COROLLARY 6. *Suppose that X and Y are as in the theorem and Y is contractible. Then every continuous mapping $f : X \to Y$ has a continuous extension to X.*

Proof. It will suffice to prove that an arbitrary continuous mapping $f : A \to Y$ is homotopic to a constant. We know that 1_Y is homotopic to a constant map k, and therefore $f = 1_Y \circ f$ is homotopic to the constant map $k \circ f$.■

I.5 : Chain homotopies

(Hatcher, § 2.1)

In this section we shall generalize a key step in the proof of that starshaped complexes have acyclic homology. The main feature of the proof is that it constructs an algebraic analog of the straight line contracting homotopy from the identity to the constant map whose value is $\mathbf{v}$.

Definition. Let (A, d) and (B, e) be chain complexes, and let f and g be chain maps from A to B. A *chain homotopy* from f to g is a sequence of mappings $d_k : A_k \to B_{k+1}$ satisfying the following condition for all integers k:

$$d^B_{k+1} \circ D_k \ + \ D_{k-1} \circ d^A_k \ \ = \ \ g_k \ - \ f_k$$

Two chain mappings f, g from A to B are said to be *chain homotopic* if there is a chain homotopy from the first to the second, and this is often written $f \simeq g$.

The proof of the following result is an elementary exercise:

PROPOSITION 1. *The relation $\simeq$ is an equivalence relation on chain maps from one chain complex (A, d) to another (B, e). Furthermore, if f and g are chain homotopic chain maps from (A, d) to (B, e), and h and k are chain homotopic chain maps from (B, e) to (C, θ), then the composites $h \circ f$ and $k \circ g$ are also chain homotopic. Finally, if f, g, h, k are chain maps from A to B and $r \in R$, then $f \simeq g$ and $h \simeq k$ imply $f + h \simeq g + k$ and $rf \simeq rg$.*

Proof. For the first part of the proof let f, g and h be chain maps from (A, d) to (B, e). The zero homomorphisms define a chain homotopy from f to itself. If D is a chain homotopy from f to g then $-D$ is a chain homotopy from g to f. Finally, if D is a chain homotopy from f to g and E is a chain homotopy from g to h, then $D + E$ is a chain homotopy from f to h.

To prove the assertion in the second sentence, let D be a chain homotopy from f to g and let E be a chain homotopy from g to h. Then one can check directly that

$$h \circ D \ + \ E \circ g$$

defines a chain homotopy from $h \circ f$ to $k \circ g$.$^{(\star)}$ The proof of the final assertion is also elementary and is left to the reader.■

Chain homotopies are useful and important because of the following result:

PROPOSITION 2. *If f and g are chain homotopic chain maps from one chain complex (A, d) to another complex (B, e), then the associated homology mappings f_* and g_* are equal.*

Proof. Suppose that $u \in H_k(A)$ and $x \in A_k$ is a cycle representing u, so that $d_k(a) = 0$. If D is a chain homotopy from f to gh, then by definition we have

$$d^B_{k+1} \circ D_k(x) \ + \ D_{k-1} \circ d^A_k \ (x) \ = \ g_k(x) \ - \ f_k(x)$$

and since $d^A_k(x) = 0$ it follows that the expression above is a boundary. Therefore $g_*(u) - f_*(u)$ must be the zero element of $H_k(B)$.■

An important example

The following basic construction gives an explicit connection between the topological notion of homotopy and the algebraic notion of chain homotopy. Let $n \geq 0$, and let $\mathbf{P}_{n+1}$ denote the *standard $(n+1)$-dimensional prism* $\Delta_n \times [0,1]$ with the simplicial decomposition given in Unit II. As in that unit, label the vertices of this prism decomposition by $\mathbf{x}_j = (\mathbf{e}_j, 0)$ and $\mathbf{y}_j = (\mathbf{e}_j, 1)$.

PROPOSITION 3. *The simplicial chain complexes $C_*(\mathbf{P}^\omega_{n+1})$ and $C_*(\mathbf{P}_{n+1})$ are acyclic.*

Proof. These follow from the isomorphism theorem and the fact that $\mathbf{P}_{n+1}$ is star shaped with respect to $\mathbf{y}_n$.■

For each integer j satisfying $0 \leq j \leq n$, let $\partial_j : \Delta_{n-1} \to \Delta_n$ be the affine map which sends Δ_{n-1} to the face opposite the vertex $\mathbf{e}_j$ and is order preserving on the vertices, and let $\partial_j \times \mathbf{I}$ denote the product of the map ∂j with the identity on $[0,1]$. It then follows immediately that we have associated injections of simplicial chain groups

$$(\partial_j)_\# : C_j(\Delta_{n-1}) \longrightarrow C_j(\Delta_n) \ , \qquad (\partial_j \times \mathbf{I})_\# : C_*(\mathbf{P}_{n-1}) \longrightarrow C_*(\mathbf{P}_n)$$

and these are chain maps. Furthermore, these chain maps send ordered chains to ordered chains.

Similarly, for $t = 0, 1$ we also have injections of simplicial chain groups

$$(i_t)_\# : C_*(\Delta_n) \longrightarrow C_*(\mathbf{P}_n)$$

which send a free generator $\mathbf{v}_0 \ \cdots \ \mathbf{v}_q$ to $i_t(\mathbf{v}_0) \ \cdots \ i_t(\mathbf{v}_q)$, where $i_t(\mathbf{v}) = (\mathbf{v}, t)$.

We then have the following result:

THEOREM 4. *For all $n \geq 0$ there are chains $P_{n+1} \in C_{n+1}(\mathbf{P}_n^\omega)$ such that*

$$d_{n+1}(P_{n+1}) \quad = \quad \mathbf{y}_0 \ \cdots \ \mathbf{y}_n \ - \ \mathbf{x}_0 \ \cdots \ \mathbf{x}_n \ - \ \sum_j \, (-1)^j \, (\partial_j \times \mathbf{I})_\#(P_{n-1}) \ .$$

Sketch of proof. Not surprisingly, the construction is inductive, with $P_0 = 0$. Suppose we have constructed the chains P_j for $j \leq n$. There is a chain P_{n+1} with the required properties if and only if the expression on the right hand side of the equation is a cycle, so we need to show that the right hand side vanishes if we apply d_n. This is a straightforward but messy calculation like several previous ones. Some key details are worked out in the bottom half of page 112 of Hatcher.■

The preceding result implies that the inclusion mappings i_t, which are topologically homotopic, determine algebraic chain maps that are chain homotopic. Specifically, if we are given a free generator $\mathbf{v}_0 \ \cdots \ \mathbf{v}_q$ of $C_q(\Delta_n)$ then we may form a chain

$$D_q(\mathbf{v}_0 \ \cdots \ \mathbf{v}_q) \quad \in \quad C_{q+1}(\Delta_n \times \mathbf{I})$$

by substituting $i_0(\mathbf{v})$ for $\mathbf{x}$ and $i_1(\mathbf{v})$ for $\mathbf{y}$. In fact, one can carry out all of this for an arbitrary simplicial complex $(P, \mathbf{K})$, and one has the following conclusion.

PROPOSITION 5. *In the setting above the maps $(i_0)_\#$ and $(i_1)_\#$ from $C_*(\mathbf{K})$ to $C_*(\mathbf{K} \times \mathbf{I})$ are chain homotopic, and hence the associated homology maps*

$$(i_0)_*, \ (i_1)_* \ : \ H_*(\mathbf{K}) \ \longrightarrow \ H_*(\mathbf{K} \times \mathbf{I})$$

are equal.■

I.6 : Cones and suspensions

(Hatcher, Ch. 0)

These two basic constructions are described on pages 8–9 of Hatcher. We shall say a little more about them and apply them to construct a homeomorphism from the standard n-disk and $(n-1)$-sphere to the standard n-simplex and its boundary.

The constructions and their properties

Definition. Let X be a topological space. The *cone on* X, usually written $\mathbf{C}(X)$, is the quotient of $X \times [0,1]$ modulo the equivalence relation whose equivalence classes are all one point subsets of the form $\{ \ (x,t) \ \}$, where $t \neq 0$, and the subset $X \times \{0\}$.

The first result explains the motivation for the name.