III.1 : Homology and the fundamental group

(Hatcher, §§ 2.A, 3.G)

Axiom (D.5) formulates a simple but important relationship between the fundamental group $\pi_1(X, x)$ of a pointed arcwise connected space and the homology group $H_1(X) \cong H_1(X, \{x\})$.

Definition. Let $[S^1] \in H_1(S^1)$ be the homology class represented by the singular 1-simplex

$$T(1-s, s) \quad = \quad \big(\cos 2\pi s,\, \sin 2\pi s\big)$$

so that T corresponds to the standard counterclockwise parametrization of the unit circle under the identification of $[0,1]$ with the 1-simplex whose vertices are $(1,0)$ and $(0,1)$. The *Hurewicz* (hoo-RAY-vich) map $h : \pi_1(X, x) \to H_1(X)$ is given by taking a representative f of $\alpha \in \pi_1(X, x)$ and setting $h(\alpha) = f_*([S^1])$. By homotopy invariance, this class does not depend upon the choice of a representative, and it is natural with respect to basepoint preserving continuous maps.

PROPOSITION 1. *The Hurewicz map h is a group homomorphism.*

Proof. The discussion on pages 166–167 of Hatcher provides a good conceptual summary of the proof. For the sake of completeness we shall add a few details.

Let $\mathfrak{p} : \Delta_2 \to [0,1]$ be the map sending $(t_0, t_1, t_2) \in \Delta_2$ to $t_2 + \frac{1}{2}t_1 \in [0.1]$. Geometrically, $\mathfrak{p}$ is the composite of the perpendicular projection from Δ_2 onto the edge $\mathbf{e}_0\mathbf{e}_2$ followed by the linear homeomorphism from the latter to $[0,1]$ sending $\mathbf{e}_0$ to 0 and $\mathbf{e}_2$ to 1. Represent $u, v \in \pi_1(X, x)$ by $f, g : [0,1] \to X$, and let $c : [0,1] \to X$ be the concatenation $f + g$. If α is the linear homeomorphism from Δ_1 to $[0,1]$ sending vertex $\mathbf{e}_t$ to t (where $t = 0, 1$), then direct calculation yields the identities

$$\partial_2 \circ \mathfrak{p} \circ h \quad = \quad f \circ \alpha, \quad \partial_0 \circ \mathfrak{p} \circ h \quad = \quad g \circ \alpha, \quad \partial_1 \circ \mathfrak{p} \circ h \quad = \quad c \circ \alpha$$

(compare the drawing on page 166 of Hatcher) so that we have

$$d_2(\mathfrak{p} \circ h) \quad = \quad f \circ \alpha + g \circ \alpha - c \circ \alpha \in S_1(X, \{x\}) \ .$$

By construction, the images of the three summands on the right hand side of this equation are $h(u)$, $h(v)$ and $-h(uv)$ respectively, and since the left hand side is a boundary it follows that $h(u) + h(v) - h(uv) = 0$, which is what we wanted to prove.■

The preceding discussion and the theorem below show that the standard construction for singular homology has extra data type (T.2) and satisfies axioms (A.6) and (D.4); by the uniqueness result in the preceding unit, the same conclusions are true for an arbitrary axiomatic singular homology theory.

THEOREM 2. *If X is arcwise connected, then h is onto and its kernel is the commutator subgroup of $\pi_1(X, x)$.*

The assertion in the first sentence of the theorem is verified on page 167 of Hatcher; the proof of the assertion in the second sentence will take the remainder of this section.

Suppose that (X, x) is a pointed space such that X is arcwise connected. The *Eilenberg subcomplex* $\overline{S_*}(X) \subset S_*(X)$ is the chain subcomplex generated by all singular simplices $T : \Delta_q \to X$ which send each vertex of Δ_q to the chosen basepoint x.

PROPOSITION 3. *Under the conditions given above, the inclusion of the Eilenberg subcomplex defines an isomorphism in singular homology.*

Sketch of proof. For each $y \in X$ there is a continuous curve joining y to x, and hence for each singular 0-simplex given by a point y there is a singular 1-simplex $P(y)$ such that $P(y) \circ \partial_1$ is the constant function with value x and $P(y) \circ \partial_0$ is the constant function with value y; clearly it is possible to choose $P(x)$ to be the constant function, and we shall do so. Starting from this, we claim by induction on q that for each singular q-simplex $T : \Delta_q \to X$ there is a continuous map

$$P(T) : \Delta_q \times [0,1] \longrightarrow X$$

with the following properties:

(i) The restriction of $P(T)$ to $\Delta_q \times \{0\}$ is given by T, and the restriction of $P(T)$ to $\Delta_q \times \{1\}$ is given by a singular simplex in the Eilenberg subcomplex.

(ii) If T lies in the Eilenberg subcomplex, then $P(T)$ is equal to $T \times \mathrm{id}_{[0,1]}$.

(iii) For each face map $\partial_i : \Delta_{q-1} \to \Delta_q$ we have $P(T \circ \partial_i) = P(T) \circ (\partial_i \times \mathrm{id}_{[0,1]})$.

To complete the inductive step, one uses (iii) and the first property in (i) to define $P(T)$ on $\Delta_q \times \{0\} \cup \partial\Delta_q \times [0,1]$, and then one extends this to all of $\Delta_q \times [0,1]$ using the Homotopy Extension Property.

Let i denote the inclusion of the Eilenberg subcomplex, and define a map ρ from $S_*(X)$ to the Eilenberg subcomplex by taking $\rho(T)$ to be the restriction of $P(T)$ to $\Delta_q \times \{1\}$. The property (iii) ensures that ρ is a chain map, and we also know that $\rho \circ i$ is the identity on the Eilenberg subcomplex. The proof of the proposition will be complete if we can show that $i \circ \rho$ is chain homotopic to the identity. The proof of this is very similar to the proof of homotopy invariance. Let $\mathbf{P}_{q+1} \in S_{q+1}(\delta_q \times [0,1])$ be the standard chain used in that proof, and define

$$E(T) = \big(P(T)\big)_{\#} \mathbf{P}_{q+1} .$$

Then the properties of $\mathbf{P}_{q+1}$ and its boundary imply this defines a chain homotopy from the identity to $i \circ \rho$.■

Conclusion of the proof of Theorem 2. We shall use the following commutative diagram:

$$\begin{array}{ccccc} F_2(X,x) & \xrightarrow{\mathbf{abel}} & \overline{S_2}\,(X) & \xrightarrow{=} & \overline{S_2}\,(X) \\ \downarrow \delta & & \downarrow d_2 & & \downarrow d_2 \\ F_1(X,x) & \xrightarrow{\mathbf{abel}} & \overline{S_1}\,(X) & \xrightarrow{=} & \overline{S_1}\,(X) \\ \downarrow \mathbf{can} & & \downarrow \mathbf{can}' & & \downarrow \mathbf{class} \\ \pi_1(X,x) & \xrightarrow{\mathbf{abel}} & \pi_1^{\mathbf{ab}}(X,x) & \xrightarrow{h'} & H_1(X) \end{array}$$

Many items in this diagram need to be explained. On the bottom line, $\pi_1^{\mathbf{ab}}$ denotes the abelianization of the fundamental group formed by factoring out the (normal) commutator subgroup, and the Hurewicz map has a unique factorization as $h' \circ \mathbf{abel}$, where **abel** refers to the canonical surjection from π_1 to its quotient modulo the commutator subgroup. The groups $F_j(X,x)$ are the free groups on the free generators for the Eilenberg subcomplexes $\overline{S_*}\,(X)$, and **abel** generically denotes the passage from free groups to the corresponding free abelian groups. The maps d_2 and class are merely the relevant maps for the Eilenberg subcomplex, the map **can**′" is the abelianization of the map **can** taking a free generator $T : \Delta_1 \to X$, which is merely a closed curve in X based at x, to its homotopy class in the fundamental group. Finally, δ is a nonabelian boundary map defined on free generators by

$$\delta(T) \quad = \quad [T \circ \partial_2] \cdot [T \circ \partial_0] \cdot [T \circ \partial_1]^{-1} \ .$$

Observe that the composite $\mathbf{can} \circ \delta$ is trivial and hence its abelianization $\mathbf{can}' \circ d_2$ is also trivial.

Proof that the Hurewicz map is onto. Suppose we are given a cycle $z = \sum_i \ n_i T_i$ in the Eilenberg subcomplex. and we let $\gamma(T_i) \in F_1(X,x)$ denote the free generator corresponding to T_i. Then it follows immediately from the commutative diagram that the homology class u represented by z satisfies

$$u \ = h(\alpha) \ , \qquad \text{where} \quad \alpha \ = \ \prod_i \left[\mathbf{can}\big(\gamma(T_i)\,\big)\right]^{n_i} \ .$$

Proof that the reduced Hurewicz map (i.e., its factorization through the abelianization of the fundamental group) is injective. Suppose that $h(\alpha) = 0$ and that the free generator $y \in F_1(X,x)$ represents α. Then it follows that $\mathbf{abel}(y) = d_2(w)$ for some 2-chain w, and if $w' \in F_2(X,x)$ projects to w then $y = \delta(w) \cdot v$, where v lies in the commutator subgroup of $F_1(X,x)$. Since $\mathbf{can} \circ \delta$ is trivial, it follows that the image of y in $\pi_1^{\mathbf{abel}}$ is trivial. Finally, since the image of y in π_1 is α, it also follows that the image of α in $\pi_1^{\mathbf{abel}}$ is trivial, or equivalently that α lies in the commutator subgroup.■

The results of this section and the normalization axioms for singular homology theories imply a strong converse to the Seifert-van Kampen Theorem for describing the fundamental group of a space X which is the union of arcwise connected open subset U and V. Namely, if the images of $\pi_1(U)$ and $\pi_1(V)$ generate $\pi_1(X)$, then the intersection is arcwise connected.

PROPOSITION 4. *Suppose that X is a topological space which is the union of arcwise connected open subsets U and V (such that the base point lies in $U \cap V$), and assume that $U \cap V$ is* not *arcwise connected, and let $\Gamma \subset \pi_1(X)$ be the subgroup generated by the images of $\pi_1(U)$ and $\pi_1(V)$. Then Γ has infinite index in $\pi_1(X)$.*

Since one of the simplest examples for Theorem 3 is the circle expressed as a union of two open arcs whose intersections are two small closed arcs, the conclusion of Theorem 3 is obvious in this special case and thus the theorem shows that something similar happens in every other example.

Proof. By Theorem 1, it will suffice to show that the image of Γ in $H_1(X)$ has infinite index in the latter group, for if a subgroup $K \subset G$ has finite index, then its image in the abelianization $G/[G, G]$ will also have finite index (verify this; it is an elementary exercise in group theory$^{(\star)}$).

Theorem 1 implies that the image of Γ in $H_1(X)$ is equal to the image of the inclusion induced homomorphism

$$H_1(U) \ \oplus \ H_1(V) \quad \longrightarrow \quad H_1(X)$$

in the Mayer-Vietoris exact sequence associated to the decomposition $X = U \cup V$:

$$H_1(U) \oplus H_1(V) \to\to H_1(X) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(X) \cong \mathbb{Z} \to 0$$

Since 0-dimensional homology groups are free abelian on their sets of arc components, this sequence is given more concretely as follows, in which Π denotes the set of arc components of $U \cap V$, the maps from $\mathbb{Z}^\Pi$ to the two $\mathbb{Z}$ factors are given up to sign by adding coordinates, and the map from $\mathbb{Z} \oplus \mathbb{Z}$ is also addition:

$$H_1(U) \oplus H_1(V) \to H_1(X) \to \mathbb{Z}^\Pi \to \mathbb{Z} \oplus \mathbb{Z} \to H_0(X) \cong \mathbb{Z} \to 0$$

Since we are assuming that Π contains at least two elements, it follows that the map $\mathbb{Z}^\Pi \to \mathbb{Z} \oplus \mathbb{Z}$ has a nontrivial kernel and hence by exactness the map $H_1(X) \to \mathbb{Z}^\Pi$ has an infinite image. One more application of exactness implies that the image of the map $H_1(U) \oplus H_1(V) \to H_1(X)$ must have infinite index, and by the remarks at the beginning of this paragraph the same is true for the image of the subgroup $\Gamma \subset \pi_1(X)$. As noted in the first paragraph of the proof, this means that Γ must have infinite index in the fundamental group of X.■