VII.3 : Separation and invariance theorems

($\mathbf{H}$, §2.B; $\mathbf{M}$, §63)

Most of this has become standard in algebraic topology texts, and we shall quote Hatcher as appropriate. The following result corresponds to the first half of Proposition 2B.1 on page 169 of that reference.

PROPOSITION 1. *If $n > 0$ and $A \subset S^n$ is homeomorphic to D^k for some $k < n$, then the $H_i(S^n - A)$ is infinite cyclic if $i = 0$ and trivial otherwise.*

Note. The hypotheses imply that A must be a proper subset of $\mathbb{R}^n$ because the homology groups of A and S^n are not isomorphic ($H_n(A) = 0$ if $n > 1$).

Proof. The proof proceeds by induction on k. If $k = 0$ then $S^n - A$ is homeomorphic to $\mathbb{R}^n$ and the conclusion in this case follows immediately. Assume now that the result is known whenever a subset A is homeomorphic to D^{k-1} for some k satisfying $1 \leq k \leq n$, and assume now that $A \subset S^n$ is homeomorphic to D^k.

The homeomorphisms of Section VII.1 imply that D^n is homeomorphic to $D^{n-1} \times [0,1]$. If $t \in [0,1]$ let $A_t \subset A$ correspond to $D^{n-1} \times \{t\}$ under some fixed homeomorphism $A \cong D^{n-1} \times [0,1]$, and if $a < b$ let $A[a,b] \subset A$ correspond to $D^{n-1} \times [a+, b-]$, where $a+$ is the larger of a and 0, and $b-$ is the smaller of b and 1.

Suppose now that $u \in H_q(S^n - A)$ lies in the kernel of the homomorphism $c_* : H_q(S^n - A) \to H_q(P)$, where P is a one point space and $c : S^n - A \to P$ is the constant map. We want to show that $u = 0$; the compact supports property implies the existence of a compact subset $L \subset S^n - A$ such that u lies in the image of the map $H_q(L) \to H_q(A)$ induced by inclusion; let u' be a class which maps to u in this fashion. If

$$j_t : S^n - A \longrightarrow S^n - A_t$$

is the inclusion mapping, then the inductive hypothesis implies that $j_{t*}(u) = 0$ for each $t \in [0,1]$.

By the compact supports property, for each t there is some compact subset K_t such that $L \subset K_t \subset S^n - A_t$ such that u maps to zero under the homology map associated to the inclusion $H_q(L) \to H_q(K_t)$. Since A_t and K_t are disjoint compact subsets of S^n, there is some $\varepsilon(t) > 0$ such that K_t and $A_{[t-\varepsilon(t), t+\varepsilon(t)]}$ are disjoint. It follows that the image of u in the homology of $H_q(A[t - \varepsilon(t), t + \varepsilon(t)])$ is zero.

The open or half open intervals $(t - \varepsilon(t), t_\varepsilon(t)) \cap [0,1]$ form an open covering of $[0,1]$, so by the Lebesgue Covering Lemma there is some $M > 0$ such that every closed interval of length $\leq 1/M$ lies in some subset of this open covering. It follows that u maps to zero in each of the sets $H_q(S^n - A[j - 1/M, j/M])$ where $j = 1, ..., M$. The next objective is to show by induction on j that u maps to zero in $H_q(S^n - A[0, j/M])$; the case $j = 1$ is known by the preceding discussion, and when $j = M$ the set $A[0, j/M]$ is all of A.

Assume now that u maps to zero in $H_q(S^n - A[0, j/M])$ where $1 \leq j \leq M - 1$. Then the identities

$$A[0,(j+1)/M] \ = \ A[0,j/M] \cup A[j/M,(j+1)/M] \ , \qquad A_{j/M} \ = \ A[0,j/M] \cap A[j/M,(j+1)/M]$$

and their complementary analogs

$$S^n - A[0,(j+1)/M] \quad = \quad (S^n - A[0,j/M]) \cap (S^n - A[j/M,(j+1)/M])$$

$$S^n - A_{j/M} \quad = \quad (S^n - A[0,j/M]) \cup (S^n - A[j/M,(j+1)/M])$$

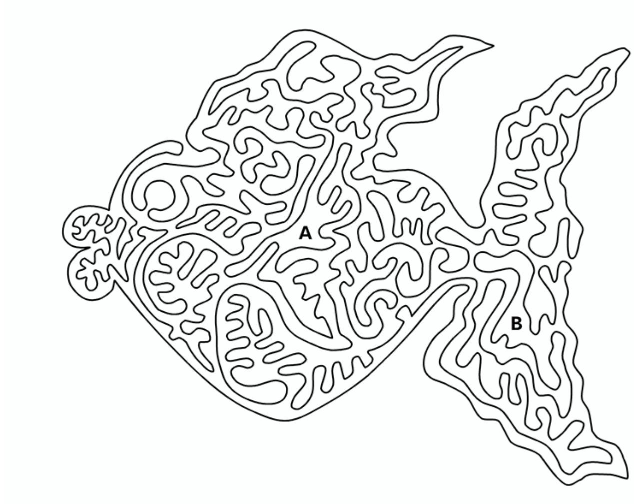
and the latter determine a long exact Mayer-Vietoris sequence. Consider the following piece of that sequence:

$$H_{q+1}(S^n - A_{j/M}) \to H_q(S^n - A[0,(j+1)/M]) \to H_q(S^n - A[0,j/M]) \oplus H_q(S^n - A[j/M,(j+1)/M])$$

If $u' \in H_q(S^n - A[0,(j+1)/M])$ is the image of u under the map induced by the inclusion $S^n - A \subset S^n - A[0,(j+1)/M]$, then the inductive hypotheses and the previous arguments show that u' maps to zero in each of the groups $H_q(S^n - A[0,j/M])$ and $H_q(S^n - A[j/M,(j+1)/M])$. Therefore by exactness u' lies in the image of $H_{q+1}(S^n - A_{j/M})$. Since the latter group vanishes, it follows that u' must be zero, completing the inductive argument with respect to j. As noted in the preceding paragraph, this implies that $u = 0$ and completes the inductive argument with respect to the dimension k such that $A \cong D^k$.■

The Jordan-Brouwer Separation Theorem

If we remove a circle from the plane, we obtain two connected regions — an interior and an exterior region — and mathematically these regions are defined by the inequalities $|x - a| < r$ and $|x - a| > r$, where a is the center of the circle and r is its radius. Similarly, if we are given a relatively simple example of a simple closed curve in the plane, it is generally easy to see that the complement is a union of two disjoint connected components, and it is natural to conjecture that the same is true for an arbitrary simple closed curve in the plane. However, as curves become more complicated it becomes increasingly difficult to verify this explicitly for examples like the maze in the drawing below.



The online article

http://en.wikipedia.org/wiki/Jordan_curve_theorem

discusses the history of this result fairly extensively (and corrects some widely circulated misinformation), and the article is definitely worth reading. Everyday experience with geometric objects in 3-space strongly suggests that there are analogous results for suitably defined closed surfaces in $\mathbb{R}^3$ which include subsets homeomorphic to S^2, and the Jordan-Brouwer Separation Theorem generalizes the Jordan Curve Theorem to subsets of $\mathbb{R}^n$ which are homeomorphic to S^{n-1} for all values of n.

Our statement the Jordan-Brouwer Theorem contains a somewhat stronger conclusion than the version in Hatcher; the case $n = 2$ is the classical Jordan Curve Theorem.

THEOREM 2. (Jordan-Brouwer Separation Theorem.) *Let $n \geq 2$, and suppose that $A \subset S^n$ is homeomorphic to S^{n-1}. Then $S^n - A$ contains two components, and A is the frontier of each component.*

Note. In the discussion preceding the statement of this theorem we have considered compact subsets $A \subset \mathbb{R}^n$ which are homeomorphic to S^{n-1}, and in fact the analogous conclusion for subsets of $\mathbb{R}^n$ follows from the theorem by passing to one point compactifications. In fact, one can say slightly more; namely, exactly one of the components of $\mathbb{R}^n - A$ must be a bounded open subset (specifically, the component not containing the point at infinity in the one point compactification of $\mathbb{R}^n$ under the identification of the latter with S^n).

It is natural to ask if one also has similar results if A is homeomorphic to some other closed surface such as the torus T^{n-1}, and the answer is that similar conclusions hold more generally. In particular this follows from results on the homology of compact manifolds and the Alexander Duality Theorem in Section 3.3 of Hatcher.

The standard textbook proof of the Jordan-Brouwer Separation Theorem involves proving the following complementary result on the homology of subsets of S^n which are homeomorphic to spheres of dimension $\leq n - 2$.

PROPOSITION 3. *Let $A \subset S^n$ be homeomorphic to S^k where $0 \leq k \leq n - 2$. Then the homology groups of $S^n - A$ are homeomorphic to the homology groups of S^{n-k-1}.*

It is important to note that the complement does not necessarily have the homotopy type of S^{n-k-1}. In particular, there are simple closed (knotted) curves K in $\mathbb{R}^3$ and S^3 for which the fundamental group is nonabelian and hence not isomorphic to $\pi_1(S^1)$; there is an extensive theory of knotted curves in 3-space which goes far beyond the scope of this course, and currently there is a high level of activity aimed at answering many open questions about such curves.

On the other hand, for the standard linear embedding of S^k in S^n corresponding to

$$S^k \quad \subset \quad \mathbb{R}^{k+1} = \mathbb{R}^{k+1} \times \{\mathbf{0}\} \quad \subset \mathbb{R}^{k+1} \times \mathbb{R}^{n-k} \;\cong\; \mathbb{R}^{n+1}$$

(note that the image is contained in S^n), the complement is homeomorphic to $S^{n-k-1} \times \mathbb{R}^{k+1}$ such that $S_{n-k-1} \times \{\mathbf{0}\}$ corresponds to the unit sphere in $\{\mathbf{0}\} \times \mathbb{R}^{n-k}$. See the file `sphere-complements.pdf` for a proof of this fact.

Proof of Proposition 3. The proof proceeds by induction on k, so we need to start by verifying the result in that case, in which A consists of two points. Since S^n is highly symmetric we can assume that one of the points is the unit vector $\mathbf{e}_{n+1} \in \mathbb{R}^{n+1}$, which implies that if one point of A is removed the complement is homeomorphic to $\mathbb{R}^n$. If we now remove the second point we are left with a subset homeomorphic $\mathbb{R}^n - \{p\}$, and since the latter is homeomorphic to $S^{n-1} \times \mathbb{R}$ the conclusion about homology groups in this case follows immediately.

Suppose now that the result is known for subsets homeomorphic to S^{k-1}, where $1 \leq k \leq n-2$. Let A be a subset which is homeomorphic to S^n, and let $A_\pm \subset A$ be the subspace corresponding to the hemisphere $D^n_\pm \subset S^n$ defined by the coordinate inequalities $x_{n+1} \geq 0$ (for D^n_+) and $x_{n+1} \leq 0$ (for D^n_-). Let $A_0 = A_+ \cap A_-$, so that A_0 is homeomorphic to S^{k-1}. Consider now the Mayer-Vietoris sequence for the decomposition

$$S^n - A_0 \quad = \quad (S^n - A_+) \cup (S^n - A_-) \ , \qquad \text{where} \quad S^n - A \quad = \quad (S^n - A_+) \cap (S^n - A_-) \ .$$

We are particularly interested in the following pieces of this exact sequence:

$$H_{q+1}(S^n - A_+) \oplus H_{q+1}(S^n - A_-) \to H_{q+1}(S^n - A_0) \to H_q(S^n - A) \to H_q(S^n - A_+) \oplus H_q(S^n - A_-)$$

If $q > 0$ then the first and last terms of this exact sequence are zero by Proposition 1, and hence the map from the second term to the third is an isomorphism. By induction we know that $H_{q+1}(S^n - A_0)$ is trivial for all $q \geq 1$ except $q+1 = n-(k-1)-1$ and is infinite cyclic in the latter case. Therefore we have $H_q(S^n - A) = 0$ if $q > 0$ and $q \neq n-k-1$, and in the latter case we have $H_q(S^n - a) \cong \mathbb{Z}$.

Suppose now that $q = 0$ in the displayed exact sequence. Then we have $H_1(S^n - A_0) = 0$ and $H_0(S^n - A_0) = \mathbb{Z}$ because $k - 2 \leq n$, and therefore the extended Mayer-Vietoris sequence reduces to

$$0 = H_1(S^n - A_0) \to H_0(S^n - A) \to H_0(S^n - A_+) \oplus H_0(S^n - A_-) \cong \mathbb{Z} \oplus \mathbb{Z} \to H_0(S^n - A_0) = Z$$

where the map at the right is surjective, so that its kernel is isomorphic to $\mathbb{Z}$. By exactness this kernel is the image of $H_0(S^n - A)$, and the mapping from the latter onto the kernel is 1–1, so that we have $H_0(S^n - A) \cong \mathbb{Z}$. This completes the proof of the inductive step.■

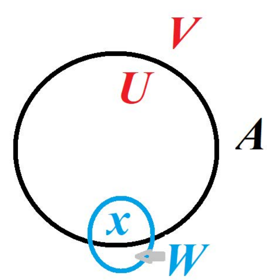
Proof of the Jordan-Brouwer Separation Theorem. The first step is to prove that the complement has exactly two components. Let $A_\pm$ and A_0 be defined as in the preceding proposition and consider the corresponding Mayer-Vietoris sequence; in particular, we are interested in the following piece:

$$0 = H_1(S^n - A_+) \oplus H_1(S^n - A_-) \to H_1(S^n - A_0) \to H_0(S^n - A) \to \ (\textit{next line})$$
$$H_0(S^n - A_+) \oplus H_0(S^n - A_-) \cong \mathbb{Z} \oplus \mathbb{Z} \to H_0(S^n - A_0) = Z$$

In this case we know that $H_1(S^n - A_0) \cong \mathbb{Z}$ and hence the latter maps injectively into $H_0(S^n - A)$. Furthermore, we can use the same argument as in Proposition 3 to conclude that the image of $H_0(S^n - A)$ in the direct sum is also isomorphic to $\mathbb{Z}$, and therefore by exactness we must have $H_0(S^n - A) \cong \mathbb{Z} \oplus \mathbb{Z}$, so that $S^n - A$ has exactly two components.

It remains to prove that points of A are limit points of each component. Suppose that $S^n - A$ is the union of the two open, connected, disjoint subsets U and V.

Assume that not every point of A is a limit point of both U and V. Without loss of generality, it is enough to consider the case where $x \in A$ is not a limit point of V. Since $x \notin V$, it follows that there is some open set W_0 in S^n such that $x \in W_0$ and $W_0 \cap V = \emptyset$.



Consider the open set $W_0 \cap A$ in A; since the latter is homeomorphic to S^{n-1}, it follows that there is a subneighborhood of the form $A - E$, where $E \subset A$ is homeomorphic to a closed $(n-1)$-

disk and $A - E$ is homeomorphic to an open $(n-1)$-disk centered at x. If $W = W_0 \cap S^n - E$, then W is still open in S^n and we still have $x \in W$ and $W \cap V = \emptyset$.

By construction we have $S^n - E = U \cup A - E \cup V$ where the pieces are pairwise disjoint. Furthermore, we have $A - E \subset W$ and hence $U \cup W$ is an open set of $S^n - E$ which is disjoint from V and contains U and $A - E$. Therefore it follows that $S^n - E$ is a union of the nonempty disjoint open sets $U \cup W$ and V and hence is disconnected. On the other hand, since E is homeomorphic to a closed disk we know that $S^n - E$ is connected, so we have a contradiction. The source of this contradiction was our assumption that x was not a limit point of V, and hence this must be false. Therefore x must be a limit point of V, and as noted above it follows that every point of A is a limit point of both U and V.■

The Mayer-Vietoris sequence in Theorem 2 also has the following implication; details of the proof are left to the reader (remember that the homology groups of $S^n - A_{\pm}$ and $S^n - A_0$ in positive dimensions are known to vanish except for $H_1(S^n - A_0)$):

COROLLARY 4. *In the setting of Theorem* 2 *the homology groups of each component of* $S^n - A$ *are zero in every positive dimension.*■

If $n = 2$ a remarkable theorem of A. Schönflies yields a much stronger conclusion: If U is a component of $S^2 - A$ then its closure $\overline{U}$ is homeomorphic to D^2 such that A corresponds to S^1 (it is also possible to use results from complex variable theory to prove the weaker result that the open set U is simply connected). On the other hand, if $n \geq 3$ then a component U of $S^n - A$ need not even be simply connected. The standard example when $n = 3$ is the Alexander Horned Sphere discussed in Example 2.B.2 on pages 170–172 of Hatcher. The following online site has an interesting video showing the recursive construction of the Alexander sphere:

`http://www.youtube.com/watch?v=Pe2mnrLUYFU`

With the preceding results at our disposal, we can prove the following basic result exactly as in Hatcher:

THEOREM 5. (Invariance of Domain, Brouwer) *Let U be an open subset of $\mathbb{R}^n$ for some $n \geq 2$, and let $h : U \to \mathbb{R}^n$ be continuous and* $1-1$. *Then h is an open mapping, the image $h[U]$ is an open subset of $\mathbb{R}^n$, and h maps U homeomorphically onto $h[U]$.*

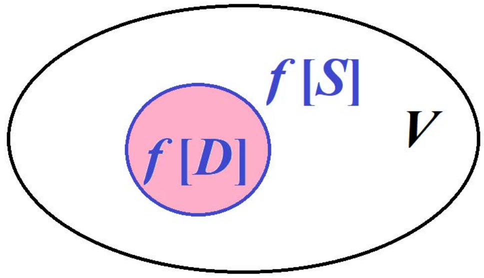
The name for the result reflects the following equivalent statement: If a subset $V \subset \mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^n$, then it must also be an open subset of $\mathbb{R}^n$.

Proof. It will suffice to prove that h is an open mapping, and to prove the latter it will suffice to show that if $D \subset U$ is an ordinary closed disk of some radius about a point of U and ∂D is the boundary sphere of D, then $f[D - \partial D]$ is an open subset of $\mathbb{R}^n$ (since every open subset of U is a union of open disks that are interiors of closed disks). Since f is 1–1 it follows that f maps D and ∂D homeomorphically onto their images.

As usual, view $\mathbb{R}^n$ as $S^n - \{p\}$ via one point compactification. Then the preceding results imply that $S^n - f[D]$ is a connected open subset and $S^n - f[\partial D]$ is an open subset with two components, say W_1 and W_2; label these so that $S^n - f[D] \subset W_1$. Now we also have

$$S^n - f[\partial D] \quad = \quad (S^n - f[D]) \ \cup \ (S^n - f[D - \partial D])$$

and the subsets on the right hand side are disjoint; since $f[D - \partial D]$ is connected it is contained in one of the components W_1, W_2.



If $f[D - \partial D]$ were contained in W_1 then we would have $S^n - f[\partial D] \subset W_1 \subset S^n - f[\partial D]$ so that the two sets would be equal, contradicting the fact that $S^n - f[\partial D]$ is disconnected. Therefore $f[D - \partial D]$ must be contained in W_2. This gives us the chain of inclusions

$$(S^n - f[D]) \ \cup \ (S^n - f[D - \partial D]) \quad \subset \quad W_1 \cup W_2 \quad \subset \quad S^n - f[\partial D]\ , \quad \text{where}$$

$$(S^n - f[D]) \ \cap \ (S^n - f[D - \partial D]) \quad = \quad \emptyset \quad = \quad W_1 \cap W_2\ .$$

Since $S^n - f[D] \subset W_1$ and $S^n - f[D - \partial D] \subset W_2$, the set-theoretic relations combine to imply that $S^n - f[D] = W_1$ and $S^n - f[D - \partial D] = W_2$. This proves that $f[D - \partial D]$ is an open subset of $\mathbb{R}^n$ (hence also of U), by the statement at the beginning of the proof this also completes the proof of the theorem.■

We shall limit ourselves to one simple consequence.

COROLLARY 6. *If $\mathbb{R}^n_+$ is defined to be the set of all points whose last coordinate is nonnegative, then $\mathbb{R}^n_+$ is not homeomorphic to $\mathbb{R}^m$ for any positive integer m.*

Proof. We first consider the cases where $n \leq m$. In these cases the sets cannot be homeomorphic by Invariance of Domain because $\mathbb{R}^n_+$ is not an open subset of $\mathbb{R}^m$ (as usual, we identify $\mathbb{R}^n$ with the set of all points in $\mathbb{R}^m$ whose last $m - n$ coordinates are all zero).

Suppose now there is a homomorphism f from $\mathbb{R}^m$ to $\mathbb{R}^n_+$ where $m < n$. If H is the hyperplane in $\mathbb{R}^n$ of all points whose last coordinate is zero and $W = \mathbb{R}^m - f^{-1}[H]$, then f defines a homeomorphism from W to $\mathbb{R}^n_+ - H \cong \mathbb{R}^n$. This is impossible by invariance of dimension, and therefore $\mathbb{R}^n_+$ cannot be homeomorphic to $\mathbb{R}^m$ if $m < n$.■

VII.4 : Nonplanar graphs

(**M**, §64)

We have already seen that every graph has a nice rectilinear embedding in $\mathbb{R}^3$. In this section we shall use homology theory to prove that some graphs do not admit any topological embeddings into $\mathbb{R}^2$. We shall treat two examples, and at the end of this section we shall explain why they are particularly important. The approach in this section is close to that in Munkres, the main difference being that we use homology theory to give simpler proofs of some key steps in the arguments.