PROPOSITION 3. *Suppose that* $(P, \mathbf{K})$ *and* $(Q, \mathbf{L})$ *are simplicial complexes, and let* $f : P \to Q$ *be continuous. Let* $r_0(f) > 0$ *be the smallest value of* r *such that* f *is homotopic to a simplicial map* $g : B^r(\mathbf{K}) \to \mathbf{L}$*. Then the following hold:*

(i) *The number* $r_0(f)$ *depends only upon the homotopy class of* f*.*

(ii) *If the set of homotopy classes* $[P, Q]$ *is infinite, then for each positive integer* M *there are infinitely many homotopy classes* $[f_n]$ *such that* $r_0(f_n) > M$*.*

Proof. The first part follows immediately from the definition, so we turn out attention to the second. Recall that a simplicial map is completely determined by its values on the vertices of the domain.

Suppose now that $\mathbf{L}$ has b vertices and $B^r(\mathbf{K})$ has a_r. There are b^{a_r} different ways of mapping the vertices of $B^r(\mathbf{K})$ to those of $\mathbf{L}$; although some of these might not arise from a simplicial map, we can still use this to obtain a finite upper bound on the number of simplicial maps from $B^r(\mathbf{K})$ to $\mathbf{L}$, and we also have a finite upper bound on the number of simplicial maps from $B^r(\mathbf{K})$ to $\mathbf{L}$ for all $r \leq M$ if M is any fixed positive integer. It follows that there are only finitely many homotopy classes for which $r_0 \leq M$.■

In particular, by the results of Section V.1 we can apply this proposition to $[P, Q]$ where P and Q are both homeomorphic to S^n for some $n \geq 1$.

III.4 : The Lefschetz Fixed Point Theorem

(Hatcher, § 2.C)

Once again the treatment in Hatcher is fairly standard, so we shall only concentrate on a few issues.

The Euler characteristic

In `algtop-notes.pdf` we discussed the Euler characteristic of a regular cell complex; our purpose here is prove extensions of the main results on Euler characteristics to finite cell complexes as defined in Section I.3 of these notes, and the crucial result is Theorem I.3.9, which shows that the singular homology of a cell complex is isomorphic to the homology of a cellular chain complex whose q-dimensional group may be viewed as a free abelian group on the set of q-cells.

Notation. Let (C, d) be a chain complex over the rationals such that only finitely many chain groups C_q are nonzero and the nonzero groups are all finite-dimensional vector spaces over the rationals.

(i) Set c_q equal to the dimension of C_q.

(ii) Set b_q equal to the rank of d_q.

(*iii*) Set z_q equal to the dimension of the kernel of d_q.

(*iv*) Set h_q equal to the dimension of $H_q(C)$.

It follows immediately that these numbers are defined for all q and are equal to zero for all but finitely many a.

The equation involving the numbers of faces for a convex linear cell depends upon the following algebraic result.

PROPOSITION 1. *In the setting above we have*

$$\sum_q (-1)^q c_q \quad = \quad \sum_q (-1)^q h_q \ .$$

Proof. The main idea of the argument is given on pages 146 – 147 of Hatcher. In analogy with the discussion there, we have $c_q - z_q = b_q$ and $z_q - b_{q+1} = h_q$, so that

$$\sum_q (-1)^q h_q \quad = \quad \sum_q (-1)^q (z_q - b_{q+1}) \quad = \quad \sum_q (-1)^q z_q \ - \ \sum_q (-1)^q b_{q+1} \quad =$$

$$\sum_r (-1)^r z_r \ + \ \sum_r (-1)^r b_r \quad = \quad \sum_q (-1)^q c_q$$

proving that the two sums in the proposition are equal.■

COROLLARY 2. *Suppose that $(X, \mathcal{E})$ is a finite cell complex with c_q cells in dimension $q \geq 0$, and suppose that $H_q(X)$ is isomorphic to a direct sum of β_q infinite cyclic groups plus a finite group. Then we have*

$$\sum_{q \geq 0} (-1)^q c_q \quad = \quad \sum_{q \geq 0} (-1)^q \beta_q \ .$$

The statement regarding convex linear cells follows immediately from Corollary 11 and Proposition 5. — In general, the topologically invariant number on the right hand side is called the **Euler characteristic** of X and is written $\chi(X)$.

Proof. Let A_* be the chain complex over the rational numbers with $A_q = C_q(X, \mathcal{E})_{(0)}$ and the differential given by rationalizing d_q. It then follows that $\dim A_q = c_q$ and $\dim H_q(A) = \beta_q$. The corollary then follows by applying Proposition 1.■

The Lefschetz number

From the viewpoint of these notes, the Lefschetz number is obtained using the traces of various maps on rational chain groups or cohomology groups. The proof that the alternating sum of traces is the same for simplicial chains and simplicial homology is a special case of the following result:

PROPOSITION 3. *Suppose that C_* is a chain complex of rational vector spaces such that each C_q is finite-dimensional and only finitely many are nontrivial, and let $T : C_* \to C_*$ be a chain map. Then*

$$\sum_q (-1)^q \operatorname{trace} T_q \quad = \quad \sum_q (-1)^q \operatorname{trace} (T_*)_q \ .$$

The proof of this combines the method of Proposition 1 with the following result:

LEMMA 4. *Let V be a finite-dimensional vector space over a field, let W be a vector subspace, and suppose that $T : V \to V$ is a linear transformation such that $T[W] \subset W$. Let T_W be the associated linear transformation from W to itself, and let $T_{V/W}$ denote the linear transformation from V/W to itself which sends $\mathbf{v} + W$ to $T(\mathbf{v}) + W$ for all $\mathbf{v} \in V$ (this is well-defined). Then* $\operatorname{trace}(T) = \operatorname{trace}(T_W) + \operatorname{trace}(T_{V/W})$.

Proof of Lemma 4. Pick a basis $\mathbf{w_1}, \ \cdots \ , \mathbf{w}_k$ for W and extend it to a basis for V by adding vectors $\mathbf{u}_{k+1}, \ \cdots \ , \mathbf{u}_n$. It follows that the vectors $\mathbf{u}_{k+1} + W, \ \cdots \ , \mathbf{u}_n + W$ form a basis for V/W. If we now let $\mathbf{v}$ denote either $\mathbf{v}$ or $\mathbf{w}$ and as usual write

$$T(\mathbf{v}_j) \quad = \quad \sum_i a_{i,j} \, \mathbf{v}_i$$

then the traces of T, T_W and $T_{V/W}$ are given by the sums of the scalars $a_{i,i}$ from 1 to n in the case of T, from 1 to k in the case of T_W, and from $k+1$ to n in the case of $T_{V/W}$.■

As noted above, Proposition 3 follows by applying the same method used in Proposition 1 with the dimensions c_q, z_q, b_q and h_q replaced by the traces of the corresponding linear transformations.■

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Vector fields on S^2

We may think of a *tangent vector field* on the sphere S^2 as a continuous map $\mathbf{X} : S^2 \to \mathbb{R}^3$ such that $\mathbf{X}(\mathbf{u})$ is perpendicular to $\mathbf{u}$ for all $\mathbf{u} \in S^2$ (in other words, the value of $\mathbf{X}$ at a point $\mathbf{u}$ in S^2 is the tangent vector to a curve passing through $\mathbf{u}$). One can use the Lefschetz Fixed Point Theorem to prove the following fundamental result on such vector fields.

THEOREM 5. *If $\mathbf{X}$ is a tangent vector field on S^2, then there is some $\mathbf{u} \in S^2$ such that* $\mathbf{X}(\mathbf{u}) = \mathbf{0}$.

Proof. Suppose that the vector field is everywhere nonzero. If we set

$$\mathbf{Y}(\mathbf{u}) \quad = \quad |\mathbf{X}(\mathbf{u})|^{-1} \cdot \mathbf{X}(\mathbf{u})$$

then $\mathbf{Y}$ is a continuous vector field such that $|\mathbf{Y}|$ is always equal to 1, so that $\mathbf{Y}$ defines a continuous map from S^2 to itself. By the perpendicularity condition we know that $\mathbf{Y}(\mathbf{u}) \neq \mathbf{u}$ for all $\mathbf{u}$, and therefore by the Lefschetz Fixed Point Theorem we know that the Lefschetz number of $\mathbf{Y}$ must be zero.

We now claim that $\mathbf{Y}$ defines a continuous map from S^2 to itself which is homotopic to the identity. Specifically, take the homotopy

$$H(\mathbf{u},t) \quad = \quad \cos\left(\frac{t\pi}{2}\right)\cdot\mathbf{Y}(\mathbf{u}) \ + \ \sin\left(\frac{t\pi}{2}\right)\cdot\mathbf{u}$$

which which moves $\mathbf{u}$ to $\mathbf{Y}(\mathbf{u})$ along a 90° great circle arc. Since $\mathbf{Y}$ is homotopic to the identity, it follows that its Lefschetz number equals the Lefschetz number of the identity, which is $\chi(S^2) = 2$. This contradicts the conclusion of the preceding paragraph; the source of this contradiction was our assumption that $\mathbf{X}(\mathbf{u}) \neq \mathbf{0}$ for all $\mathbf{u}$, and therefore it follows that there is some $\mathbf{u}_0 \in S^2$ such that $\mathbf{X}(\mathbf{u}_0) = \mathbf{0}$.■

In fact, *the same argument goes through virtually unchanged for all even-dimensional spheres*. On the other hand, every odd-dimensional sphere does admit a tangent vector field which is everywhere nonzero. One quick way to construct an example is to take the vector field on $S^{2n+1} \subset \mathbb{R}^{2n+2}$ given by the formula

$$\mathbf{X}(x_1, x_2, x_3, x_4, \ \cdots \ , x_{2n+1}, x_{2n+2}) \quad = \quad (-x_2, x_1, -x_4, x_3, \ \cdots \ , -x_{2n+2}, x_{2n+1}) \ ;$$

if we view $\mathbb{R}^{2n+2}$ as $\mathbb{C}^{n+1}$, then the vector field sends a vector $\mathbf{z} = (z_1, \ \cdots \ , z_{n+1})$ to $i\,\mathbf{z}$.

Geometric interpretation of the Lefschetz number. Suppose that P is a polyhedron which is homeomorphic to a compact smooth manifold M (without boundary), and let $f : M \to M$ be a smooth self-map. Basic results on approximating mappings on smooth manifolds imply that f is homotopic to a smooth map $g : M \to M$ such that g has only finitely many fixed points and for each fixed point $x \in M$ the associated linear map of the tangent space $T(x)$ at x

$$L_f(x) \quad = \quad \mathbf{T}(g)_x : T(x) \ \longrightarrow \ T(x)$$

has the property that $L_f(x) - \mathrm{id}_{T(x)}$ is an isomorphism (in such cases the fixed point set is said to be isolated and nondegenerate). For each fixed point x one can define a *local fixed point index* $\Lambda(g)_x$ to be the sign of the determinant of $L_f(x) - \mathrm{id}_{T(x)}$. Under these conditions the Lefschetz number of g turns out to be given by

$$\Lambda(g) \quad = \quad \sum_{g(x)=x} \Lambda(g)_x \ .$$

Proving this is beyond the scope of these notes and requires the notion of *local fixed point index*. In the paper cited below, a set of axioms for fixed point indices of smooth maps is given, and Chapter 7 of the text by Dold explains how such indices are related to the Lefschetz number as described here:

M. Furi, M. P. Pera, and M. Spadini. *On the uniqueness of the fixed point index on differentiable manifolds*. Fixed point theory and its applications **2004**, 251–259.

Generalizations of the Brouwer Fixed Point Theorem. One can view the Brouwer Fixed Point Theorem as a special case of the Lefschetz Fixed Point Theorem in which the polyhedron P is homeomorphic to a disk or simplex. More generally, we have the following:

THEOREM 6. *Suppose that* P *is a connected polyhedron such that* $H_i(P, \mathbb{Q}) = 0$ *for all* $i > 0$, *and let* $f : P \to P$ *be a continuous mapping. Then the Lefschetz number of* f *is equal to* 1 *and hence* f *has a fixed point.*

Proof. Since P is connected it follows that f induces the identity on $H_0(P; \mathbb{Q}) \cong \mathbb{Q}$, and since all higher dimensional rational homology groups vanish it follows that the Lefschetz number must be 1. The conclusion regarding fixed points now follows from the Lefschetz Fixed Point Theorem.■

A very similar argument yields another generalization in a somewhat different direction.

THEOREM 7. *Suppose that* P *is a connected polyhedron, and let* $f : P \to P$ *be a nullhomotopic continuous mapping. Then the Lefschetz number of* f *is equal to* 1 *and hence* f *has a fixed point.*

Proof. Since P is connected it follows that f induces the identity on $H_0(P; \mathbb{Q}) \cong \mathbb{Q}$, and since f is nullhomotopic all self maps of higher dimensional (rational) homology groups are trivial, so that the Lefschetz number of f must be 1. The conclusion regarding fixed points now follows from the Lefschetz Fixed Point Theorem.■

Finally, we should also mention an infinite-dimensional generalization of the Brouwer Fixed Point Theorem.

THEOREM 8. (Schauder Fixed Point Theorem) *Let* C *be a closed convex subset of the Banach space* X, *and suppose that* $f : C \to C$ *is a continuous self-map Which is also compact (i.e., the image of a bounded subset in* C *has compact closure). Then* f *has a fixed point in* C.■

Here is an online reference for a proof of this result:

`http://www.math.unl.edu/∼s-bbockel1/933-notes/node5.html`

III.5 : Dimension theory

(Munkres, § 50)

In this section, we are interested in the following basic question:

> *Is there some purely topological way to describe the intuitive notion of* $n-$*dimensionality, at least for spaces that are relatively well-behaved?*