If ω is a linear ordering for the vertices of $\mathbf{K}$, then the unordered simplicial chain complex $C_*(P, \mathbf{K})$ contains the ordered simplicial chain complex $C_*(P, \mathbf{K}^\omega)$ as a chain subcomplex, and we shall let i denote the resulting inclusion map of chain complexes. If we can show that the associated homology maps i_* are isomorphisms, then it will follow that the homology groups for the ordered simplicial chain complex agree with the corresponding groups for the unordered simplicial chain complex, and therefore the homology groups do not depend upon choosing a linear ordering of the vertices.

One major difference between the unordered and ordered simplicial chain groups is that the latter are nontrivial in every positive dimension. In particular, if $\mathbf{v}$ is a vertex of $\mathbf{K}$, then the free generator $\mathbf{v}\ \cdots\ \mathbf{v} = \mathbf{u}_0\ \cdots\ \mathbf{u}_k$, with $\mathbf{u}_j = \mathbf{v}$ for all j, represents a nonzero element of $C_k(P, \mathbf{K})$. On the other hand, the ordered simplicial chain groups are nonzero for only finitely many values of k.

In order to analyze the mappings i_*, we shall introduce yet another definition of homology groups.

Third Definition. In the setting above, define the subgroup $C'_k(P, \mathbf{K})$ of *degenerate* simplicial k-chains to be the subgroup generated by

(a) all elements $\mathbf{v}_0\ \cdots\ \mathbf{v}_k$ such that $\mathbf{v}_i = \mathbf{v}_{i+1}$ for some (at least one) i,

(b) all sums $\quad \mathbf{v}_0 \cdots \mathbf{v}_i\mathbf{v}_{i+1} \cdots \mathbf{v}_k + \mathbf{v}_0 \cdots \mathbf{v}_{i+1}\mathbf{v}_i \cdots \mathbf{v}_k, \quad$ where $0 \leq i < k$.

We claim these subgroups define a chain subcomplex, and to show this we need to verify the following.

LEMMA 1. *The boundary homomorphism d_k sends elements of $C'_k(P, \mathbf{K})$ to $C'_{k-1}(P, \mathbf{K})$.* $\blacksquare$

It suffices to prove that the boundary map sends the previously described generators into degenerate chains, and checking this is essentially a routine calculation.

Here are the details: Suppose we have a generator of type (a), say $\mathbf{v}_0\ \cdots\ \mathbf{v}_k$ such that $\mathbf{v}_i = \mathbf{v}_{i+1}$ for some i. Then the generator's boundary is

$$\sum_{j=0}^{k} (-1)^j \partial_j \mathbf{v}_0\ \cdots\ \widehat{\mathbf{v}_j}\ \cdots\ \mathbf{v}_k \ = \sum_{j \neq i, i+1} (-1)^j \partial_j \mathbf{v}_0\ \cdots\ \widehat{\mathbf{v}_j}\ \cdots\ \mathbf{v}_k\ +$$

$$(-1)^i \mathbf{v}_0\ \cdots\ \widehat{\mathbf{v}_i}\ \cdots\ \mathbf{v}_k\ + (-1)^{i+1}\mathbf{v}_0\ \cdots\ \widehat{\mathbf{v}_{i+1}}\ \cdots\ \mathbf{v}_k$$

and since $\mathbf{v}_i = \mathbf{v}_{i+1}$ the last two terms cancel each other. Each term in the remaining sum has the form $\mathbf{v}_0\ \cdots\ \mathbf{v}_i\mathbf{v}_i\ \cdots\ \mathbf{v}_k$, and therefore this sum lies in $C'_{k-1}(P, \mathbf{K})$. Therefore the boundary takes a generator of type (a) to a chain in $C'_{k-1}(P, \mathbf{K})$.—You should try working this out for some relatively small value of k such as $k = 3$ or 4.

Suppose now that we have a generator of type (b), say

$$\mathbf{v}_0\ \cdots\ \mathbf{v}_i\mathbf{v}_{i+1}\ \cdots\ \mathbf{v}_k\ +\ \mathbf{v}_0\ \cdots\ \mathbf{v}_{i+1}\mathbf{v}_i\ \cdots\ \mathbf{v}_k\ .$$

If we take the boundary of this chain, as in the previous paragraph we see that is the sum of a chain in $C'_{k-1}(P, \mathbf{K})$ with the following sum of four terms:

$$(-1)^i \mathbf{v}_0 \ \cdots \ [\text{omit } \mathbf{v}_i] \ \mathbf{v}_{i+1} \ \cdots \ \mathbf{v}_k \ + \ (-1)^{i+1} \mathbf{v}_0 \ \cdots \ \mathbf{v}_i \ [\text{omit } \mathbf{v}_{i+1}] \ \cdots \ \mathbf{v}_k \ +$$

$$(-1)^i \mathbf{v}_0 \ \cdots \ [\text{omit } \mathbf{v}_{i+1}] \ \mathbf{v}_i \ \cdots \ \mathbf{v}_k \ + \ (-1)^{i+1} \mathbf{v}_0 \ \cdots \ \mathbf{v}_{i+1} \ [\text{omit } \mathbf{v}_i] \ \cdots \ \mathbf{v}_k$$

and in this case the first and last terms cancel, and the second and third terms also cancel. Therefore the boundary of a type (b) generator also lies in $C'_{k-1}(P, \mathbf{K})$.■

We now define the complex of *alternating simplicial chains* $C^{\text{alt}}_*(P, \mathbf{K})$ to be the quotient complex $C_*(P, \mathbf{K})/C'_*(P, \mathbf{K})$ with the associated differential or boundary map.

PROPOSITION 2. *The composite* $\varphi : C_*(P, \mathbf{K}^\omega) \to C_*(P, \mathbf{K}) \to C^{\text{alt}}_*(P, \mathbf{K})$ *is an isomorphism of chain complexes.*

COROLLARY 3. *The morphism* $i_* : H_*(P, \mathbf{K}^\omega) \to H_*(P, \mathbf{K})$ *is injection onto a direct summand.*

Proof that Proposition 2 implies Corollary 3. Let q be the projection map from unordered to alternating chains, so that $\varphi_* = q_* \circ i_*$. General considerations imply that φ_* is an isomorphism.

Suppose now that $i_*(a) = i_*(b)$. Applying q_* to each side we obtain

$$\varphi_*(a) \quad = \quad q_* \circ i_*(a) \quad = \quad q_* \circ i_*(b) \quad = \quad \varphi_*(b)$$

and since φ_* is bijective it follows that $a = b$.

Now let B_* be the kernel of q_*. We shall prove that every element of $H_*(P, \mathbf{K})$ has a unique expression as $i_*(a) + c$, where $c \in B_*$. Given $u \in H_*(P, \mathbf{K})$, direct computation implies that

$$u \ - \ i_*(\varphi_*)^{-1} q_*(u) \quad \in \quad B_*$$

and thus yields existence. Suppose now that $u = i_*(a) + c$, where $c \in B_*$. It then follows from the definitions that

$$i_*(a) \quad = \quad i_*(\varphi_*)^{-1} q_*(u)$$

and hence we also have

$$c \quad = \quad u \ - \ i_*(a) \quad = \quad u \ - \ i_*(\varphi_*)^{-1} q_*(u)$$

which proves uniqueness.■

Proof of Proposition 2. Analogs of standard arguments for determinants yield the following observations:

1. *The generator* $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k \in C_k(P, \mathbf{K})$ *lies in the subgroup of degenerate chains if two vertices are equal.*
2. *If* σ *is a permutation of* $\{0, \ \cdots \ , k\}$, *then* $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k - (-1)^{\text{sgn}(\sigma)} \mathbf{v}_{\sigma(0)} \ \cdots \ \mathbf{v}_{\sigma(k)}$ *is a degenerate chain.*