**Proof.** Suppose first that $x \in \mathbb{R}^n$ for some $n$, so that $\{x\}$ is naturally a 0-dimensional polyhedron. We have already noted that the simplicial and singular chains on $X$ are isomorphic. Since the conclusion of the proposition holds for (unordered) simplicial chains by the results of the preceding unit, it follows that the same holds for singular chains. To prove the general case, note that if $\{x\}$ is an arbitrary space consisting of a single point and $\mathbf{0} \in \mathbb{R}^n$, then $\{\mathbf{0}\}$ is homeomorphic to $\{x\}$ and in this case the conclusion follows from the special case because homeomorphic spaces have isomorphic homology groups.■

*The compact supports property*

Our next result verifies (C.2) and is often summarized with the phrase, *singular homology is compactly supported*. This was not one of the original Eilenberg-Steenrod axioms, but its importance for using singular homology was already clear when Eilenberg and Steenrod developed singular homology.

**THEOREM 8.** *Let $X$ be a topological space, and let $u \in H_q(X)$. Then there is a compact subspace $A \subset X$ such that $u$ lies in the image of the associated map from $H_q(A)$ to $H_q(X)$. Furthermore, if $A$ is a compact subset of $X$ and $u, v \in H_q(A)$ are two classes with the same image in $H_q(X)$, then there is a compact subset $B$ satisfying $A \subset B \subset X$ such that the images of $u$ and $v$ are equal in $H_q(B)$.*

**Proof.** If $c$ is a singular $q$-chain and

$$c \;=\; \sum_j n_j\, T_j$$

define the *support* of $c$, written $\mathsf{Supp}\,(c)$, to be the compact set $\cup_j\; T_j(\Delta_q)$. Note that this subset is compact.

If $u \in H_q(X)$ is represented by the chain $z$ and if $A = \mathsf{Supp}\,(z)$, then since $S_*(A) \to S_*(X)$ is 1–1 it follows that $z$ represents a cycle in $A$ and hence $u$ lies in the image of $H_q(A) \to H_q(X)$.

Suppose now that $A$ is a compact subset of $X$ and $u, v \in H_q(A)$ are two classes with the same image in $H_q(X)$. Let $z$ and $w$ be chains in $S_q(A)$ representing $u$ and $v$ respectively, and let $b \in S_{q+1}(X)$ be such that $d(b) = i_{\#}(z) - i_{\#}(w)$. If we set $B = A \cup \mathsf{Supp}\,(b)$, then it follows that the images of $z - w$ bounds in $S_q(B)$, and therefore it follows that $u$ and $v$ have the same image in $H_q(B)$.■

## II.2 : Exactness and homotopy invariance

(Hatcher, §§ 2.1, 2.3)

We have seen that long exact sequences and homotopy invariance yield a great deal of information about homology groups. The next step is to verify some of the properties