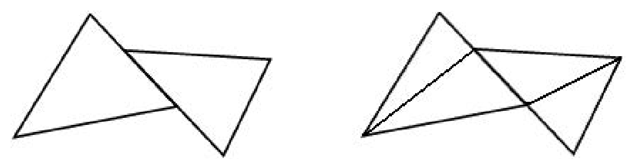
I.2 : Subdivisions

(Hatcher, § 2.1)

For many purposes it is convenient or necessary to replace a simplicial decomposition **K** of a polyhedron P by another decomposition **L** with smaller simplices. More precisely, we would like the smaller simplices in **L** to determine simplicial decompositions for each of the simplices in **K**.

The need for working with subdivisions arises in many contexts. For example, as in the figure below, the union of two solid triangular regions in the plane usually does not satify the conditions for a simplicial decomposition, but it is possible to subdivide the union and obtain a simplicial decomposition such that each of the original regions is a subcomplex.



not a simplicial complex *simplicial complex*

Similar considerations apply to arbitrary polyhedra. We shall not attempt to state this precisely or prove it because we do not need such results in this course, but here are some references:

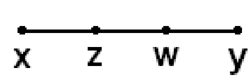
> **J. F. P. Hudson.** *Piecewise Linear Topology. W. A. Benjamin, New York*, 1969. (Online: `http://www.maths.ed.ac.uk/∼aar/surgery/hudson.pdf`)
>
> **C. P. Rourke and B. J. Sanderson.** *Introduction to Piecewise-Linear Topology (Ergeb. Math. Bd.* 69*). Springer-Verlag, New York–etc.*, 1972.

A few topics are also discussed in [MunkresEDT]. An extremely detailed study of some topics in this section also appears in the following online book:

`http://www.cis.penn.edu/∼jean/gbooks/convexpoly.html`

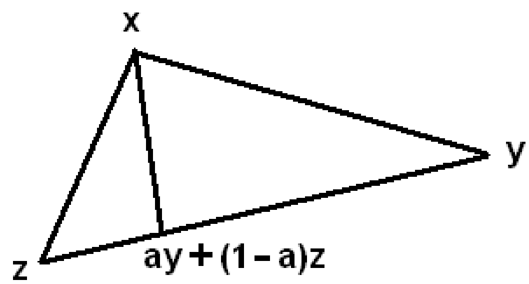
Explicit simple examples

1. If P is a 1-simplex with vertices $\mathbf{x}$ and $\mathbf{y}$, and **K** is the standard decomposition given by P and the endpoints, then there is a subdivision **L** given by trisecting P; specifically, the vertices are given by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} = \frac{2}{3}\mathbf{x} + \frac{1}{3}\mathbf{y}$, and $\mathbf{w} = \frac{1}{3}\mathbf{x} + \frac{2}{3}\mathbf{y}$, and the 1-simplices are $\mathbf{xw}$, $\mathbf{wz}$ and $\mathbf{zy}$. This is illustrated in the figure below.



2. Similarly, if $[a,b]$ is a closed interval in the real line and we are given a finite sequence $a = t_0 < \;\cdots\; < t_m = b$, then these points and the intervals $[t_{j-1}, t_j]$, where $1 \leq j \leq n$, form a subdivision of the standard decomposition of $[a,b]$.

3. If P is the 2-simplex with vertices $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, and $\mathbf{K}$ is the standard decomposition given by P and its faces, then there is an obvious decomposition $\mathbf{L}$ which splits P into two simplices $\mathbf{xyz}$ and $\mathbf{xyw}$, where $\mathbf{w} = \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}$ is the midpoint of the 1-simplex $\mathbf{yz}$. Similar eamples exist if we take $\mathbf{z} = a\mathbf{y} + (1-a)\mathbf{z}$, where a is an arbitrary number such that $0 < a < 1$ (see the figure below).



Formal definition of subdivisions

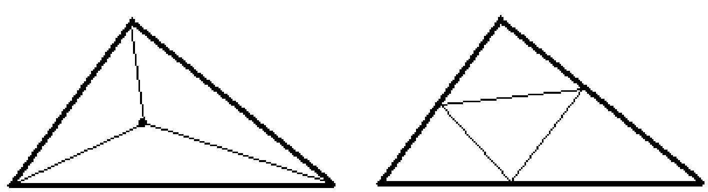
Each of the preceding examples is consistent with the following general concept.

Definition. Let $(P, \mathbf{K})$ be a simplicial complex, and let $\mathbf{L}$ be a simplicial decomposition of P. Then $\mathbf{L}$ is called a (linear) *subdivision* of $\mathbf{K}$ if every simplex of $\mathbf{L}$ is contained in a simplex of $\mathbf{K}$.

The following observation is very elementary, but we shall need it in the discussion below.

PROPOSITION 0. *Suppose P is a polyhedron with simplicial decompositions $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{M}$ such that $\mathbf{L}$ is a subdivision of $\mathbf{K}$ and $\mathbf{M}$ is a subdivision of $\mathbf{L}$. Then $\mathbf{M}$ is also a subdivision of $\mathbf{K}$.*■

The first figure below depicts two subdivisions of a 2-simplex that are different from the one in Example 3 above. As indicated by the second figure, in general if we have two simplicial decompositions of a polyhedron then neither is necessarily a subdivision of the other.



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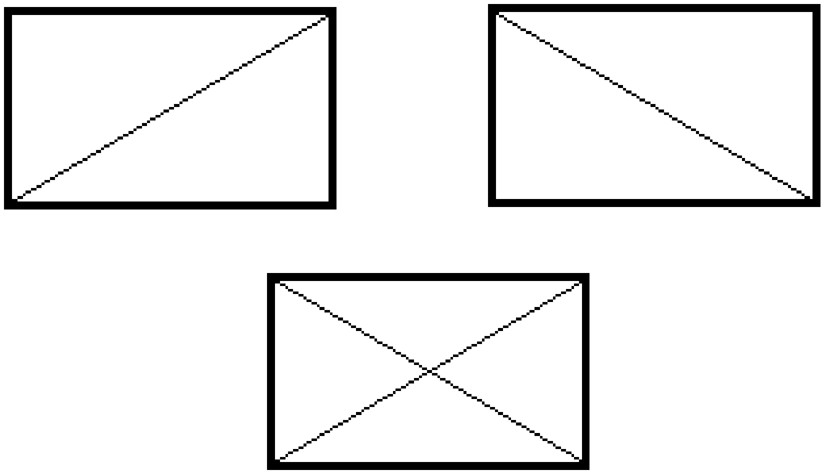


Note that the bottom figure is a common subdivision of the top two.

However, it is possible to prove the following compatibilty result for linear subdivisions:

> *If* $\mathbf{K}$ *and* $\mathbf{L}$ *are simplicial decompositions of the same polyhedron* P*, then there is a third decomposition which is a subdivision of both* $\mathbf{K}$ *and* $\mathbf{L}$.

Proving this requires more machinery than we need for other purposes, and since we shall not need the existence of such subdivisions in this course we shall simply note that one can prove this result using methods from the second part of [MunkresEDT]:

SUBDIVISION AND SUBCOMPLEXES. These two concepts are related by the following elementary results.

PROPOSITION 1. *Suppose that* $(P, \mathbf{K})$ *is a simplicial complex and that* $(P_1, \mathbf{K}_1)$ *is a subcomplex of* $(P, \mathbf{K})$*. If* $\mathbf{L}$ *is a subdivision of* $\mathbf{K}$ *and* $\mathbf{L}_1$ *is the set of all simplices in* $\mathbf{L}$ *which are contained in* P_1*, then* $(P_1, \mathbf{L}_1)$ *is a subcomplex of* $(P, \mathbf{L})$.■

Recall our Default Hypothesis (at the end of Section I.2) that all simplicial decompositions should be closed under taking faces unless specifically stated otherwise.

COROLLARY 2. *Let* P, $\mathbf{K}$ *and* $\mathbf{L}$ *be as above, and let* $A \subset P$ *be a simplex of* $\mathbf{K}$*. Then* $\mathbf{L}$ *determines a simplicial decomposition of* A.■

We are particularly interested in describing a systematic construction for subdivisions that works for all simplicial complexes and allows one to form decompositions for which the diameters of all the simplices are very small. This will generalize a standard method for partitioning an interval $[a, b]$ into small intervals by first splitting the interval in half at the midpoint, then splitting the two subintervals in half similarly, and so on. If this is done n times, the length of each interval in the subdivision is equal to $(b-a)/2^n$, and if $\varepsilon > 0$ is arbitrary then for sufficiently large values of n the lengths of the subintervals will all be less than ε.

The generalization of this to higher dimensions is called the **barycentric subdivision**.

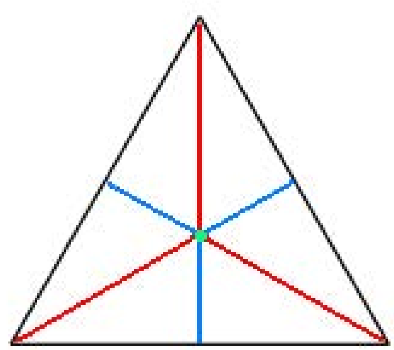
Definition. Given an n-simplex $A \subset \mathbb{R}^m$ with vertices $\mathbf{v}_0, \cdots, \mathbf{v}_n$, the *barycenter* $\mathbf{b}_A$ of A is given by

$$\mathbf{b}_A \quad = \quad \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{v}_i \ .$$

If $n \leq m \leq 3$, this corresponds to the physical center of mass for A, assuming the density in A is uniform.

Definition. If $P \subset \mathbb{R}^m$ is a polyhedron and $(P, \mathbf{K})$ is a simplicial complex, then the *barycentric subdivision* $\mathbf{B(K)}$ consists of all simplices having the form $\mathbf{b}_0 \ \cdots \ \mathbf{b}_k$, where (i) each $\mathbf{b}_j$ is the barycenter of a simplex $A_j \in \mathbf{K}$, (ii) for each $j > 0$ the simplex A_{j-1} is a face of A_j.

Here is the barycentric subdivision of a 2-simplex:



In order to justify this definition and show that we actually have a simplicial decomposition, we first need to prove the following result:

PROPOSITION 3. *Let A be an n-simplex, suppose that we are given simplices $A_j \subset A$ such that A_{j-1} is a face of A_j for each j, and let $\mathbf{b}_j$ be the barycenter of A_j. Then the set of vertices $\{\, \mathbf{b}_0, \ \cdots \ , \mathbf{b}_q\}$ is affinely independent.*

Proof. We can extend the sequence of simplices $\{\, A_j \,\}$ to obtain a new sequence $C_0 \subset \ \cdots \ \subset C_n = A$ such that each C_k is obtained from the preceding one C_{k-1} by adding a single vertex, and it suffices to prove the result for the corresponding sequence of barycenters. Therefore we shall assume henceforth in this proof that each A_j is obtained from its predecessor by adding a single vertex and that A is the last simplex in the list.

It suffices to show that the vectors $\mathbf{b}_j - \mathbf{b}_0$ are linearly independent. For each j let $\mathbf{v}_{j_i}$ be the vertex in A_j that is not in its predecessor. Then for each $j > 0$ we have

$$\mathbf{b}_j - \mathbf{b}_0 \quad = \quad \left(\frac{1}{j+1} \sum_{k \leq j} \mathbf{v}_{i_k} \right) \ - \ \mathbf{v}_0 \quad = \quad \frac{1}{j+1} \sum_{k \leq j} (\mathbf{v}_{i_k} - \mathbf{v}_{i_0}) \ .$$

which is a linear combination of the linearly independent vectors $\mathbf{v}_{i_1} - \mathbf{v}_{i_0}, \ \cdots \ , \mathbf{v}_{i_j} - \mathbf{v}_{i_0}$ such that the coefficient of the last vector in the set is nonzero.

If we let $\mathbf{u}_k = \mathbf{v}_{i_k} - \mathbf{v}_{i_0}$, then it follows that for all $k > 0$ we have $\mathbf{b}_k - \mathbf{b}_0 = a_k \mathbf{u}_k + \mathbf{y}_k$, where $\mathbf{y}_k$ is a linear combination of $\mathbf{u}_1, \ \cdots \ , \mathbf{u}_{k-1}$ and $a_k \neq 0$. Since the vectors $\mathbf{u}_j$ are linearly independent, it follows that the vectors $\mathbf{b}_k - \mathbf{b}_0$ (where $0 < k \leq n$) are linearly independent and hence the vectors $\mathbf{b}_0, \ \cdots \ , \mathbf{b}_n$ are affinely independent.∎

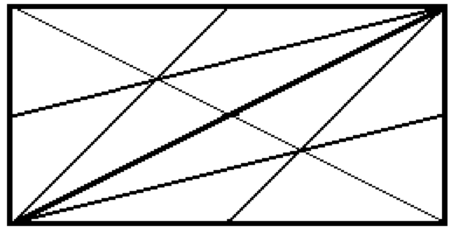
The simplest nontrivial examples of barycentric subdivisions are given by 2-simplices. For the sake of definiteness, we shall call the simplex P and the vertices $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$.

- (i) The 0-simplices are merely the barycenters $\mathbf{b}_A$, where A runs through all the nonempty faces of P and P itself. There are 7 such simplices and hence 7 vertices in $\mathbf{B(K)}$.
- (ii) The 1-simplices have the form $\mathbf{b}_A\mathbf{b}_C$, where A is a face of C. There are three possible choices for the ordered pair $(\dim A, \dim C)$; namely, $(0,1)$, $(0,2)$ and $(1,2)$. The number of pairs $\{A, \ C\}$ for the case $(0,1)$ is equal to 6, the number for the case $(0,2)$ is equal to 3, and the number for the case $(0,1)$ is also equal to 3, so there are 12 different 1-simplices in $\mathbf{B(K)}$.
- (iii) The 2-simplices have the form $\mathbf{b}_A\mathbf{b}_C\mathbf{b}_E$, where A is a face of C and C is a face of E. There are 6 possible choices for $\{A, \ C, \ E\}$.

Obviously one could carry out a similar analysis for a 3-simplex but the details would be more complicated.

Of course, it is absolutely essential to verify the that barycentric subdivision construction actually defines simplicial decompositions.

THEOREM 4. *If $(P, \mathbf{K})$ is a simplicial complex and $\mathbf{B(K)}$ is the barycentric subdivision of $\mathbf{K}$, then $(P, \mathbf{B(K)}\,)$ is also a simplicial complex (in other words, the collection $\mathbf{B(K)}$ determines a simplicial decomposition of P).*



> By their second year of graduate studies students must make the transition from understanding simple proofs line-by-line to understanding the overall structure of proofs of [long or] difficult theorems. [Of course it is still necessary to understand simple proofs in detail, but as one progresses it is necessary to begin the study of more complicated arguments by having some grasp of the main steps and how they are studied.]

Several steps in the proof of this result are fairly intricate, and the following remark from Davis and Kirk, *Lecture Notes in Algebraic Topology*, are worth remembering:

Proof. We shall concentrate on the special case where P is a simplex. The general case can be recovered from the special case and Lemma IV.2.6 in `algtop-notes.pdf` (see p. 51).

Suppose now that P is a simplex with vertices vertices $\mathbf{v}_0, \cdots, \mathbf{v}_n$. We first show that P is the union of the simplices in $\mathbf{B(K)}$. Given $\mathbf{x} \in P$, write $\mathbf{x}$ as a convex combination $\sum_j \; t_j \, \mathbf{v_j}$, and rearrange the scalars into a sequence

$$t_{k_0} \;\; \geq \;\; t_{k_1} \;\; \cdots \;\; \geq \;\; t_{k_n}$$

(this is not necessarily unique, and in particular it is not so if $t_u = t_v$ for $u \neq v$). For each i between 0 and n, let A_i be the simplex whose vertices are $\mathbf{v}_{k_0}, \cdots, \mathbf{v}_{k_i}$. We CLAIM that $x \in \mathbf{b}_0 \;\cdots\; \mathbf{b}_n$, where $\mathbf{b}_i$ is the barycenter of A_i.

Let $s_i = t_{k_i} - t_{k_{i+1}}$ for $0 \leq i \leq n-1$ and set $s_n = t_{k_n}$. Then $s_i \geq 0$ for all i, and it is elementary to verify that

$$\mathbf{x} \;\; = \;\; \sum_{i=0}^{n} \, (i+1) \, s_i \, \mathbf{b}_i \;, \qquad \text{where} \qquad \sum_{1=0}^{n} (i+1) \, s_i \;\; = \;\; \sum_{i=0}^{n} \; t_{k_i} \;\; = \;\; 1 \;.$$

Therefore $\mathbf{x} \in \mathbf{b}_0 \;\cdots\; \mathbf{b}_n$, so that every point in A lies on one of the simplices in the barycentric subdivision.

To conclude the proof, we must show that the intersection of two simplices in $\mathbf{B(K)}$ is a common face. First of all, it suffices to show this for a pair of n-dimensional simplices; this follows from the argument following the Default Hypothesis at the end of Section IV.2 in `algtop-notes.pdf`.

Suppose now that α and γ are n-simplices in $\mathbf{B(K)}$. Then the vertices of α are barycenters of simplices $A_0, \cdots, A_n$ where A_j has one more vertex than A_{j-1} for each j, and the vertices of γ are barycenters of simplices $C_0, \cdots, C_n$ where C_j has one more vertex than C_{j-1} for each j. Label the vertices of the original simplex as $\mathbf{v}_{i_0}, \cdots, \mathbf{v}_{i_n}$ where $A_j = \mathbf{v}_{i_0} \;\cdots\; \mathbf{v}_{i_j}$ and also as $\mathbf{v}_{k_0}, \cdots, \mathbf{v}_{k_n}$ where $C_j = \mathbf{v}_{k_0} \;\cdots\; \mathbf{v}_{k_j}$. The key point is to determine how $(i_0, \cdots, i_n)$ and $(k_0, \cdots, k_n)$ are related.

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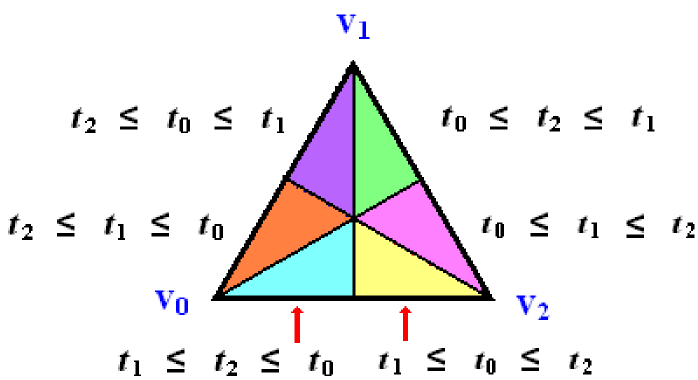

If $\mathbf{x}$ lies on the original simplex and $\mathbf{x}$ is written as a convex combination $\sum_j \ t_j \, \mathbf{v}_j$, then we have shown that $\mathbf{x} \in A$ if $t_{i_0} \le \ \cdots \ \le t_{i_n}$. In fact, we can reverse the steps in that argument to show that if $\mathbf{x} \in A$ then conversely we have $t_{i_0} \le \ \cdots \ \le t_{i_n}$. Similarly, if $\mathbf{x} \in C$ then $t_{k_0} \le \ \cdots \ \le t_{k_n}$. Therefore if $\mathbf{x} \in A \cap C$ then $t_{i_j} = t_{k_j}$ for all j. Choose $m_0, \ \cdots \ , m_q \in \{0, \ \cdots \ , n\}$ such that $t_{m_j} > t_{m_{j+1}}$, with the convention that $t_{n+1} = 0$, and split $\{0, \ \cdots \ ,_n \}$ into equivalence classes $\mathcal{M}_0, \ \cdots \ , \mathcal{M}_q$ such that $\mathcal{M}_j$ is the set of all u such that $t_u = t_{m_j}$. It follows that $\mathbf{x}$ lies on the simplex $\mathbf{z}_0 \cdots \mathbf{z}_q$, where $\mathbf{z}_j$ is the barycenter of the simplex whose vertices are $\mathcal{M}_0 \cup \ \cdots \ \cup \mathcal{M}_j$. The vertices of this simplex are vertices of both A and C. Since $A \cap C$ is convex, this implies that it is the simplex whose vertices are those which lie in $A \cap C$, and thus $A \cap C$ is a face of both A and C.■

Terminology. Frequently the complex $(P, \mathbf{B}(\mathbf{K}))$ is called the *derived complex* of $(P, \mathbf{K})$. The barycentric subdivision construction can be iterated, and thus one obtains a sequence of decompositions $\mathbf{B}^r(\mathbf{K})$. The latter is often called the r^{th} barycentric subdivision of $\mathbf{K}$ and $(P, \mathbf{B}^r(\mathbf{K}))$ is often called the r^{th} derived complex of $(P, \mathbf{K})$.

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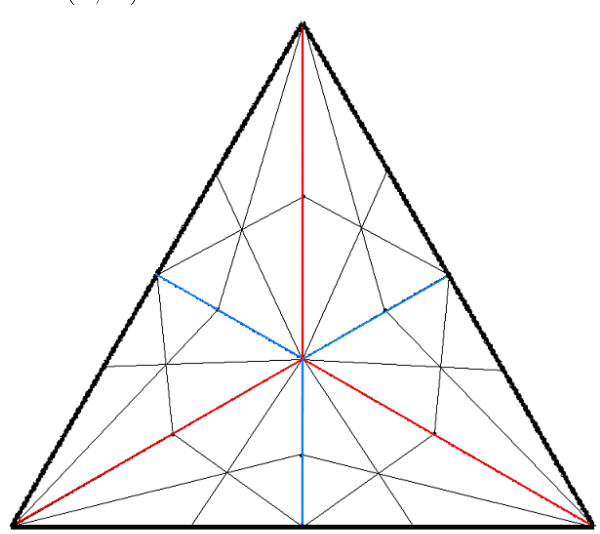
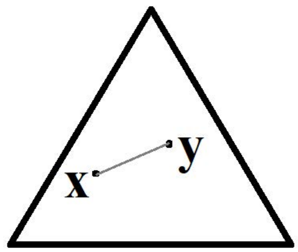
Drawing of the second barycentric subdivision for a 2 - simplex:



Given a metric space $(X, \mathbf{d})$, its *diameter* is the least upper bound of the distances $\mathbf{d}(y, z)$, where $y, z \in X$; if the set of distances is unbounded, we shall follow standard usage and say that the diameter is infinite or equal to ∞.

PROPOSITION 5. *Let $A \subset \mathbb{R}^n$ be an n-simplex with vertices $\mathbf{v}_0, \cdots, \mathbf{v}_n$. Then the diameter of A is the maximum of the distances $|\mathbf{v}_i - \mathbf{v}_j|$, where $0 \le i, j \le n$.*



Proof. Let $\mathbf{x}, \mathbf{y} \in A$, and write these as convex combinations $\mathbf{x} = \sum_j \ t_j \, \mathbf{v}_j$ and $\mathbf{y} = \sum_j \ s_j \, \mathbf{v}_j$. Then

$$\mathbf{x} - \mathbf{y} \ = \ \left(\sum_i \ s_i \right) \mathbf{x} \ - \ \left(\sum_j \ t_j \right) \mathbf{y} \ = \ \sum_{i,j} \ s_i t_j \, \mathbf{v}_j \ - \ \sum_{i,j} \ s_i t_j \, \mathbf{v}_i \ .$$

Since $0 \le s_i, t_j \le 1$ for all i and j, we have $0 \le s_i t_j \le 1$ for all i and j, so that

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) \ = \ |\mathbf{x} - \mathbf{y}| \ \le \ \Big| \sum_{i,j} \ s_i t_j \, (\mathbf{x}_j - \mathbf{x}_i) \Big| \ \le$$

$$\sum_{i,j} \ s_i t_j \, |\mathbf{v}_i - \mathbf{v}_j| \ \le \ \sum_{i,j} s_i t_j \max |\mathbf{v}_k - \mathbf{v}_\ell| \ = \ \max |\mathbf{v}_k - \mathbf{v}_\ell|$$

as required.■

Definition. If $\mathbf{K}$ is a simplicial decomposition of a polyhedron P, then the *mesh* of $\mathbf{K}$, written $\mu(\mathbf{K})$, is the maximum diameter of the simplices in $\mathbf{K}$.

PROPOSITION 6. *In the preceding notation, the mesh of $\mathbf{K}$ is the maximum distance $|\mathbf{v} - \mathbf{w}|$, where $\mathbf{v}$ and $\mathbf{w}$ are vertices of some simplex in $\mathbf{K}$.*■

The main result in this discussion is a comparison of the mesh of $\mathbf{K}$ with the mesh of $\mathbf{B(K)}$.

PROPOSITION 7. *Suppose that $(P, \mathbf{K})$ be a simplicial complex and that all simplices of $\mathbf{K}$ have dimension $\le n$. Then*

$$\mu(\,\mathbf{B}(\mathbf{K})\,) \ \le \ \frac{n}{n+1} \cdot \mu(\mathbf{K}) \ .$$

Before proving this result, we shall derive some of its consequences.

COROLLARY 8. *In the preceding notation, if $r \geq 1$ then*

$$\mu(\mathbf{B}^r(\mathbf{K})) \quad \leq \quad \left(\frac{n}{n+1}\right)^r \cdot \mu(\mathbf{K}) \;. \blacksquare$$

COROLLARY 9. *In the preceding notation, if $\varepsilon > 0$ then there exists an r_0 such that $r \geq r_0$ implies $\mu(\mathbf{B}^r(\mathbf{K})) \quad < \quad \varepsilon$.*

Corollary 9 follows from Corollary 8 and the fact that

$$\lim_{r\to\infty} \left(\frac{n}{n+1}\right)^r \quad = \quad 0 \;. \blacksquare$$

Proof of Proposition 7. By Proposition 5 and the definition of barycentric subdivision we know that $\mu(\mathbf{B}(\mathbf{K}))$ is the maximum of all distances $|\mathbf{b}_A - \mathbf{b}_C|$, where $\mathbf{b}_A$ and $\mathbf{b}_C$ are barycenters of simplices $A,\ C \in \mathbf{K}$ such that $A \subset C$. Suppose that A is an a-simplex and C is a c-simplex, so that $0 \leq a < c \leq n$. We then have

$$|\mathbf{b}_A - \mathbf{b}_C| \quad = \quad \left| \frac{1}{a+1} \sum_{\mathbf{v}\in A} \mathbf{v} \; - \; \frac{1}{c+1} \sum_{\mathbf{w}\in C} \mathbf{w} \right|$$

and as in the proof of Proposition 5 we have

$$\frac{1}{a+1} \sum_{\mathbf{v}\in A} \mathbf{v} \; - \; \frac{1}{c+1} \sum_{\mathbf{w}\in C} \mathbf{w} \quad = \quad \frac{1}{(a+1)(c+1)} \sum_{\mathbf{v},\mathbf{w}} (\mathbf{v} - \mathbf{w}) \;.$$

There are $(a+1)$ terms in this summation which vanish (namely, those for which $\mathbf{w} = \mathbf{v}$), and therefore we have

$$|\mathbf{b}_A - \mathbf{b}_C| \quad = \quad \left| \frac{1}{(a+1)(c+1)} \sum_{\mathbf{v}\neq\mathbf{w}} (\mathbf{v} - \mathbf{w}) \right| \quad \leq \quad \frac{1}{(a+1)(c+1)} \sum_{\mathbf{v}\neq\mathbf{w}} |\mathbf{v} - \mathbf{w}| \quad \leq$$

$$\frac{1}{(a+1)(c+1)} \cdot \left(\max_{\mathbf{v},\mathbf{w}}\right) |\mathbf{v} - \mathbf{w}| \cdot \Big[(a+1)(c+1) - (a+1) \Big] \quad =$$

$$\left(\max_{\mathbf{v},\mathbf{w}} |\mathbf{v} - \mathbf{w}|\right) \cdot \left(1 \; - \; \frac{1}{c+1}\right) \quad \leq \quad \left(1 \; - \; \frac{1}{n+1}\right) \;.$$

At the last step we use $c \leq n$ and the fact that the function $1 - (1/x)$ is an increasing function of x if $x > 1$. The inequality in the corollary follows directly from the precedng chain of inequalities.$\blacksquare$

One further consequence of Proposition 7 will be important for our purposes.

COROLLARY 10. *Let $(P, \mathbf{K})$ be a simplicial complex, and let $\mathcal{W}$ be an open covering of P. Then there is a positive integer r_0 such that $r \geq r_0$ implies that every simplex of $\mu(\,\mathbf{B}^r(\mathbf{K})\,)$ is contained in an element of $\mathcal{W}$.*

Proof. By construction, P is a compact subset of a the metric space $\mathbb{R}^m$. Therefore the Lebesgue Covering Lemma implies the existence of a real number $\eta > 0$ such that every subset of diameter $< \eta$ is contained in an element of $\mathcal{W}$. If we choose $r_0 > 0$ such that $r \geq r_0$ implies $\mu(\,\mathbf{B}^r(\mathbf{K})\,) \quad < \quad \eta$, then $\mathbf{B}^r(\mathbf{K})$ will have the required properties.■

Homology and barycentric subdivisions

We shall now use the preceding results to show that the homology groups of a barycentric subdivision $B(\mathbf{K})$ are isomorphic to the homology groups of the original complex $\mathbf{K}$. In this case the homology theories will be $H_*(\mathbf{K}^\omega, \mathbf{L}^\omega)$ and $H_*\big(\,B(\mathbf{K})^\tau, B(\mathbf{L})^\tau\,\big)$, and the natural transformation will be associated to maps defined on the chain level. It will suffice to define these chain maps for a simplex and to extend to arbitrary complexes and pairs by putting things together in an obvious manner.

PROPOSITION 11. *Given a nonnegative integer n, let $\partial_j : \Delta_{n-1} \to \Delta_n$ be the order preserving affine map sending Δ_{n-1} to the face of Δ_n opposite the $j^{\,\mathrm{th}}$ vertex, and let $(\delta_j)_\#$ generically denote an associated chain map. Then there are classes $\beta_n \in C_n(\Delta_n^\omega)$ such that β_0 is just the standard generator and if $n > 0$ then*

$$d_n(\beta_n) \quad = \quad \sum_{j=0}^{n} (-1)^j (\partial_j)_\#(\beta_{n-1}) \ .$$

Proof. Since Δ_n is acyclic, it suffices to show that the right hand side lies in the kernel of d_{n-1} if $n > 1$ and in the kernel of ε if $n = 1$. Both of these are routine (but tedious) calculations.■

Using the chains β_n one can piece together chain maps

$$C_*(\mathbf{K}^\omega, \mathbf{L}^\omega) \ \longrightarrow \ C_*\big(\,B(\mathbf{K})^\tau, B(\mathbf{L})^\tau\,\big) \ .$$

We claim these define a natural transformation of homology theories, but in order to do this we must first show that $H_*\big(\,B(\mathbf{K})^\tau, B(\mathbf{L})^\tau\,\big)$ actually defines a homology theory. Properties (a), (c) and (e) follow directly from the construction. Property (b) follows because $B(\Delta_n)$ is star shaped with respect to the vertex $\mathbf{b}$ given by the barycenter of Δ_n. Thus it only remains to verify property (d); in fact, direct inspection similar to an argument in the proof of Theorem 1.6 shows that the map on the chain level is an isomorphism.

By Theorem 1.7, it suffices to check that the natural transformation of homology theories is an isomorphism for a simplicial complex consisting of a single vertex; in fact, for such complexes the map is already an isomorphism on the chain level. Therefore the barycentric subdivision chain maps determine isomorphism of homology groups as asserted in the proposition.■