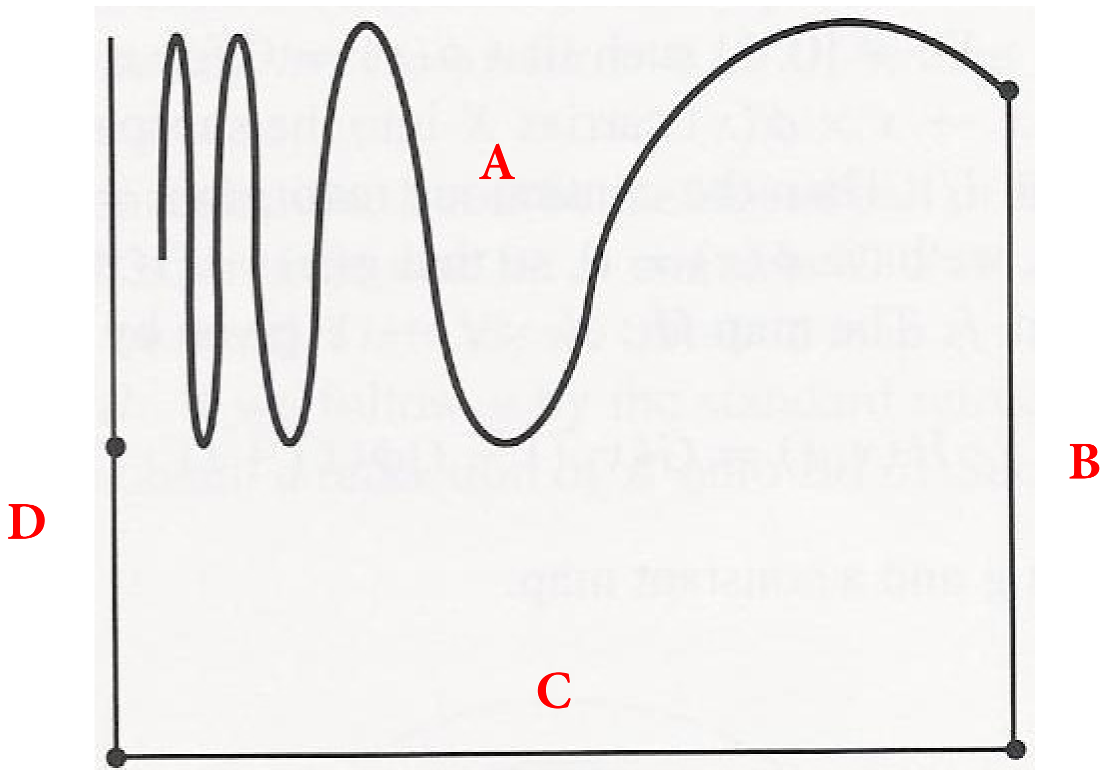
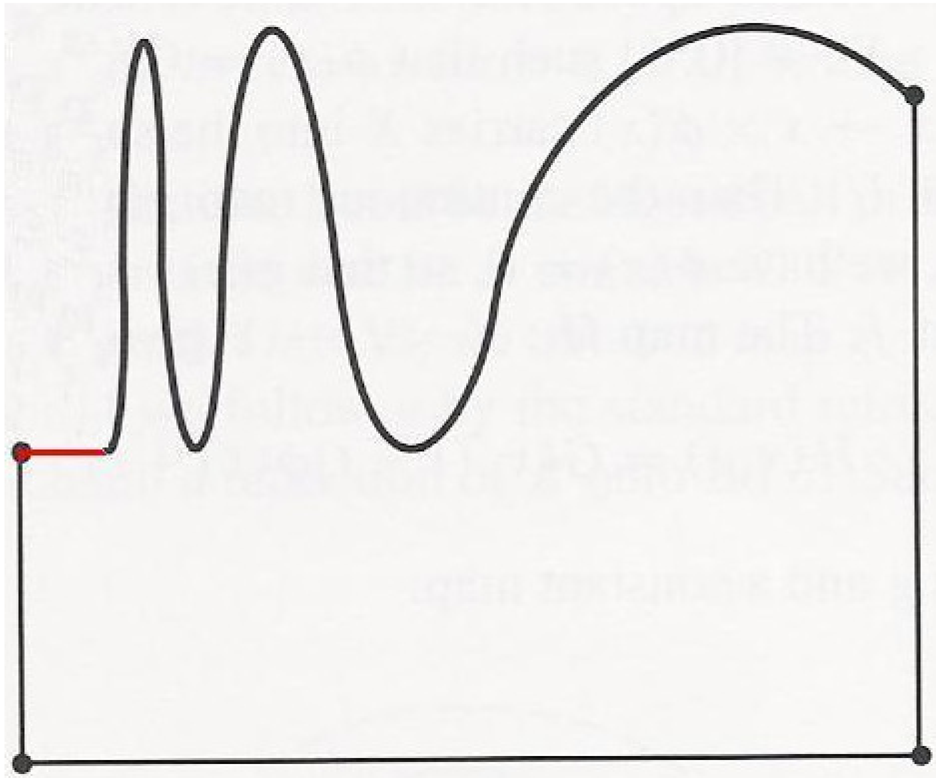
Constructions on the Polish Circle

A drawing of the Polish Circle **P** appears on page 381 of Munkres (Figure 61.4). It is constructed using vertical and horizontal line segments and the portion of the graph of the function $y = \sin(1/x)$ with x in the half – open interval $(0, 1]$; the name is given because this is one of many unusual subsets of the plane that were studied by Polish topologists during the period between the First and Second World Wars.

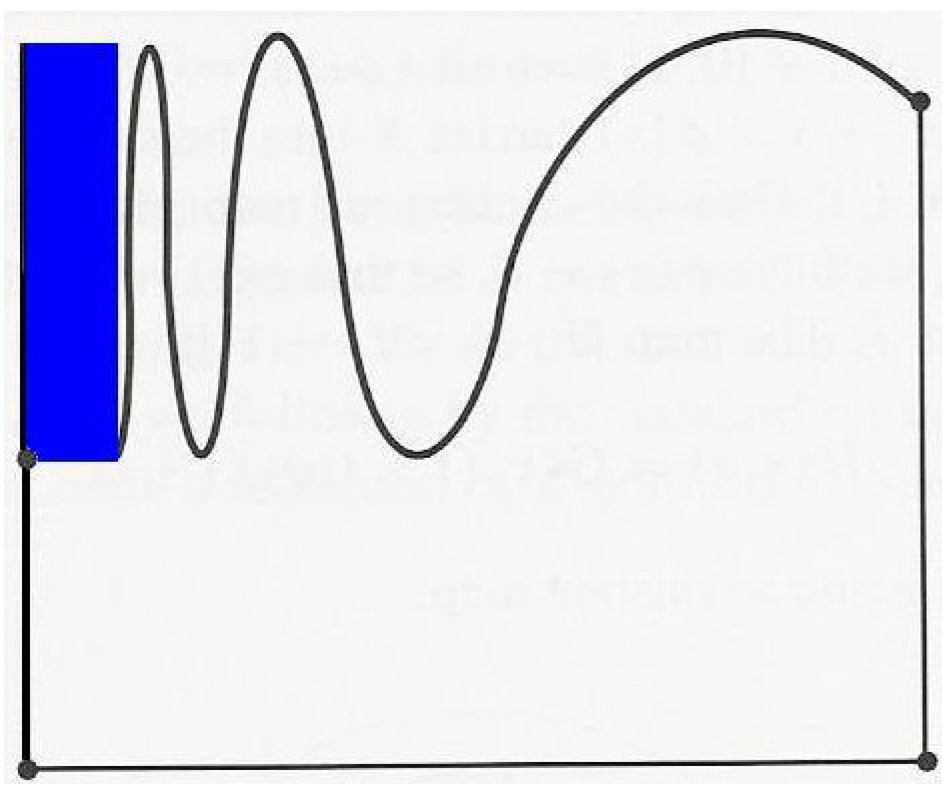


The left, right and bottom pieces of this space **P** are given by segments on the vertical lines $x = 0$, $x = 1$, and the horizontal line $y = -2$.

For each nonnegative integer n we may define a crude approximation $\mathbf{C}_n$ to **P** as follows: Remove the piece of the graph over the open interval from 0 to $2/(4n+3)\pi$ along with its limit points on the y – axis, and replace it with a horizontal line segment joining the points $(0, -1)$ and $(2/(4n+3)\pi, -1)$. A drawing of this subset appears on the next page. By construction it is a piecewise smooth simple closed curve.



Finally, we may construct another crude approximation $\mathbf{B}_n$ to **P** by taking the union of **P** with the closed rectangular region of all points (x, y) such that x lies in the closed interval from 0 to $2/(4n+3)\pi$ and y lies in the closed interval from -1 to 1.



Several basic properties of these crude approximations are worth noting and important for solving the Exercise 61.2 in Munkres. By construction the sets $\mathbf{C}_n$ and $\mathbf{B}_n$ are compact for all n. Furthermore, for all n we have $\mathbf{C}_n \subset \mathbf{B}_n$ and $\mathbf{B}_{n+1} \subset \mathbf{B}_n$ for all n, and the intersection of the decreasing family of sets $\mathbf{B}_n$ is equal to $\mathbf{P}$.