One can use the preceding discussion to place the proof of Euler's Formula $E + 2 = V + F$ into a more general setting (see the exercises).

I.4 : The Homotopy Extension Property

(Hatcher, Ch. 0, § 2.1)

In this section we shall bring together several concepts from the preceding sections. The basis is the following central Extension Question stated at the beginning of this unit, and our first result describes a condition under which this question always has an affirmative answer.

PROPSITION 1. *Suppose that X and Y are topological spaces, that $A \subset X$ is a retract, and that $g : A \to Y$ is continuous. Then there is an extension of g to a continuous mapping $f : X \to Y$.*

Proof. Let $r : X \to A$ be a continuous function such that $r|A$ is the identity, and define $f = g \circ r$. Then if $a \in A$ we have $f(a) = g \circ r(a) = g(\, r(a)\,)$, and the latter is equal to $g(a)$ because $r|A$ is the identity.■

The hypothesis of the proposition is fairly rigid, but the result itself is a key step in proving a general result on the Extension Question.

THEOREM 2. (HOMOTOPY EXTENSION PROPERTY) *Let $(X, \mathcal{E})$ be a cell complex, and suppose that A determines a subcomplex. Suppose that Y is a topological space, that $g : A \to Y$ is a continuous map, and $f : X \to Y$ is a continuous map such that $f|A$ is homotopic to g. Then there is a continuous map $G : X \to Y$ such that $G|A = g$.*

COROLLARY 3. *Suppose that X and A are as above and that $g : A \to Y$ is homotopic to a constant map. Then g extends to a continuous function from X to Y.*

COROLLARY 4. *Suppose that X and A are as above and that $g : A \to X$ is homotopic to the inclusion map. Then g extends to a continuous function from X to itself.*

Corollary 3 follows because every constant map from A to Y extends to the analogous constant map from X to Y, and Corollary 4 follows because the inclusion of A in X extends continuously to the identity map from X to itself.■

One important step in the proof of the Homotopy Extension Property relies upon the following result:

PROPOSITION 5. *For all $k > 0$ the set $D^k \times \{0\} \cup S^{k-1} \times [0, 1]$ is a strong deformation retract of $D^k \times [0, 1]$.*

Proof. This argument is outlined in Proposition 0.16 on page 15 of Hatcher, and there is a drawing to illustrate the proof (when $n = 1$) on the next page.

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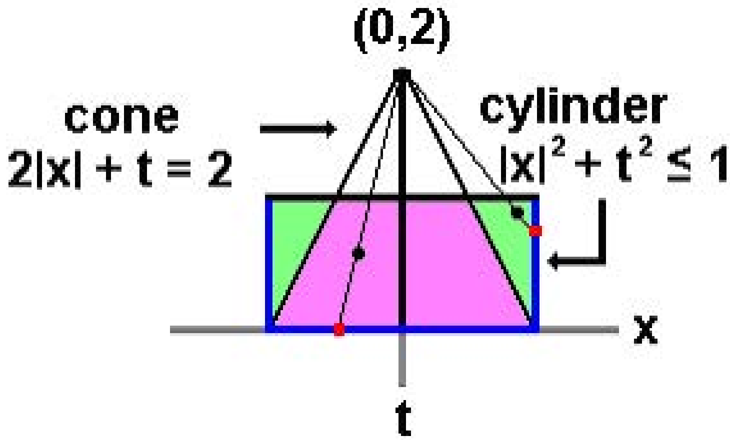

The retraction $r : D^k \times [0,1] \to D^k \times \{0\} \ \cup \ S^{k-1} \times [0,1]$ is defined by a radial projection with center $(\mathbf{0}, 2) \in D^k \times \mathbb{R}$. As indicated by the drawing, the formula for r depends upon whether $2|\mathbf{x}| + t \geq 2$ or $2|\mathbf{x}| + t \leq 2$. Specifically, if $2|\mathbf{x}| + t \geq 2$ then

$$r(\mathbf{x},\, t) \quad = \quad \frac{1}{|\mathbf{x}|}\left(\mathbf{x},\, 2|\mathbf{x}| + t - 2\right)$$

while if $2|\mathbf{x}| + t \leq 2$ then we have

$$r(\mathbf{x},\, t) \quad = \quad \frac{1}{2}\left((2-t)\mathbf{x},\, 0\right)$$

and these are consistent when $2|\mathbf{x}| + t = 2$ then both formulas yield the value $|\mathbf{x}|^{-1}(\mathbf{x},\, 0)$. Elementary but slightly tedious calculation then implies that $r(\mathbf{x},\, t)$ always lies in $D^k \times [0,1]$, and likewise that r is the identity on $D^k \times \{0\} \ \cup \ S^{k-1} \times [0,1]$.$^{(\star)}$ The homotopy from inclusion$\circ r$ to the identity is then the straight line homotopy

$$H(\mathbf{x},\, t;\, s) \quad = \quad (1-s) \cdot r(\mathbf{x},\, t) \ + \ s \cdot (\mathbf{x},\, t)$$

and this completes the proof of the proposition.∎

Proof of Theorem 2. In the course of the proof we shall need the following basic fact: *If A and B are compact Hausdorff spaces and $\varphi : A \to B$ is a quotient map in the sense of Munkres' book, then for each compact Hausdorff space C the product map $\varphi \times 1_C : A \times C \to B \times C$ is also a quotient map.* — This follows because $\varphi \times 1_C$ is closed, continuous and surjective; as noted in Exercise 11 on page 186 of Munkres, the same conclusion also holds with weaker hypotheses on φ and C.

Since the homotopy relation on continuous functions is transitive, a standard recursive argument reduces the proof of the theorem to the special cases of subcomplex inclusions

$$X_{k-1} \ \cup \ A \quad \subset \quad X_k \ \cup \ A\ .$$