to prove a few topological results — for example, the fact that open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $m \neq n$, the Jordan Curve Theorem which states that a simple closed curve in S^2 separates its complement into two connected components, the Brouwer Fixed Point Theorem, and the fact that certain graphs are not homeomorphic to subsets of $\mathbb{R}^2$. It is often easier to work slowly through some complicated mathematical constructions if their ultimate benefits are understood.

As noted at the beginning of this unit, the first step in constructing a singular homology theory satisfying the axioms in 205B is to formulate and prove results about simplicial complexes that are needed in the construction or are useful in some other respect, and the present unit is devoted to this process. The construction of singular homology will be given in the next unit.

I.1 : Ordered simplicial chains

(Hatcher, § 2.1)

We have already mentioned the topological invariance question, and in fact there is another issue along these lines which is even more basic. The definition of simplicial chains in 205B required the choice of a linear ordering for the vertices, so the first step is to prove that different orderings yield isomorphic homology groups. In order to show this, we have to go back and give alternate definitions of simplicial homology groups which by construction do not involve any choices of vertex orderings. As noted in the 205B notes, this need to redo fundamental definitions frequently is typical of the subject, and it sometimes makes algebraic topology seem like a real-life parody of the film *Groundhog Day* (see `http://www.imdb.com/title/t0107048`).

> Well, it's Groundhog Day ... **again**. ... I was in the Virgin Islands once ... **That** was a pretty good day. Why couldn't I get **that** day over and over and over?
>
> Phil Connors, in the film *Groundhog Day*

Definition. Suppose that $(P, \mathbf{K})$ is a simplicial complex The *unordered simplicial chain group* $C_k(P, \mathbf{K})$ is the free abelian group on all symbols $\mathbf{u}_0 \ \cdots \ \mathbf{u}_k$, where the $\mathbf{u}_j$ are all vertices of some simplex in $\mathbf{K}$ and repetitions of vertices are allowed. A family of differential or boundary homomorphisms d_k is defined as before, and the k-dimensional simplicial homology $H_k(P, \mathbf{K})$ is defined to be the k-dimensional homology of this chain complex.

If ω is a linear ordering for the vertices of $\mathbf{K}$, then the unordered simplicial chain complex $C_*(P, \mathbf{K})$ contains the ordered simplicial chain complex $C_*(P, \mathbf{K}^\omega)$ as a chain subcomplex, and we shall let i denote the resulting inclusion map of chain complexes. If we can show that the associated homology maps i_* are isomorphisms, then it will follow that the homology groups for the ordered simplicial chain complex agree with the corresponding groups for the unordered simplicial chain complex, and therefore the homology groups do not depend upon choosing a linear ordering of the vertices.

One major difference between the unordered and ordered simplicial chain groups is that the latter are nontrivial in every positive dimension. In particular, if $\mathbf{v}$ is a vertex of $\mathbf{K}$, then the free generator $\mathbf{v} \ \cdots \ \mathbf{v} = \mathbf{u}_0 \ \cdots \ \mathbf{u}_k$, with $\mathbf{u}_j = \mathbf{v}$ for all j, represents a nonzero element of $C_k(P, \mathbf{K})$. On the other hand, the ordered simplicial chain groups are nonzero for only finitely many values of k.

In order to analyze the mappings i_*, we shall introduce yet another definition of homology groups.

Third Definition. In the setting above, define the subgroup $C'_k(P, \mathbf{K})$ of *degenerate* simplicial k-chains to be the subgroup generated by

(*a*) all elements $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$ such that $\mathbf{v}_i = \mathbf{v}_{i+1}$ for some (at least one) i,

(*b*) all sums $\mathbf{v}_0 \cdots \mathbf{v}_i \mathbf{v}_{i+1} \cdots \mathbf{v}_k + \mathbf{v}_0 \cdots \mathbf{v}_{i+1} \mathbf{v}_i \cdots \mathbf{v}_k$, where $0 \leq i < k$.

We claim these subgroups define a chain subcomplex, and to show this we need to verify the following.

LEMMA 1. *The boundary homomorphism d_k sends elements of $C'_k(P, \mathbf{K})$ to $C'_{k-1}(P, \mathbf{K})$.*

It suffices to prove that the boundary map sends the previously described generators into degenerate chains, and checking this is essentially a routine calculation.■

See notes2022-03a.pdf for the details.

We now define the complex of *alternating simplicial chains* $C^{\mathrm{alt}}_*(P, \mathbf{K})$ to be the quotient complex $C_*(P, \mathbf{K})/C'_*(P, \mathbf{K})$ with the associated differential or boundary map.

PROPOSITION 2. *The composite $\varphi : C_*(P, \mathbf{K}^\omega) \to C_*(P, \mathbf{K}) \to C^{\mathrm{alt}}_*(P, \mathbf{K})$ is an isomorphism of chain complexes.*

COROLLARY 3. *The morphism $i_* : H_*(P, \mathbf{K}^\omega) \to H_*(P, \mathbf{K})$ is injection onto a direct summand.*

Proof that Proposition 2 implies Corollary 3. Let q be the projection map from unordered to alternating chains, so that $\varphi_* = q_* \circ i_*$. General considerations imply that φ_* is an isomorphism.

Suppose now that $i_*(a) = i_*(b)$. Applying q_* to each side we obtain

$$\varphi_*(a) \quad = \quad q_* \circ i_*(a) \quad = \quad q_* \circ i_*(b) \quad = \quad \varphi_*(b)$$

and since φ_* is bijective it follows that $a = b$.

Now let B_* be the kernel of q_*. We shall prove that every element of $H_*(P, \mathbf{K})$ has a unique expression as $i_*(a) + c$, where $c \in B_*$. Given $u \in H_*(P, \mathbf{K})$, direct computation implies that

$$u \; - \; i_*(\varphi_*)^{-1} q_*(u) \quad \in \quad B_*$$

and thus yields existence. Suppose now that $u = i_*(a) + c$, where $c \in B_*$. It then follows from the definitions that

$$i_*(a) \quad = \quad i_*(\varphi_*)^{-1} q_*(u)$$

and hence we also have

$$c \quad = \quad u \; - \; i_*(a) \quad = \quad u \; - \; i_*(\varphi_*)^{-1} q_*(u)$$

which proves uniqueness.■

Proof of Proposition 2. Analogs of standard arguments for determinants yield the following observations:

(1) *The generator* $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k \in C_k(P, \mathbf{K})$ *lies in the subgroup of degenerate chains if two vertices are equal.*

(2) *If* σ *is a permutation of* $\{0, \ \cdots \ , k\}$*, then* $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k - (-1)^{\mathrm{sgn}(\sigma)} \mathbf{v}_{\sigma(0)} \ \cdots \ \mathbf{v}_{\sigma(k)}$ *is a degenerate chain.*

Define a map of graded abelian groups Ψ from $C_*(P, \mathbf{K})$ to $C_*(P, \mathbf{K}^\omega)$ which sends $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$ to zero if there are repeated vertices and sends $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$ to $(-1)^{\mathrm{sgn}(\sigma)} \mathbf{v}_{\sigma(0)} \ \cdots \ \mathbf{v}_{\sigma(k)}$ if the vertices are distinct and σ is the unique permutation which puts the vertices in the proper order:

$$\mathbf{v}_{\sigma(0)} \quad < \quad \cdots \quad < \quad \mathbf{v}_{\sigma(k)}$$

It follows that Ψ passes to a map ψ of quotients from $C_*^{\mathrm{alt}}(P, \mathbf{K})$ to $C_*(P, \mathbf{K}^\omega)$ such that $\psi \circ \varphi$ is the identity. In particular, it follows that φ is injective. To prove it is surjective, note that (1) and (2) imply that $C_k^{\mathrm{alt}}(P, \mathbf{K})$ is generated by the image of φ and hence φ is also surjective. It follows that φ determines an isomorphism of chain complexes as required.■

Acyclic complexes

Definition. An *augmented chain complex* over a ring R consists of a chain complex (C_*, d) and a homomorphism $\varepsilon : C_0 \to R$ (the augmentation map) such that ε is onto and $\varepsilon \circ d_1 = 0$.

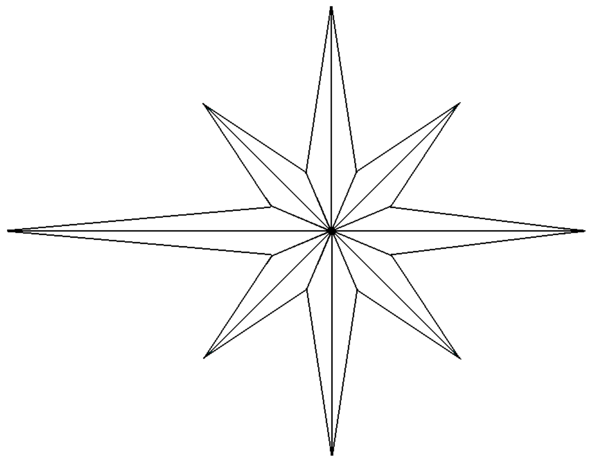
All of the simplicial chain complexes defined above have canonical augmentations given by sending expressions of the form $\sum \, n_{\mathbf{v}} \, \mathbf{v}$ to the corresponding integers $\sum \, n_{\mathbf{v}}$.

Definition. A simplicial complex is said to be *acyclic* ("has no nontrivial cycles") if $H_j(P, \mathbf{K}) = 0$ for $j \neq 0$ and $H_0 \cong \mathbb{Z}$, with the generator in homology represented by an arbitrary free generator of $C_0(P, \mathbf{K})$.

There is a simple geometric criterion for a simplicial chain complexe to be acyclic.

Definition. A simplicial complex $(P, \mathbf{K})$ is said to be *star shaped* with respect to some vertex $\mathbf{v}$ in $\mathbf{K}$ if for each simplex A in $\mathbf{K}$ either $\mathbf{v}$ is a vertex of A or else there is a simplex $\mathbf{B}$ in $\mathbf{K}$ such that $\mathbf{A}$ is a face of $\mathbf{B}$ and $\mathbf{v}$ is a vertex of $\mathbf{B}$.

One particularly important example for the time being is the standard simplex Δ_n with its standard decomposition. Another is illustrated below. It is an eight pointed star which is star shaped with respect to the vertex in the center but not to any other vertex.



PROPOSITION 4. *If the simplicial complex $(P, \mathbf{K})$ is star shaped with respect to some vertex, then it is acyclic, and the map $i_* : H_*(P, \mathbf{K}^\omega) \to H_*(P, \mathbf{K})$ is an isomorphism.*

Proof. Define a map of graded abelian groups $\eta : C_*(P, \mathbf{K}) \to C_*(P, \mathbf{K})$ such that $\eta_q : C_q(P, \mathbf{K}) \to C_q(P, \mathbf{K})$ is zero if $q \neq 0$ and η_0 sends a chain y to $\varepsilon(y)\,\mathbf{v}$. Then η is a chain map because $\varepsilon \circ d_1 = 0$.

We next define homomorphisms $D_q : C_q(P, \mathbf{K}) \to C_{q+1}(P, \mathbf{K})$ such that

$$d_{q+1} \circ D_q \quad = \quad \text{identity} \; - \; d_q \circ D_{q-1}$$

if q is positive and

$$d_1 \circ D_0 \quad = \quad \text{identity} \; - \; \eta_0$$

on C_0. We do this by setting $D_q(\mathbf{x}_0 \, \cdots \, \mathbf{x}_q) = \mathbf{v}\mathbf{x}_0 \, \cdots \, \mathbf{x}_q$ and taking the unique extension which exists since the classes $\mathbf{x}_0 \, \cdots \, \mathbf{x}_q$ are free generators for C_q. Elementary calculations show that the mappings D_q satisfy the conditions given above.

To see that $H_q(P, \mathbf{K}) = 0$ if $q > 0$, suppose that $d_q(z) = 0$. Then the first formula implies that $z = d_{q+1} \circ D_q(z)$. Therefore $H_q = 0$ if $q > 0$. On the other hand, if $z \in C_0$, then the second formula implies that $d_1 \circ D_0(z) = z - \varepsilon(z)\,\mathbf{v}$. Furthermore, since $\varepsilon \circ d_1 = 0$ and $d_0 = 0$, it follows that

(i) the map ε passes to a homomorphism from H_0 to $\mathbb{Z}$,

(ii) since $\varepsilon(\mathbf{v}) = 1$ this homomorphism is onto, (*iii*) the multiples of the class $[\mathbf{v}]$ give all the classes in H_0.

Taken together, these imply that $H_0(P, \mathbf{K}) \cong \mathbb{Z}$, and it is generated by $[\mathbf{v}]$. This completes the computation of $H_*(P, \mathbf{K})$.

By Corollary 3 we know that $H_q(P, \mathbf{K}^\omega)$ is isomorphic to a direct summand of $H_q(P, \mathbf{K})$ and since the latter is zero if $q > 0$ it follows that the former is also zero if $q > 0$. Similarly, we know that $H_0(P, \mathbf{K}^\omega)$ is isomorphic to a direct summand of $H_0(P, \mathbf{K}) \cong \mathbb{Z}$. By construction we know that the generating class $[\mathbf{v}]$ for the latter lies in the image of i_*, and therefore it follows that the map from $H_0(P, \mathbf{K}^\omega)$ to $H_0(P, \mathbf{K})$ must also be an isomorphism.■

COROLLARY 5. *If Δ is a simplex with the standard simplicial decomposition, then*

$$H_q(\Delta, \mathbf{K}^\omega) \quad \cong \quad H_q(\Delta, \mathbf{K})$$

is trivial if $q \neq 0$ and infinite cyclic if $q = 0$.■

Clearly we would like to "leverage" this result into a proof for an arbitrary simplicial complex $(P, \mathbf{K})$. This will require some additional algebraic tools.

Extension to pairs

Let $\big(\, (P, \mathbf{K}),\, (Q, \mathbf{L})\, \big)$ be a simplicial complex pair consisting of a simplicial complex $(P, \mathbf{K})$ and a subcomplex $(Q, \mathbf{L})$. To simplify notation, we shall often denote such a pair by $(\mathbf{K}, \mathbf{L})$. The unordered simplicial chain complex $C_*(\mathbf{K}, \mathbf{L})$ is defined to be the quotient chain complex $C_*(\mathbf{K})/C_*(\mathbf{L})$, and the unordered relative simplicial homology groups, denoted by $H_*(\mathbf{K}, \mathbf{L})$, are the homlogy groups of these chain complexes. As in the absolute case, we have canonical homomorphisms from the relative homology groups for ordered chains to the relative homology groups for unordered chains. We should also note that the previously defined absolute chain groups may be viewed as special cases of this definition where $\mathbf{L} = \emptyset$.

By the preceding discussion and Theorem V.3.2 from `algtopnotes2012.tex`; (*i.e.*, short exact sequences of chain complexes determine long exact sequences of homology groups), we have the following result:

THEOREM 6. (Long Exact Homology Sequence Theorem — Simplicial Version). *Let $i : \mathbf{L} \to \mathbf{K}$ denote a simplicial subcomplex inclusion, and let ω be a linear ordering of the vertices. Then there are long exact sequences of homology groups, and they fit into the following commutative diagram, in which the rows are exact and the horizontal arrows represent the canonical maps from ordered to unordered chains:*

$$\begin{array}{ccccccccccc}
\cdots & H_{k+1}(\mathbf{K}^\omega, \mathbf{L}^\omega) & \xrightarrow{\partial} & H_k(\mathbf{L}^\omega) & \xrightarrow{i_*} & H_k(\mathbf{K}^\omega) & \xrightarrow{j_*} & H_k(\mathbf{K}^\omega, \mathbf{L}^\omega) & \xrightarrow{\partial} & H_{k-1}(\mathbf{L}^\omega) & \cdots \\
 & \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \varphi_* & \\
\cdots & H_{k+1}(\mathbf{K}, \mathbf{L}) & \xrightarrow{\partial} & H_k(\mathbf{L}) & \xrightarrow{i_*} & H_k(\mathbf{K}) & \xrightarrow{j_*} & H_k(\mathbf{K}, \mathbf{L}) & \xrightarrow{\partial} & H_{k-1}(\mathbf{L}) & \cdots
\end{array}$$

Sketch of proof. The definitions of simplicial chain groups imply that one has a commutative diagram of short exact sequences which goes from the ordered chain complex short exact sequence

$$0 \;\rightarrow\; C_*(\mathbf{L}^\omega) \;\rightarrow\; C_*(\mathbf{K}^\omega) \;\rightarrow\; C_*(\mathbf{K}^\omega, \mathbf{L}^\omega) \;\rightarrow\; 0$$

to the unordered chain complex short exact sequence

$$0 \;\rightarrow\; C_*(\mathbf{L}) \;\rightarrow\; C_*(\mathbf{K}) \;\rightarrow\; C_*(\mathbf{K}, \mathbf{L}) \;\rightarrow\; 0\ .$$

The theorem follows by taking the associated long exact homology sequences and using the naturality of these sequences with respect to maps of short exact sequences of chain complexes.■

At this point it is also appropriate to recall another result on diagrams with exact sequences from `algtopnotes2012.tex`; namely, the Five Lemma (Theorem V.3.4).

The isomorphism theorem

Here is the result that has been our main objective:

THEOREM 7. *If* $(\mathbf{K}, \mathbf{L})$ *is a simplicial complex pair, then the canonical map*

$$\varphi_* : H_*(\mathbf{K}^\omega, \mathbf{L}^\omega) \;\rightarrow\; H_*(\mathbf{K}, \mathbf{L})$$

is an isomorphism.

Proof. Consider the following statements:

> $(\mathbf{X}_n)$ *The map* φ *above is an isomorphism for all simplicial complex pairs* $(\mathbf{K}, \mathbf{L})$ *such that* $\dim \mathbf{K} \leq n$.
>
> $(\mathbf{Y}_{n+1})$ *The map* φ *above is an isomorphism for all* $(\mathbf{K}, \mathbf{L})$ *such that* $\dim \mathbf{K} \leq n$ *and also for* $(\Delta_{n+1}, \partial\Delta_{n+1})$.
>
> $(\mathbf{W}_{n+1,m})$ *The map* φ *above is an isomorphism for all* $(\mathbf{K}, \mathbf{L})$ *such that* $\dim \mathbf{K} \leq n$ *and also for all* $(\mathbf{K}, \mathbf{L})$ *such that* $\dim \mathbf{K} \leq n+1$ *and* $\mathbf{K}$ *has at most* m *simplices of dimension equal to* $n+1$.

The theorem is then established by the following double inductive argument:

- [F] The statement $(\mathbf{X}_0)$ and the equivalent statement $(\mathbf{W}_{1,0})$ are true.
- [G] For all nonnegative integers n, the statement $(\mathbf{X}_n)$ implies $(\mathbf{Y}_{n+1})$.
- [K] For all nonnegative integers n and m, the statements $(\mathbf{W}_{n+1,m})$ and $(\mathbf{Y}_{n+1})$ imply $(\mathbf{W}_{n+1,m+1})$.

Since statement $(\mathbf{X}_n)$ is true if and only if $(\mathbf{W}_{n,m})$ is true for all m, and the latter are all true if and only if $(\mathbf{W}_{n+1,0})$ is true, we also have the following:

[L] For all n the statements $(\mathbf{X}_n) \Longleftrightarrow (\mathbf{W}_{n+1,0})$ and $(\mathbf{Y}_{n+1})$ imply $(\mathbf{W}_{n+1,m})$ for all m, and hence $(\mathbf{X}_n)$ implies $(\mathbf{X}_{n+1})$.

Therefore $(\mathbf{X}_n)$ is true for all n, and this is the conclusion of the theorem.

Proof of [F]. By the Five Lemma it suffices to prove the result when $\mathbf{L}$ is empty. Since the 0-dimensional complex determined by $\mathbf{K}$ is merely a finite set of vertices, write these vertices as $\mathbf{w}_1, \cdots \mathbf{w}_m$. We then have canonical chain complex isomorphisms

$$\bigoplus_{j=1}^{m} C_*(\{\mathbf{w}_j\}^\omega) \longrightarrow C_*(\mathbf{K}^\omega) , \qquad \bigoplus_{j=1}^{m} C_*(\{\mathbf{w}_j\}) \longrightarrow C_*(\mathbf{K})$$

and these pass to homology isomorphisms

$$\bigoplus_{j=1}^{m} H_*(\{\mathbf{w}_j\}^\omega) \longrightarrow H_*(\mathbf{K}^\omega) , \qquad \bigoplus_{j=1}^{m} H_*(\{\mathbf{w}_j\}) \longrightarrow H_*(\mathbf{K}) .$$

These maps commute with the homomorphisms φ_* sending ordered to unordered chains. and since the maps φ_* are isomorphisms for one point complexes (= 0-simplices), it follows that φ defines an isomorphism from $H_*(\mathbf{K}^\omega)$ to $H_*(\mathbf{K})$. The completes the proof of $(\mathbf{X}_0)$.

Proof of [G]. By $(\mathbf{X}_n)$ we know that φ_* is an isomorphism for the complex $\partial\Delta_{n+1}$. Since φ_* is also an isomorphism for Δ_{n+1} by Corollary III.3.6. Therefore the Five Lemma implies that φ_* is an isomorphism for $(\Delta_{n+1}, \partial\Delta_{n+1})$.

Proof of [K]. This is the crucial step. Let $\mathbf{K}$ be an $(n+1)$-dimensional complex, and let $\mathbf{M}$ be a subcomplex obtained by removing exactly one $(n+1)$-simplex from $\mathbf{K}$, so that φ_* is an isomorphism for $\mathbf{M}$ by the inductive hypothesis. If we can show that φ_* is an isomorphism for $(\mathbf{K}, \mathbf{M})$, then it will follow that φ_* is an isomorphism for $\mathbf{K}$, and the relative case will the follow from the Five Lemma.

Let $\mathbf{S}$ be the extra simplex of $\mathbf{K}$ and let $\partial\mathbf{S}$ be its boundary. Then there are canonical isomorphism from the chain groups of Δ_{n+1}, $\partial\Delta_{n+1}$ and $(\Delta_n, \partial\Delta_{n+1})$ to the chain groups of $\mathbf{S}$, $\partial\mathbf{S}$ and $(\mathbf{S}, \partial\mathbf{S})$. We then have the following commutative diagram, in which the morphisms α and β are determined by subcomplex inclusions:

$$\begin{array}{ccc}
C_*(\mathbf{S}^\omega, \partial\mathbf{S}^\omega) & \xrightarrow{\ \alpha\ } & C_*(\mathbf{K}^\omega, \mathbf{M}^\omega) \\
\big\downarrow \varphi(\mathbf{S}, \partial\mathbf{S}) & & \big\downarrow \varphi(\mathbf{K}, \mathbf{M}) \\
C_*(\mathbf{S}, \partial\mathbf{S}) & \xrightarrow{\ \beta\ } & C_*(\mathbf{K}, \mathbf{M})
\end{array}$$

We CLAIM that α and β are isomorphisms of chain complexes. For the mapping α, this follows because the relative ordered chain groups of a pair $(\mathbf{T}, \mathbf{T}_0)$ are free abelian groups on the simplices in $\mathbf{T} - \mathbf{T}_0$, and each of the sets $\mathbf{S} - \partial\mathbf{S}$ and $\mathbf{K} - \mathbf{M}$ is given by the same $(n+1)$-simplex. For the mapping β, this follows because the relative unordered chain groups of a pair $(\mathbf{T}, \mathbf{T}_0)$ are free abelian groups on the generators $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$, where the $\mathbf{v}_j$ are vertices of a simplex that is in $\mathbf{T}$ but not in $\mathbf{T}_0$ (with repetitions allowed as usual), and once again these free generators are identical for te pairs $(\mathbf{S}, \partial\mathbf{S})$ and $(\mathbf{K}, \mathbf{M})$ because $\mathbf{S} - \partial\mathbf{S}$ and $\mathbf{K} - \mathbf{M}$ are the same.

By $(\mathbf{Y}_{n+1})$ we know that $\varphi(\mathbf{S}, \partial\mathbf{S})$ defines an isomorphism in homology, and therefore it follows that the homology map

$$\varphi(\mathbf{K}, \mathbf{M})_* \quad = \quad \beta_* \circ \varphi(\mathbf{S}, \partial\mathbf{S})_* \circ \alpha_*^{-1}$$

also defines an isomorphism in homology. We can now use the Five Lemma and $(\mathbf{W}_{n+1,m})$ to conclude that the map $\varphi(\mathbf{K})$ defines an isomorphism in homology, and finally we can use the Five Lemma once more to see that the statement $(\mathbf{W}_{n+1,m+1})$ is true. This completes the proof of [K], and as noted above it also yields [L] and the theorem.■

The preceding result can be reformulated in an abstract setting that will be needed later. We begin by defining a category **SCPairs** whose objects are pairs of simplicial complexes $(\mathbf{K}, \mathbf{K}_0)$ and whose morphisms are given by subcomplex inclusions $(\mathbf{L}, \mathbf{L}_0) \subset (\mathbf{K}, \mathbf{K}_0)$; in other words, $\mathbf{L}_0$ is a subcomplex of both $\mathbf{L}$ and $\mathbf{K}_0$ while $\mathbf{L}$ is also a subcomplex of $\mathbf{K}$. A *homology theory* on this category is a covariant functor h_* valued in some category of modules together with a natural transformation

$$\partial(\mathbf{K}, \mathbf{L}) : h_*(\mathbf{K}, \mathbf{L}) \;\longrightarrow\; h_{*-1}(\mathbf{L})$$

such that

(a) one has long exact homology sequences,

(b) if $\mathbf{K}$ is a simplex and $\mathbf{v}$ is a vertex of $\mathbf{K}$ then $h_*(\{\mathbf{v}\}) \to h_*(\mathbf{K})$ is an isomorphism,

(c) if $\mathbf{K}$ is 0-dimensional with vertices $\mathbf{v}_j$ then the associated map from $\oplus_j \; h_j(\{\mathbf{v}_j\})$ to $h_*(\mathbf{K})$ is an isomorphism,

(d) if $\mathbf{K}$ is obtained from $\mathbf{M}$ by adding a single simplex $\mathbf{S}$, then $h_*(\mathbf{S}, \partial\mathbf{S}) \to h_*(\mathbf{M}, \mathbf{K})$ is an isomorphism,

(d) if $\mathbf{K}$ is complex consisting only of a single vertex then $h_0(\mathbf{K})$ is the underlying ring R and $h_j(\mathbf{K}) = 0$ if $j \neq 0$.

A *natural transformation* from one such theory (h_*, ∂) to another (h'_*, ∂') is a natural transformation of θ of functors that is compatible with the mappings ∂ and ∂'; specifically, we want

$$\theta(\mathbf{L}) \circ \partial \quad = \quad \partial' \circ \theta(\mathbf{K}, \mathbf{L}) \; .$$

These conditions imply the existence of a commutative ladder diagram as in Theorem 6, where the rows are the long exact sequences determined by the two abstract homology theories. The definition is set up so that the proof of the next result is formally parallel to the proof of Theorem 7:

THEOREM 8. *Suppose we are given a natural transformation of homology theories θ as above such that $\theta(\mathbf{K})$ is an isomorphism if $\mathbf{K}$ consists of just a single vertex. Then $\theta(\mathbf{K}, \mathbf{L})$ is an isomorphism for all pairs $(\mathbf{K}, \mathbf{L})$.*■