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III.3 : Simplicial approximation

(Hatcher, § 2.C)

The treatment in Hatcher is fairly standard, so we shall only discuss a few issues here$^{(\star)}$.

PROPOSITION 1. *Let $g : \mathbf{K} \to \mathbf{L}$ be a simplicial map, let $|g|$ be the associated continuous map of underlying topological spaces, and let λ_* denote the standard natural transformation obtained from the chain complex inclusion $C_*(\mathbf{K}) \to S_*(P)$, where P is the polyhedron with simplicial decomposition $\mathbf{K}$. Then $\lambda_* \circ g_* = |g|_* \circ \lambda_*$.*

This follows immediately from the construction of λ, for if $\mathbf{v}_0 \cdots \mathbf{v}_q$ is one of the free generators for $C_q(\mathbf{K})$, then its image under the associated simplicial chain map associated to g is $g(\mathbf{v}_0) \cdots g(\mathbf{v}_q)$i, and under the chain map $\lambda(\mathbf{L})_\#$ this goes to $|g|_\# \circ \lambda(\mathbf{K})_\# (\mathbf{v}_0 \cdots \mathbf{v}_q)$.■

COROLLARY 2. *Suppose that $(P, \mathbf{K})$ and $(Q, \mathbf{L})$ are simplicial complexes, and let $f : P \to Q$ be continuous. Suppose that $r > 0$ and $g : B^r(\mathbf{K}) \to \mathbf{L}$ are such that g is a simplicial approximation to f, and let $\beta_r : C_*(\mathbf{K}) \to C_*(B^r(\mathbf{K}))$ be the iterated barycentric subdivision map. Then $f_* \circ \lambda_* = \lambda_* \circ g_* \circ (\beta_r)_*$.*

Sketch of proof. We have an analog of β_r defined from $S_*(P)$ to itself, and by the results leading to the proof of the Excision Property this map is chain homotopic to the identity. From this it follows that $|g|_* \circ \lambda_* = \lambda_* \circ g_* \circ (\beta_r)_*$. Since g is a simplicial approximation to f we know that $f_* = |g|_*$, and if we make this substitution into the equation in the preceding sentence we obtain the assertion in the corollary.■

Of course, the point of the corollary is that one can compute the map in homology associated to f using the simplicial approximation g.

Given a continuous function f as above, one natural question about simplicial approximations is to find the value(s) of r for which there is a simplicial approximation $g : B^r(\mathbf{K}) \to \mathbf{L}$. The result below shows that in many cases we must take r to be very large.

PROPOSITION 3. *Suppose that* $(P, \mathbf{K})$ *and* $(Q, \mathbf{L})$ *are simplicial complexes, and let* $f : P \to Q$ *be continuous. Let* $r_0(f) > 0$ *be the smallest value of* r *such that* f *is homotopic to a simplicial map* $g : B^r(\mathbf{K}) \to \mathbf{L}$*. Then the following hold:*

(i) *The number* $r_0(f)$ *depends only upon the homotopy class of* f.

(ii) *If the set of homotopy classes* $[P, Q]$ *is infinite, then for each positive integer* M *there are infinitely many homotopy classes* $[f_n]$ *such that* $r_0(f_n) > M$.

Proof. The first part follows immediately from the definition, so we turn out attention to the second. Recall that a simplicial map is completely determined by its values on the vertices of the domain.

Suppose now that $\mathbf{L}$ has b vertices and $B^r(\mathbf{K})$ has a_r. There are b^{a_r} different ways of mapping the vertices of $B^r(\mathbf{K})$ to those of $\mathbf{L}$; although some of these might not arise from a simplicial map, we can still use this to obtain a finite upper bound on the number of simplicial maps from $B^r(\mathbf{K})$ to $\mathbf{L}$, and we also have a finite upper bound on the number of simplicial maps from $B^r(\mathbf{K})$ to $\mathbf{L}$ for all $r \leq M$ if M is any fixed positive integer. It follows that there are only finitely many homotopy classes for which $r_0 \leq M$.■

In particular, by the results of Section V.1 we can apply this proposition to $[P, Q]$ where P and Q are both homeomorphic to S^n for some $n \geq 1$.

III.4 : The Lefschetz Fixed Point Theorem

(Hatcher, § 2.C)

Once again the treatment in Hatcher is fairly standard, so we shall only concentrate on a few issues.

The Euler characteristic

In `algtop-notes.pdf` we discussed the Euler characteristic of a regular cell complex; our purpose here is prove extensions of the main results on Euler characteristics to finite cell complexes as defined in Section I.3 of these notes, and the crucial result is Theorem I.3.9, which shows that the singular homology of a cell complex is isomorphic to the homology of a cellular chain complex whose q-dimensional group may be viewed as a free abelian group on the set of q-cells.

Notation. Let (C, d) be a chain complex over the rationals such that only finitely many chain groups C_q are nonzero and the nonzero groups are all finite-dimensional vector spaces over the rationals.

(i) Set c_q equal to the dimension of C_q.

(ii) Set b_q equal to the rank of d_q.