.

If $f[D - \partial D]$ were contained in $W_1$ then we would have $S^n - f[\partial D] \subset W_1 \subset S^n - f[\partial D]$ so that the two sets would be equal, contradicting the fact that $S^n - f[\partial D]$ is disconnected. Therefore $f[D - \partial D]$ must be contained in $W_2$. This gives us the chain of inclusions

$$(S^n - f[D]) \cup (S^n - f[D - \partial D]) \quad \subset \quad W_1 \cup W_2 \quad \subset \quad S^n - f[\partial D] \, , \quad \text{where}$$

$$(S^n - f[D]) \cap (S^n - f[D - \partial D]) \quad = \quad \emptyset \quad = \quad W_1 \cap W_2 \, .$$

Since $S^n - f[D] \subset W_1$ and $S^n - f[D - \partial D] \subset W_2$, the set-theoretic relations combine to imply that $S^n - f[D] = W_1$ and $S^n - f[D - \partial D] = W_2$. This proves that $f[D - \partial D]$ is an open subset of $\mathbb{R}^n$ (hence also of $U$), by the statement at the beginning of the proof this also completes the proof of the theorem.■

We shall limit ourselves to one simple consequence.

**COROLLARY 6.** *If $\mathbb{R}^n_+$ is defined to be the set of all points whose last coordinate is nonnegative, then $\mathbb{R}^n_+$ is not homeomorphic to $\mathbb{R}^m$ for any positive integer $m$.*

**Proof.** We first consider the cases where $n \leq m$. In these cases the sets cannot be homeomorphic by Invariance of Domain because $\mathbb{R}^n_+$ is not an open subset of $\mathbb{R}^m$ (as usual, we identify $\mathbb{R}^n$ with the set of all points in $\mathbb{R}^m$ whose last $m - n$ coordinates are all zero).

Suppose now there is a homomorphism $f$ from $\mathbb{R}^m$ to $\mathbb{R}^n_+$ where $m < n$. If $H$ is the hyperplane in $\mathbb{R}^n$ of all points whose last coordinate is zero and $W = \mathbb{R}^m - f^{-1}[H]$, then $f$ defines a homeomorphism from $W$ to $\mathbb{R}^n_+ - H \cong \mathbb{R}^n$. This is impossible by invariance of dimension, and therefore $\mathbb{R}^n_+$ cannot be homeomorphic to $\mathbb{R}^m$ if $m < n$.■

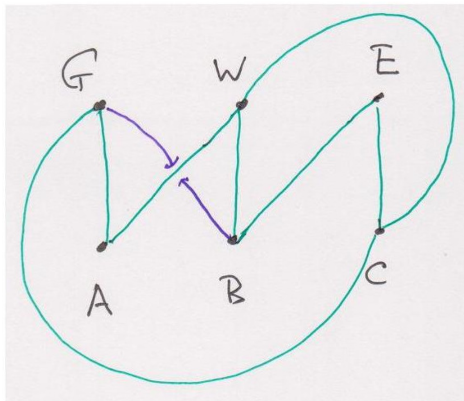
## VII.4 : Nonplanar graphs

(**M**, §64)

We have already seen that every graph has a nice rectilinear embedding in $\mathbb{R}^3$. In this section we shall use homology theory to prove that some graphs do not admit any topological embeddings into $\mathbb{R}^2$. We shall treat two examples, and at the end of this section we shall explain why they are particularly important. The approach in this section is close to that in Munkres, the main difference being that we use homology theory to give simpler proofs of some key steps in the arguments.

*The utilities network*

This is a fairly well-known example with three vertices $a, b, c$ representing houses and another three vertices $g, w, e$ representing gas, water and electricity utilities. There are nine edges which they join the individual houses to each of the three utilities, and the question is whether this can be done on a flat surface with none of the lines crossing over or under each other.




This graph is often called $K_{3,3}$. In mathematical terms, here is what we what to prove:

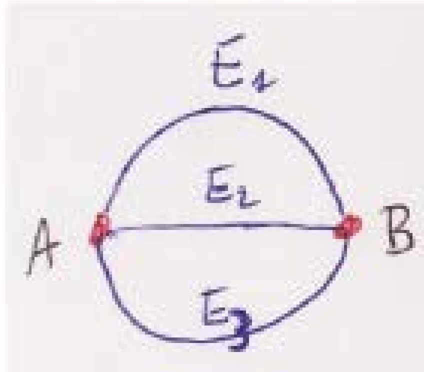
**THEOREM 1.** *The utilities network $K_{3,3}$ is not homeomorphic to a subset of $S^2$.*

In fact, one has the same conclusion if $S^2$ is replaced by $\mathbb{R}^2$ because $K_{3,3}$ and $S^2$ are not homeomorphic — the quickest way to see this is to note that $H_2(K_{3,3}) = 0$ for dimensional reasons but $H_2(S^2) \cong \mathbb{Z}$.

As suggested by the figure below, it is fairly easy to embed the subgraph of $K_{3,3}$ by removing one edge; the point of the proof is that there cannot be some clever way of inserting the remaining edge.

**(Remove edge *BG* from the previoius drawing.)**

The proof of Theorem 1 involves separation theorems that are similar to the Jordan Curve Theorem but are somewhat more complicated to state and prove. The first of these involves *theta spaces* which can be expressed as unions of three subsets $E_1, E_2, E_3$ which are all homeomorphic to $[0, 1]$ and whose intersections are given by their endpoints.




We want to prove that every theta space in $S^2$ has the separation properties which are apparent in the figure. This can be stated formally as follows:

**PROPOSITION 2.** *If $X \subset S^2$ is a theta space with edges $E_1, E_2, E_3$ meeting at the common endpoints $\{A, B\}$, then $S^2 - X$ has three connected components $U, V, W$ such that*

> *the boundary of $U$ is $E_1 \cup E_2$,*
>
> *the boundary of $V$ is $E_2 \cup E_3$,*
>
> *the boundary of $U$ is $E_1 \cup E_3$.*

Note that we can make $X$ into a graph by taking the derived decomposition that we defined in `graphs.pdf`.

**Proof.** There are three main steps. First, we prove that $S^2 - X$ has exactly three components. Next, we prove that $E_1 \cup E_3$ is the boundary of one of these components. Finally, we use the same sort of argument to obtain similar conclusions for $E_2 \cup E_3$ and $E_1 \cup E_2$. Since the simple closed curves given by $E_1 \cup E_2$, $E_2 \cup E_3$, are distinct, it follows that they bound distinct components of $S^2 - X$, and since there are exactly three components in the latter, it follows that each is the boundary of one of the given simple closed curves.

By the preceding discussion, we need only show the assertions that $S^2 - X$ has three components and one component of $S^2 - X$ has $E_1 \cup E_3$ as its boundary. We shall begin by proving the first statement, and it will be convenient to introduce some notation for certain open subsets of $S^2$. For $i = 1, 2, 3$ let $U_i = S^2 - E_i$ and if $i \neq j$ let $U_{i,j}$ be

$$S^2 \; - \; (E_i \cup E_j) \quad = \quad U_i \cap U_j \; .$$

Finally, let $U_{1,2,3}$ be $S^2 - X$ and note that the latter is equal to $U_1 \cap U_2 \cap U_3$. Consider the Mayer-Vietoris exact sequence associated to the decomposition $U_3 = U_{1,3} \cup U_{2,3}$, noting that $U_{1,2,3} = U_{1,3} \cap U_{2,3}$. Since $U_3$ has the homology of a point by Proposition 1 of the preceding section, the final nontrival terms in the Mayer-Vietoris sequence are given as follows:

$$0 = H_1(U_3) \to H_0(U_{1,2,3}) \to H_0(U_{1,2}) \oplus H_0(U_{2,3}) \to H_0(U_3) \cong \mathbb{Z}$$

By the Jordan Curve Theorem the direct sum isomorphic to $\mathbb{Z}^4$, and the axiom regarding 0-dimensional homology implies that the standard free generators for this direct sum all map to the standard free generator of $H_0(U_3) \cong \mathbb{Z}$. Therefore the kernel of the map from the direct sum into $H_0(U_3)$ is isomorphic to a free abelian group on three generators, and by exactness this group is isomorphic to the image of the map $\varphi$ from $H_0(U_{1,2,3})$ to $H_0(U_{1,2}) \oplus H_0(U_{2,3})$. Since $H_1(U_3) = 0$ it also follows that $\varphi$ is 1–1, and therefore we have shown that $H_0(U_{1,2,3} = S^2 - X)$ is a free abelian group on three generators. This computation implies that the open set $U_{1,2,3}$ has exactly three components, so we have completed the first step of the proof.

The only point remaining is to prove that $E_1 \cup E_3$ is the boundary of one component in $S^2 - X$. By the Jordan Curve Theorem, the set $U_{1,3} = S^2 - (E_1 \cup E_3)$ has two components and $E_1 \cup E_2$ is the boundary of each one. Denote these components by $V$ and $W$, and notice that one of them must contain the connected set $E_2 - \{A, B\}$. Without loss of generality, we may assume that this component is $W$ (if not, reverse the roles of $V$ and $W$ in the discussion which follows). We then have

$$U_{1,2,3} \quad = \quad V \ \cup (W - (E_2 - \{A, B\})) \ .$$

Each of the summands on the right is an open and closed subset of $U_{1,2,3}$, and therefore each component of $U_{1,2,3}$ is contained in $V$ or $W$. Now we know that $V \subset U_{1,2,3}$, and $V$ must be a component of $U_{1,2,3}$ because $V$ is a maximal connected subset of $U_{1,3}$, which contains $U_{1,2,3}$, and hence $V$ is also a maximal connected subset of $U_{1,2,3}$. By construction the boundary of $V$ is $E_1 \cup E_3$, and thus we have shown that the latter bounds one component of $U_{1,2,3} = S^2 - X$. Since we have already noted that this assertion (plus the one about three components) imply the conclusion of the proposition, this completes the proof.■

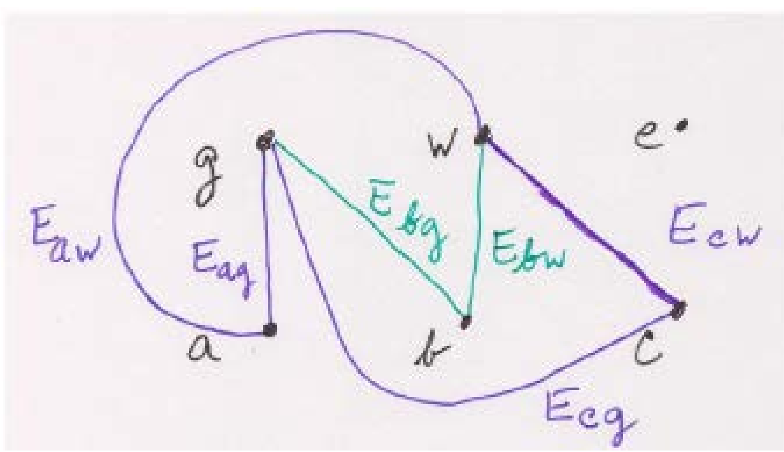
At a later point we shall also need information about the higher homology groups of the (components of the) space $S^2 - X$ when $X$ is a theta space. The result is analogous to Corollary VII.3.4.

**COROLLARY 3.** *If $X \subset S^2$ is a theta space and $U$ is a component of $S^2 - X$, then $H_i(U) = 0$ for all $i > 0$, and likewise for $H_i(S^2 - X)$.*

**Proof.** The proof of the proposition shows that the boundary of each component is a simple closed curve, and thus we can apply Corollary VII.3.4 directly to find the higher dimensional homology of $U$. The statement about $S^2 - X$ follows because this space is locally arcwise connected and hence its homology is the direct sum of the homology of its components.■

We are now ready to prove that the graph $K_{3,3}$ is not topologically embeddable in $\mathbb{R}^2$.

**Proof of Theorem 1.** We shall assume that there is a topological embedding of the graph in $S^2$ and derive a contradiction. It may be worthwhile to look at the figure belowin order to visualize the steps in the argument.




Let $X$ be a graph, and let $X_0 \subset X$ be the subgraph consisting of all edges that do not have $e$ as a vertex. If $p$ and $q$ are vertices which are endpoints of some edge, denote that edge by $E_{pq}$. Then $X_0$ is a theta space with edges

$$L_1 = E_{ag} \cup E_{aw} , \qquad L_2 = E_{bg} \cup E_{bw} , \qquad L_3 = E_{cg} \cup E_{cw} .$$

Then Proposition 2 implies that $S^n - X_0$ has three components, and the remaining vertex $e \in X$ must lie in one of them, say $U$. It follows that each of the half-open intervals

$$E_{ae} - \{a\} , \quad E_{be} - \{b\} , \quad E_{ce} - \{c\}$$

must be contained in the component $U$ because each is connected and contains $e$. Therefore each of $a, b, c$ must lie in the closure $\overline{U}$ of $U$.

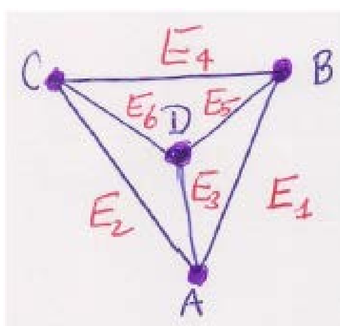
Trial and error suggests that the conclusion of the preceding sentence is impossible, and we shall now give mathematical reasons for this. The endpoints of $L_1, L_2$ and $L_3$ are $g$ and $w$, and we also know that $a \in L_1$, $b \in L_2$ and $c \in L_3$ but none of these points can be endpoints of an edge $L_i$. Proposition 2 implies that the boundary of $U$ is the union of exactly two of these edges, so only two of the points in $\{a, b, c\}$ can lie in $\overline{U}$, and thus we have derived a contradiction. The source of this contradiction was the assumption that $X$ could be topologically embedded in $S^2$, and therefore we know this assumption is false. As noted earlier, this suffices to complete the proof of the theorem.■

*The complete graph on 5 vertices*

We now proceed to the next example. Recall that the complete graph on $n$ vertices is a graph with $n$ vertices such that for each pair of vertices $\{p, q\}$ there is an edge whose endpoints are $p$ and $q$.

**THEOREM 3.** *The complete graph on 5 vertices is not homeomorphic to a subset of $S^2$.*

We have already noted that the complete graph on 4 vertices can be embedded in $S^2$, and the standard embedding is given below.




The vertices of this graph are denoted by $A, B, C, D$, and the 6 edges will be labeled lexicographically (alphabetical order) as follows:

$$E_1 = AB , \quad E_2 = AC , \quad E_3 = AD , \quad E_4 = BC , \quad E_5 = BD , \quad E_6 = CD$$

The first step in the proof of Theorem 3 is to prove that an arbitrary topological embedding of the complete graph on 4 vertices into $S^2$ has the same separation properties that evidently hold in previous drawing:

**THEOREM 4.** *Let $X \subset \S^2$ be homeomorphic to the graph described above, and label its edges and vertices as in the preceding discussion. Then $S^2 - X$ has four components $U, V, W, \mathcal{O}$ such that*

*the boundary of $U$ is $E_1$, $E_3$ and $E_5$,*

*the boundary of $V$ is $E_2$, $E_3$ and $E_6$,*

*the boundary of $W$ is $E_4$, $E_5$ and $E_6$,*

*the boundary of $\mathcal{O}$ is $E_1$, $E_2$ and $E_4$.*

**Proof of Theorem 4.** The strategy is similar to the method for proving Proposition 2:

(1) Prove that $S^2 - X$ has four components.

(2) Prove that $\Gamma = E_1 \cup E_2 \cup E_4$ is the boundary of one component.

(3) Use similar arguments to show that the other three triangular graphs in the theorem statement bound components of $S^2 - X$. As before, each of the four triangular graphs bounds a component, and since there are exactly four components it follows that each component is the boundary of one such graph,

To prove the first step, let $X_0$ be obtained from $X$ by deleting the interior points of the edge $E_4$ (see Figure 5 in `graphpix4.pdf`), and let $X_1$ be the triangle graph whose edges are $E_4$, $E_5$ and $E_6$. Then $X_0 \cup X_1 = X$ and $X_0 \cap X_1 = E_5 \cup E_6$; note that the latter is homeomorphic to a closed interval. Consider the Mayer-Vietoris exact sequence for the decomposition

$$S^2 - (E_5 \cup E_6) \;=\; (S^2 - X_0) \;\cup\; (S^2 - X_1) \;, \qquad S^2 - X \;=\; (S^2 - X_0) \;\cap\; (S^2 - X_1) \;.$$

By construction $X_0$ is a theta space and $X_1$ is a simple closed curve, so the homology groups of $S^2 - X_0$ and $S^2 - X_1$ are known (and likewise for the homology groups of $S^2 - (E_5 \cup E_6)$ by Proposition VII.3.2). If we feed this into the long exact Mayer-Vietoris sequence, we find that the final nontrivial terms of the Mayer-Vietoris sequence are given as follows:

$$0 = H_1(S^2 - (E_5 \cup E_6)) \to H_0(S^2 - X) \to H_0(S^2 - X_0) \oplus H_0(S^2 - X_1) \to H_0(S^2 - (E_5 \cup E_6)) = \mathbb{Z}$$

The results of this section and the preceding ones imply that the direct sum is isomorphic to $\mathbb{Z}^3 \oplus \mathbb{Z}^2 \cong \mathbb{Z}^5$; furthermore, the standard free generators of this group map to the standard free generator of $H_0(S^2 - (E_5 \cup E_6)) = \mathbb{Z}$. As before, it follows that $H_0(S^2 - X)$ is a free abelian group on 4 generators, which means that $S^2 - X$ has four components, completing the proof of the first step.

In the second step we are interested in the complement of the triangular subgraph $\Gamma = E_1 \cup E_2 \cup E_4$. By the Jordan Curve Theorem $S^2 - \Gamma$ has two components, say $G$ and $H$. One of these components contains the remaining vertex $D$; as before, without loss of generality we might as well assume that $D \in H$.

The half open intervals $E_3 - \{A\}, E_5 - \{B\}, E_6 - \{C\}$ are all connected, disjoint from $\Gamma$, and contain $D$, so they are all contained in $H$. Then we have

$$S^2 \;-\; X \;\;=\;\; G \;\cup\; (\,H - (E_1 \cup E_2 \cup E_3)\,)$$

where $G$ and $(\,H - (E_1 \cup E_2 \cup E_3)\,)$ are nonempty, open and disjoint (hence both are also closed in $S^2 - X$).

Let $Q_1, Q_2, Q_3, Q_4$ be the components of $S^2 - X$, and number them such that $G \subset Q_1$. Since $G$ is open and closed in $S^2 - X$, every connected subset of the latter is either contained in $G$ or disjoint from it. In particular, since $G$ is contained in $Q_1$ we know that these two sets are not disjoint and hence the connected component $Q_1$ must be contained in $G$, so that the two sets are equal. Since the boundary of $G$ is $E_1 \cup E_2 \cup E_4$, this proves the statement needed to complete the second step of the argument. As noted at the beginning of this proof, the third step follows once we have completed the first two, and therefore we have completed the proof of the theorem.■

**Proof of Theorem 3.** Assume that the complete graph on 5 vertices is homeomorphic to some subset $Y \subset S^2$, and let $a, b, c, d, e$ be its vertices. Let $X \subset Y$ be the subgraph of all edges which do not have $e$ as an endpoint, so that $X$ is homeomorphic to a complete graph on 4 vertices. We shall now use Theorem 4 to analyze $S^2 - X$.

Let $E_{uv}$ be the edge joining the vertices $u$ and $v$ in $Y$. Without loss of generality, we can assume that $e$ lies in the component of $S^2 - X$ whose boundary is $E_{ab} \cup E_{bc} \cup E_{ac}$. In any case, the vertex $e$ lies in one component of $S^2 - X$, and we can treat the other cases by permuting the roles of $a, b, c, d$. Note that $d$ does not lie in the closure $\overline{U}$ of $U$ by the proof of Theorem 4.

Now each of the sets $(E_{xe} - \{x\})$ — where $x = a, b, c, d$ — is connected and contains $e$, so each of these connected sets must be contained in $U$. This implies that each boundary endpoint $x$ of $E_{xe} - \{x\}$ must be contained in $\overline{U}$. However, we have already observed that $d$ does not lie in this subset, and therefore we have a contradiction. The problem arises from our assumption that $Y \subset S^2$ is homeomorphic to a complete graph on 5 vertices, and consequently no such subset can exist.■

*Kuratowski's Theorem*

The results of this section lead to the more general question of determining which connected graphs are not topologically embeddable in $\mathbb{R}^2$. Clearly a graph which contains a subgraph isomorphic to the utilities network or the complete graph on 5 vertices cannot be homeomorphic to a subset of $\mathbb{R}^2$. The end of Section 64 in Munkres mentions a remarkable converse to this result attributed to C. Kuratowski (1896–1980): *Every graph which is not homeomorphic to subset of* $\mathbb{R}^2$ *must contain a subgraph homeomorphic to* either *the utilities network* or *the complete graph on five vertices.* Here is an online reference for the proof:

`http://cs.princeton.edu/∼ymakaryc/papers/kuratowski.pdf`

The file `kuratowski.pdf` contains clickable links to other proofs and further information, including independent discoveries of this result by others.