Intuitively it is clear that a closed interval is not homeomorphic to a Y shaped graph because the latter has a vertex which lies on exactly three edges. Similarly, in Munkres it is noted that a figure eight space (8) is not homeomorphic to a figure theta (θ) space even though they are homotopy equivalent, and one expects this because the first space has a graph decomposition for which there is a vertex lying on four edges and the analogous statement for the second space appears to be false. These statements appear to reflect something about the topological nature of neighborhoods of points in a space. Local homology provides an efficient means for handling such problems.

Definition. Let X be a Hausdorff topological space, and let $x \in X$. The *local homology groups of x in X* are given by $H_*(X, X - \{x\})$.

These groups have the following important properties:

PROPOSITION 3. (Localization property) *Let $x \in X$ where X is Hausdorff, and let U be an open neighborhood of x. Then the inclusion map of pairs induces isomorphisms from $H_q(U, U - \{x\})$ to $H_q(X, X - \{x\})$ for all integers q.*

Proof. This is an immediate consequence of the excision axiom (E.1) to U and $V = X - \{x\}$, for then $X = U \cup V$ and $U \cap V = U - \{x\}$.■

PROPOSITION 4. (Topological invariance) *If X and Y are Hausdorff spaces with $x \in X$, and if $f : X \to Y$ is a homeomorphism, then there is an isomorphism of local homology groups from $H_*(X, X - \{x\})$ to $H_*(Y, Y - \{f(x)\})$.*

Proof. The homeomorphism f induces a homeomorphism of pairs $(X, x - \{x\}) \cong (Y, Y - \{y\})$, so the associated homology groups must be isomorphic.■

For computational purposes the following result is very helpful when working with local homology groups:

LEMMA 5. *Suppose that $B \subset A \subset X$ and B is a deformation retract of A. Then the inclusion map of pairs induces isomorphisms from $H_*(X, B)$ to $H_*(X, A)$.*

Proof. By the exactness axiom (B.2) we have the following commutative diagram in which the rows are long exact homology sequences:

$$\begin{tikzcd}
\cdots & H_{q+1}(X,B) \arrow[r, "\partial"] \arrow[d, "g_*"] & H_q(B) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_q(X) \arrow[r, "j_*"] \arrow[d, "="] & H_q(X,B) \arrow[r, "\partial"] \arrow[d, "g_*"] & H_{q-1}(B) \arrow[d, "f_*"] & \cdots \\
\cdots & H_{q+1}(X,A) \arrow[r, "\partial"] & H_q(A) \arrow[r, "i_*"] & H_q(X) \arrow[r, "j_*"] & H_q(X,A) \arrow[r, "\partial"] & H_{q-1}(A) & \cdots
\end{tikzcd}$$

The mappings f and g are the associated inclusions of spaces or pairs. Since B is a deformation retract of A the maps f_* are isomorphisms, and of course the identity maps on $H_*(X)$ are also isomorphisms. Therefore the Five Lemma implies that the mappings g_* are also isomorphisms.■

Example. The hypotheses of the proposition do not imply that the inclusion $(X, B) \subset (X, A)$ is a homotopy equivalence of pairs; as noted in Hatcher, the inclusion $(D^n, S^{n-1}) \subset (D^n, D^n - \{0\})$ satisfies the hypothesis but this map is not a homotopy equivalence of pairs.

Application to graphs

If $(X, \mathcal{E})$ is a graph then it is easy to compute the local homology of X at all points.

See graphs.pdf for background

THEOREM 6. *Let $(X, \mathcal{E})$ be a connected graph, and let $x \in X$. Then the local homology group $H_1(X, X - \{x\})$ is given as follows:*

(i) If x is not a vertex for $\mathcal{E}$ then $H_1(X, X - \{x\}) \cong \mathbb{Z}$.

(ii) If x is a vertex which lies on exactly n edges, then $H_1(X, X - \{x\}) \cong \mathbb{Z}^{n-1}$.

In particular, if $n_k(X, \mathcal{E})$ is the number of vertices which lie on k vertices and $k \neq 2$, then Theorem 6 implies that $n_k(X, \mathcal{E})$ depends only upon the topological space X because it is the number of points in X for which the 1-dimensional local homology is isomorphic to $\mathbb{Z}^{k-1}$. Stated differently, if the underlying spaces of the connected graphs $(X, \mathcal{E})$ and $(X', \mathcal{E}')$ are homeomorphic, then $n_k(X, \mathcal{E})$ and $n_k(X', \mathcal{E}')$. One can apply this very easily to determine whether graphs corresponding to various letters of the alphabet are homeomorphic to each other, and there is a problem of this type in the exercises.

Proof of Theorem 6. By the Localization Property it suffices to compute the relative groups $H_1(U, U - \{x\})$ where U is some open neighborhood of x.

(i) Suppose that x lies in the edge E but is not an endpoint, and let F be the union of all the edges except E together with the vertices of E, and let $U = X - F$; then U is open and contains x, and the pair $(U, U - \{x\})$ is homeomorphic to $(V, V - \{t\})$ where V is the open unit interval $(0, 1)$ and $t \in V$. The local homology of the latter pair can be studied using the tail end of the long exact homology sequence:

$$\cdots \to 0 = H_1(V) \to H_1(V, V - \{t\}) \to H_0(V - \{t\}) \to H_0(V) = \mathbb{Z}$$

The homology groups of V are given as in this sequence because V is convex and hence contractible. Since the space $V - \{t\}$ has two components, axiom (D.2) and Proposition VI.3.1 implies that $H_0(V - \{t\}) \cong \mathbb{Z}^2$ and each free generator of the latter maps onto a free generator of $H_0(V)$. It follows immediately that the local homology group must be isomorphic to $\mathbb{Z}$.

(ii) Let V be the open star on the vertex x as defined in Unit III. Then $\{x\}$ is a deformation retract of V by Proposition III.1.7 and $V - \{x\}$ is homeomorphic to a union of pairwise disjoint subsets $V_j = E_j - [\mathit{endpoints}]$, where E_j runs through the n edges which have x as one of their endpoints. Since V is contractible one has an exact sequence for computing $H_1(V, V - \{t\})$ just like the one in the preceding paragraph. However, in this case we know that that $H_0(V - \{t\}) \cong \mathbb{Z}^n$ and each free generator of the latter maps onto a free generator of $H_0(V)$. It follows immediately that the local homology group must be isomorphic to $\mathbb{Z}^{n-1}$.■

Local homology also yields the following strengthening of Theorem 2.

THEOREM 7. (Invariance of dimension, L. E. J. Brouwer) *Let U and V be nonempty open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. If U is homeomorphic to V, then $m = n$.*

Proof. It suffices to prove that if W is a nonempty open subset of $\mathbb{R}^k$ then $H_k(W, W - \{p\}) \cong \mathbb{Z}$ and $H_j(W, W - \{p\}) = 0$ if $j \neq k$. Thus the local homology groups at points $x \in U$ and $y \in V$ are not isomorphic if $m \neq n$, and accordingly U and V cannot be homeomorphic in that case.

By the localization property it suffices to prove that the local homology groups $H_*(\mathbb{R}^k, \mathbb{R}^k - \{p\})$ are given as in the preceding paragraph. Furthermore, it suffices to consider the case where $p = \mathbf{0}$, for the translation map $T(x) = x + p$ is a homeomorphism which induces isomorphisms $H_*(\mathbb{R}^k, \mathbb{R}^k - \{\mathbf{0}\}) \cong H_*(\mathbb{R}^k, \mathbb{R}^k - \{p\})$.

It will be convenient to treat the case $k = 1$ separately. We can use the argument in the first part of Theorem 6 to prove that $H_1(\mathbb{R}, \mathbb{R} - \{0\}) \cong \mathbb{Z}$. Since $\mathbb{R} - \{0\}$ has the homotopy type of S^0

it follows that its homology vanishes in all dimensions except zero, so for each $q \geq 2$ we have the following exact sequence in which all terms except the relative group are known to be zero:

$$0 = H_q(\mathbb{R}) \to H_q(\mathbb{R}, \mathbb{R} - \{0\}) \to H_{q-1}(\mathbb{R} - \{0\}) \to H_{q-1}(\mathbb{R}) = 0$$

It follows that $H_q(\mathbb{R}, \mathbb{R} - \{0\})$ must also be zero if $q \geq 2$. Finally the 0-dimensional relative homology is given by the following piece of the long exact homology sequence

$$H_0(\mathbb{R} - \{0\}) \to H_0(\mathbb{R}) \to H_0(\mathbb{R}, \mathbb{R} - \{0\}) \to 0$$

since homology groups vanish in negative dimensions. We already know that the map at the left of this exact sequence is onto, and by exactness it follows that the second map is zero and the third is 1–1. These combine to imply that $H_0(\mathbb{R}, \mathbb{R} - \{0\}) = 0$.

Since S^{k-1} is a deformation retract of $\mathbb{R}^k - \{\mathbf{0}\}$ it follows that the homology groups of the latter are $\mathbb{Z}$ in dimensions $0, k-1$ and zero otherwise. If $q > 0$, then we have the exact homology sequence

$$0 = H_q(\mathbb{R}^k) \to H_q(\mathbb{R}^k, \mathbb{R}^k - \{\mathbf{0}\}) \to H_{q-1}(\mathbb{R}^k - \{\mathbf{0}\}) \to H_{q-1}(\mathbb{R}^k) \ .$$

If $q \geq 2$ then the groups at the end of this sequence are both zero, and therefore we have

$$H_q(\mathbb{R}^k, \mathbb{R}^k - \{\mathbf{0}\}) \quad \cong \quad H_{q-1}(\mathbb{R}^k - \{\mathbf{0}\})$$

for $q \geq 2$. This yields the conclusion of the theorem except in the cases $q = 0, 1$. The 0-dimensional case can be established by the same argument employed in the previous paragraph, so we are left with the 1-dimensional case, for which we have the following exact sequence:

$$0 = H_1(\mathbb{R}^k) \to H_1(\mathbb{R}^k, \mathbb{R}^k - \{\mathbf{0}\}) \to H_0(\mathbb{R}^k - \{\mathbf{0}\}) \to H_0(\mathbb{R}^k) = \mathbb{Z}$$

Since $k \geq 2$ the space $\mathbb{R}^k - \{\mathbf{0}\}$ is connected and hence the map at the right is an isomorphism. This implies that the map in the middle is zero and hence the map on the left is onto. Since the domain of the latter map is zero, it follows that $H_1(\mathbb{R}^k, \mathbb{R}^k - \{\mathbf{0}\}) = 0$.■

Further nonhomeomorphism theorems

We know that two spheres and two Euclidean spaces of different dimensions cannot be homeomorphic, and it is natural to ask similar questions about other familiar pairs like D^m and D^n or $\mathbb{R}^m_+$ and $\mathbb{R}^n_+$ where $\mathbb{R}^k_+$ denotes the points in $\mathbb{R}^k$ whose first coordinate is nonnegative. Invariance of domain provides effective criteria for dealing with such questions, and the following result will show that the paired spaces cannot be diffeomorphic if their dimensions are unequal:

THEOREM 8. *Suppose that X_m and X_n are subspaces of some $\mathbb{R}^p$ and for $k = m, n$ the set X_k has an open dense subset which is homeomorphic to an open subset of $\mathbb{R}^k$. If X_m and X_n are homeomorphic, then $m = n$.*

Proof. Let $U_k \subset X_k$ be the open dense subset, and let $h : X_m \to X_n$ be the homeomorphism. Then $h^{-1}[U_n]$ is dense in X_m because h is a homeomorphism, and it is open by the continuity of h, and $h[U_m]$ has analogous properties. It follows that the intersections $U_m \cap h^{-1}[U_n]$ and $h[U_m] \cap U_n$ are open subsets of U_m and U_n respectively and hence are homeomorphic to open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. These sets are homeomorphic because h maps the first to the second, and therefore Invariance of Dimension implies that $m = n$.■