# VII. Some elementary applications

The motivation for developing delicate and abstract topological machinery like singular homology is that such constructions are useful for answering mathematical questions that were interesting but difficult to handle with previously existing tools. One of the most obvious examples is the Jordan Curve Theorem, which states that the complement of a simple closed curve in the plane has two connected components, and the curve is the boundary of each component. Experience strongly suggests that such a result is true, but even in simple cases like regular smooth curves the proof is challenging (for example, see the proof in M. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976). There is a proof of this result in Munkres which does not use homology theory, but it is long and delicate. We shall use homology theory to give a fairly short proof of the Jordan Curve Theorem and its higher dimensional generalizations; one needs the full force of homology theory for the latter, for they cannot be proved using the concepts in Munkres' book.

Likewise, homology theory provides a very simple proof that the coordinate spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic if $m \neq n$. Once again, the material in Munkres yields these results if $n = 1$ or 2 but cannot be used to draw any conclusions if $m, n \geq 3$. We shall also use homology theory to give alternate proofs for two results of Munkres about graphs which are not topologically embeddable in $\mathbb{R}^2$ (although we know that all graphs are nicely embeddable in $\mathbb{R}^3$). Finally, if time permits we shall use homology theory to derive a classical formula of R. Descartes and L. Euler relating the numbers of edges, vertices and faces in a polyhedron which bounds a convex linear cell in $\mathbb{R}^3$:

$$E + 2 = V + F$$

Many additional uses of homology theory are mentioned very briefly in `morgan-lamberson.pdf`.

## VII.1 : Consequences of the axioms

(**H**, §§2.1–2.3, 2.B)

Our first objective is to show that the coordinate spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $m \neq n$. For the sake of clarity and convenience we begin by showing that certain convex sets in $\mathbb{R}^n$ are homeomorphic. We have already stated more general results and referred to files for their proofs, but it seems worthwhile to give direct, simple proofs for the specific examples of interest to us here.

*Semi-explicit homeomorphisms of various convex sets*

The sets of interest to us are the $n$-simplex $E_n$ in $\mathbb{R}^n$ given by the inequalities

$$x_i \geq 0 \ , \qquad \sum_i x_i \geq 1$$

the hypercubes $[a, b]^n$ which are homeomorphic to each other because all closed intervals in $\mathbb{R}$ are homeomorphic, and the usual unit disk $D^n$.

**THEOREM 1.** *All of the sets listed above are homeomorphic such that interior points of one correspond to interior points of the other and boundary points of one correspond to boundary points of the other.*

**Proof.** We begin with the easiest pair; namely, the disk and the hypercube $[-1,1]^n$. Given a vector $x \in \mathbb{R}^n$, let $|x|_2$ denote its length with respect to the usual inner product and let $|x|_\infty$ be the maximum of the absolute values of the coordinates ($= \max_i |x_i|$). Both of these define norms on $\mathbb{R}^n$, and the unit disks with respect to these norms are $D^n$ and $[-1,1]^n$ respectively. If one defines a map $f$ of $\mathbb{R}^n$ to itself by $f(\mathbf{0}) = \mathbf{0}$ and by

$$f(x) \quad = \quad \frac{|x|_\infty}{|x|_2} \cdot x$$

if $x \neq \mathbf{0}$, then it follows that $f$ is 1–1 onto and a homeomorphism except possibly at $\mathbf{0}$, and that for each $r > 0$ the map $f$ sends points satisfying $|x|_2 = r$ to points satisfying $|x|_\infty = r$; one can check continuity of $f$ and its inverse at $\mathbf{0}$ using the elementary inequalities

$$|x|_\infty \quad \leq \quad |x|_2 \quad \leq \quad n \cdot |x|_\infty \ .$$

It follows that $f$ defines a homeomorphism from $D^n$ to $[-1,1]^n$.

Since all $n$-dimensional hypercubes are homeomorphic, it will suffice to show that $E_n$ is homeomorphic to the hypercube $[0,1]^n$ such that their boundaries correspond. For this we need the "taxicab norm" $|x|_1 = \sum_i |x_i|$. Let $F_n$ be the unit disk with respect to this norm. Then $E_n$ and $[0,1]^n$ are the intersections of the unit disks $F_n$ and $[-1,1]^n$ with the *closed first orthant* in $\mathbb{R}^n$ defined by the inequalities $x_n \geq 0$. In analogy with the previous paragraph define a mapping $g$ by $g(\mathbf{0}) = \mathbf{0}$ and

$$g(x) \quad = \quad \frac{|x|_\infty}{|x|_1} \cdot x$$

if $x \neq \mathbf{0}$. Then $g$ has similar properties to $f$, with continuity at $\mathbf{0}$ is true because of the inequalities

$$|x|_\infty \quad \leq \quad |x|_1 \quad \leq \quad n \cdot |x|_\infty \ .$$

By construction, both $f$ and $g$ map the first orthant into itself such that the boundary points (those for which some coordinate $x_i = 0$) are sent to themselves. The boundaries of $E_n$ and $[0,1]^n$ are given by their intersections with the orthants and their intersections with the sets $|x|_p = 1$ where $p = 1$ and $\infty$ respectively, and therefore it follows that $g$ defines the desired homeomorphism from $E_n$ to $[0,1]^n$.■

*Some nonhomeomorphic spaces*

**THEOREM 2.** *If $m$ and $n$ are distinct positive integers, then $S^m$ and $S^n$ are not homeomorphic, and similarly $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.*

**Proof.** By Theorem 1 we know that $S_k$ is homeomorphic to the boundary of the simplex $E_{k+1}$, and hence $H_q(S^k) = \mathbb{Z}$ if $q = 0, k$ and zero otherwise. In particular, this means that the homology groups of $S^m$ and $S^n$ are not isomorphic if $m \neq n$, so the spaces cannot be homeomorphic.

If $\mathbb{R}^m$ and $\mathbb{R}^n$ were homeomorphic, then it follows that their one point compactifications would also be homeomorphic (verify this as a general statement about locally compact Hausdorff spaces!). Since these one point compactificatons are homeomorphic to $S^m$ and $S^n$ respectively, it follows that $\mathbb{R}^m$ and $\mathbb{R}^n$ cannot be homeomorphic if $m \neq n$.■