Simplicial approximation

Prop. 0 K = simplicial complex, P = underlying space, $\lambda : C_*(K) \longrightarrow S_*(P)$ the map previously called θ. Then the simplicial and singular barycentric subdivision maps are related by the following commutative diagram:

$$\begin{array}{ccc} C_*(K) & \xrightarrow{\beta} & C_*(BK) \\ \lambda \downarrow & & \downarrow \lambda \\ S_*(P) & \xrightarrow{\beta} & S_*(P). \end{array}$$

This follows from the constructions of the maps.

<u>Simplicial Approximation Thm.</u>

Let (P, K) and (Q, L) be simplicial complexes and let $f: P \to Q$ be continuous. Then $\exists\, r > 0$ and a simplicial map $g: B^r(K) \to L$ such that for all $x \in P$, if σ is a minimal simplex containing x, then $f(x) \in g[\sigma]$.

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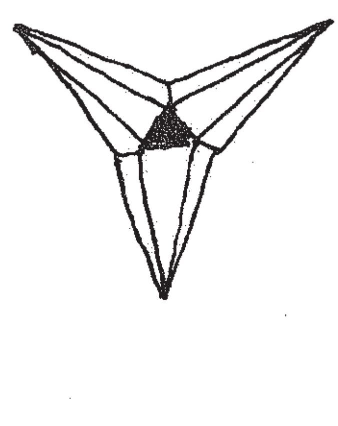
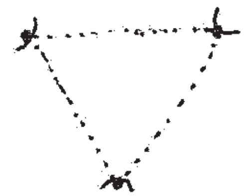
To some extend this corrects the definitions etc. on page 178 of Hatcher.

v vertex, σ simplex in K

The <u>(closed) star</u> $\operatorname{Star}\sigma$ = all simplices τ in K such that $\sigma \cap \tau \neq \emptyset$



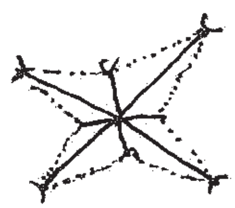
The <u>open simplex</u> $\mathring{\sigma} = \sigma - \partial\sigma$



[If σ has vertices v_i, then $\mathring{\sigma}$ = all $\sum t_i v_i$ s.t. $t_i > 0 \ \forall i$.

<u>Note</u> Every point lies on a unique open simplex.

The <u>open star</u> of a vertex v = all open simplices $\mathring{\sigma}$ s.t. $v \in \sigma$. $st(v)$ or Open star(v)



(v = vertex in the middle)

<u>Claim</u> Open star (v) is open in $|K|$ = underlying space of K.

In fact, $|K|$ - Open star (v) is the union of all (closed) simplices τ such that $v \notin \tau$.

Proof: Call the described set F. Then $x \in F \Rightarrow x \in \tau$ where $v \notin \tau$. Let $\tau' \subseteq \tau$ such that $x \in \mathring{\tau}'$. Then v is also not a vertex of τ', so $x \notin \text{Openstar}(v)$. Conversely, $x \notin \text{Openstar}(v) \Rightarrow x \in \mathring{\tau}$ where v is not a vertex of $\tau \Rightarrow x \in \tau \subseteq F$.

Key Lemma $v_0, \dots, v_q$ are vertices of a simplex in $K \Leftrightarrow \bigcap_i \text{Openstar}(v_i) \neq \phi$.

Proof ($\Rightarrow$) Let $v_0, \dots, v_q$ be the vertices of σ, and let $y \in \mathring{\sigma}$. Then $y \in \cap \text{Openstar}(v_i)$ and in fact $\mathring{\sigma} \subseteq$ intersection.

($\Leftarrow$) Suppose $y \in \bigcap_i \text{openstar}(v_i)$ and let σ be the unique simplex such that $y \in \mathring{\sigma}$. Then for each i, v_i must be a vertex of σ, so there is a face of σ with vertices v_i (in fact, it's σ, but we don't need this).

NOTE In the lemma, duplications of v_i's are allowed.

Proof of Thm. Let $\mathcal{U}_0$ be the open covering of Q by sets Open star (w) where w runs through the vertices of L, and let $\mathcal{U} = f^{-1}\mathcal{U}_0$. Using Lebesgue #s and barycentric subdivision, can find some $B^r K$ s.t. (i) each subset of P with diam $< \varepsilon$ lies in an element of $\mathcal{U}$, (ii) all simplices of $B^r K$ have diameter less than $\varepsilon/3$. Then each Star(v) has diam $\leq 2\varepsilon/3 < \varepsilon$, so $\exists$ vertex $g(v)$ in L s.t. $f[\text{Star}(v)] \subseteq$ Open star $g(v)$.

Define g: Vertices of $B^r K \longrightarrow$ Vertices of L using these choices.

<u>CLAIM</u> If $x \in v_0 \dots v_q$, where the latter is minimal, then $f(x) \in$ simplex with vertices $g(v_0) \dots g(v_q)$. $[x = \sum t_i v_i$, all $t_i > 0]$

This follows because $x \in \cap$ Open star(v_i), so $f(x) \in \bigcap_i f[\text{Openstar}(v_i)] \subseteq \bigcap_i \text{Openstar}\, g(v_i)$. This shows $g(v_i)$ are the vertices of a simplex, and the latter contains $f(x)$. Let $g(x) = \sum t_i g(v_i)$.

It follows that the image of the straight line homotopy

$$H(x,t) = t\,g(x) + (1-t)\,f(x)$$

lies in P so the latter defines a homotopy from f to g.