathematical Notes No.* 4. *Princeton University Press, Princeton,* 1967.

Convex linear cells

In elementary geometry, the terms *polygon* and *polyhedron* are often used to denote frontiers of bounded open sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ that are defined by finitely many linear equations and inequalities. For example, one has the standard isosceles right triangle in the plane which bounds the compact convex set defined by the inequalities

$$x \;\geq\; 0\,, \qquad y \;\geq\; 0\,, \qquad x + y \;\leq\; 1$$

while standard squares and cubes in the plane and 3-space are defined by

$$0 \;\leq\; x,\, y \;\leq 1\,, \qquad 0 \;\leq\; x,\, y,\, z \;\leq\; 1$$

and the octagon in the plane with vertices

$$(2, \pm 1),\ (-2, \pm 1),\ (1, \pm 2),\ (-1, \pm 2)$$

is defined by the eight inequalities

$$-2 \;\leq\; x,\, y \;\leq\; 2\,, \qquad -3 \;\leq\; x + y \;\leq\; 3\,, \qquad -3 \;\leq\; x - y \;\leq\; 3\,.$$

Convex sets in $\mathbb{R}^n$ defined by finitely many linear equations and inequalities are basic objects of study in the usual theory of linear programming. In particular, it turns out that the sorts of sets we consider are given by all convex combinations of a finite subset of *extreme points* which correspond to the usual geometric notion of vertices. The reference below is the text for Mathematics 120, which covers linear programming and provides some background on the sets considered here, (particularly in Sections 15.4 – 15.8 on pages 264 – 285).

E. K. P. Chong and S. Zak. An Introduction to Optimization. Wiley, New York, 2001. ISBN: 0-471-39126-3.

We defined convex linear cells in Section I.2; recall that a bounded subset $E \subset \mathbb{R}^n$ is a *convex linear cell* (or also as a *rectilinear cell*) if it is defined by finitely many linear equations and inequalities. It follows immediately that such a set is compact and convex.

The main properties of such cells that we shall need are formulated and proved in Section 7 of [MunkresEDT]. Here is a summary of what we need: If we define a k-plane in a real vector space V to be a set of the form $\mathbf{x} + W$, where W is a k-dimensional vector subspace of V, then the *dimension* of a convex linear cell E is equal to the least k such that E lies in a k-plane. If V is an n-dimensional vector space, this dimension is a nonnegative integer which is less than or equal to n. Suppose now that E is k-dimensional in this sense and $\mathbf{P} = \mathbf{x} + W$ is a k-plane containing E; it follows fairly directly that $\mathbf{P}$ is the unique such k-plane. Less obvious is the fact that the interior of E with respect to $\mathbf{P}$ is nonempty.

[For the sake of completeness, here is a sketch of the proof: The cell E must contain a set of $k+1$ points that are affinely independent, for otherwise it would lie in a $(k-1)$-plane. Since a convex linear cell is a closed convex set, it must contain the k-simplex whose vertices are these points, and this set has a nonempty interior in the k-plane $\mathbf{P}$.]

It is convenient to describe a minimal and irredundant set of equations and inequalities which define a convex linear cell E. The unique minimal k-plane containing E can be defined as the set of solutions to a system of $n - k$ independent linear equations, and to describe E it is enough to add a MINIMAL set of inequalities which define E.

Definition. If E is a k-dimensional convex linear cell and we are given an efficient set of defining linear equations and inequalities as in the preceding paragraph, then a $(k-1)$-*dimensional face* of E is obtained by taking the subset for which one of the listed inequalities is replaced by an equation.

For example, in the square the four faces are given by adding one of the four conditions

$$x = 0, \qquad x = 1, \qquad y = 0, \qquad y = 1$$

to the equations and inequalities defining the square, and for the 2-simplex whose vertices are $(0,0)$, $(1,0)$ and $(0,1)$ one has the three faces defined by strengthening one of the defining inequalities to one of the three equations $x = 0$, $y = 0$ or $x + y = 1$.

It follows immediately that each $(k-1)$-face of E is a convex linear cell, and Lemmas 7.3 and 7.5 on pages 72 – 74 of [MunkresEDT] show that each face described in this manner is $(k-1)$-dimensional. — One can iterate the process of taking faces and define q-faces of E where $-1 \leq q \leq k$; more details appear on page 75 of the book by Munkres (by definition, the empty set is a (-1)-face).

The geometric boundary of E, written $\mathbf{Bdy}(E)$, may be described in two equivalent ways: It is the union of all the lower dimensional faces ofr E, and it is also the point set theoretic frontier of E in $\mathbf{P}$. We shall need the following theorem, which is discussed on pages 71 – 74 of the Munkres book:

PROPOSITION 10. *If $E \subset \mathbb{R}^n$ is a convex linear cell, then the pair $\big(E, \mathbf{Bdy}(E)\big)$ is homeomorphic to (D^k, S^{k-1}).*

We have already shown this result when E is a simplex by constructing a *radial projection homeomorphism*, and as noted on page 71 of Munkres' book a similar construction proves the corresponding result for an arbitrary convex linear k-cell.■

If we combine this proposition with the remaining material on convex linear cells, we obtain the following basic consequence.

PROPOSITION 11. *If E is a convex linear k-cell and $\mathbf{Bdy}(E)$ is its boundary, then these spaces have cell decompositions such that (i) the cells of $\mathbf{Bdy}(E)$ are the faces of dimension less than k, (ii) the cells of E are the cells of $\mathbf{Bdy}(E)$ together with E itself.*■

If we combine the preceding result with Theorem 3, we obtain the following conclusion relating the geometry and algebraic topology of E and its boundary.

COROLLARY 12. *If E and $\mathbf{Bdy}(E)$ are as above, then there exist chain complexes A_* and B_* such the groups A_q are free abelian groups on the sets of nonempty faces of dimension less than k, the groups B_q are free abelian groups on the sets of nonempty faces of dimension $\leq k$, and the homology groups of A_* and B_* are isomorphic to $H_*(S^{k-1})$ and $H_*(D^k)$ respectively.*■