Generalizations of the Brouwer Fixed Point Theorem. One can view the Brouwer Fixed Point Theorem as a special case of the Lefschetz Fixed Point Theorem in which the polyhedron P is homeomorphic to a disk or simplex. More generally, we have the following:

THEOREM 6. *Suppose that P is a connected polyhedron such that $H_i(P, \mathbb{Q}) = 0$ for all $i > 0$, and let $f : P \to P$ be a continuous mapping. Then the Lefschetz number of f is equal to 1 and hence f has a fixed point.*

Proof. Since P is connected it follows that f induces the identity on $H_0(P; \mathbb{Q}) \cong \mathbb{Q}$, and since all higher dimensional rational homology groups vanish it follows that the Lefschetz number must be 1. The conclusion regarding fixed points now follows from the Lefschetz Fixed Point Theorem.■

A very similar argument yields another generalization in a somewhat different direction.

THEOREM 7. *Suppose that P is a connected polyhedron, and let $f : P \to P$ be a nullhomotopic continuous mapping. Then the Lefschetz number of f is equal to 1 and hence f has a fixed point.*

Proof. Since P is connected it follows that f induces the identity on $H_0(P; \mathbb{Q}) \cong \mathbb{Q}$, and since f is nullhomotopic all self maps of higher dimensional (rational) homology groups are trivial, so that the Lefschetz number of f must be 1. The conclusion regarding fixed points now follows from the Lefschetz Fixed Point Theorem.■

Finally, we should also mention an infinite-dimensional generalization of the Brouwer Fixed Point Theorem.

THEOREM 8. (Schauder Fixed Point Theorem) *Let C be a closed convex subset of the Banach space X, and suppose that $f : C \to C$ is a continuous self-map Which is also compact (i.e., the image of a bounded subset in C has compact closure). Then f has a fixed point in C.*■

Here is an online reference for a proof of this result:

`http://www.math.unl.edu/∼s-bbockel1/933-notes/node5.html`

III.5 : Dimension theory

(Munkres, § 50)

In this section, we are interested in the following basic question:

> *Is there some purely topological way to describe the intuitive notion of n−dimensionality, at least for spaces that are relatively well-behaved?*

Of course, in linear algebra there is the standard notion of dimension, and this concept has far-reaching consequences for understanding dimensions in geometry. A topological approach to describing the dimensions of at least some spaces is implicit in our proof for Invariance of Dimension (see Proposition IV.2.16), which can be used to define a notion of dimension for topological spaces which locally look like an open subset of $\mathbb{R}^n$ for some fixed $n \geq 0$. There is an extensive literature on topological approaches to defining the dimensions of spaces. Our purpose here is to discuss one particularly important example known as the *Lebesgue covering dimension*; for reasonably well-behaved classes of spaces this is equivalent to other frequently used concepts of dimension. Here are some printed and online references for topological dimension theory:

W. Hurewicz and H. Wallman. *Dimension Theory* (Revised Edition, Princeton Mathematical Series, Vol. 4). *Princeton University Press, Princeton*, 1996.

K. Nagami. *Dimension Theory* (with an appendix by Y. Kodama, Pure and Applied Mathematics Series, Vol. 37). *Academic Press, New York*, 1970.

J. Nagata. *Modern Dimension Theory* (Second Edition, revised and extended; Sigma Series in Pure Mathematics, Vol. 2). *Heldermann-Verlag, Berlin*, 1983.

`http://en.wikipedia.org/wiki/Lebesgue_covering_dimension`

`http://en.wikipedia.org/wiki/Dimension`

`http://en.wikipedia.org/wiki/Inductive_dimension`

FRACTAL DIMENSIONS. There are several notions of *fractal dimension* for subsets of $\mathbb{R}^n$ which depend on the way in which an object is embedded in $\mathbb{R}^n$ and not just the subset's underlying topological structure; for example, various standard examples of nonrectifiable curves in the plane have fractal dimensions which are numbers strictly between 1 and 2. Such objects are interesting for a variety of reasons, but they are beyond the scope of this course so we shall only give two online references here:

`http://en.wikipedia.org/wiki/Fractal_dimension`

`http://www.warwick.ac.uk/~masdbl/dimension-total.pdf`

The basic setting

We shall base our discussion upon the material in Section 50 of Munkres. For the sake of clarity we shall state the main definition and mention some standard conventions.

Definition. Let X be a topological space, let n be a nonnegative integer, and let $\mathcal{U}$ be an indexed open covering of X. Then we shall say that *the open covering $\mathcal{U}$ has order at most n* provided every intersection of the form

$$U_{\alpha(0)} \ \cap\ \cdots\ \cap\ U_{\alpha(n)}$$

is empty, and we shall say that *the space X has Lebesgue covering dimension* $\leq n$ provided every open covering $\mathcal{U}$ of X has a refinement $\mathcal{V}$ of order $\leq n$. Frequently we shall write $\dim X \leq n$ if the Lebesgue covering dimension is at most n.

We shall say that $\dim X = n$ (the Lebesgue covering dimension is equal to n) if $\dim X \leq n$ is true but $\dim X \leq n-1$ is not. By convention, the Lebesgue covering dimension of the empty set is taken to be -1, and we shall write $\dim X = \infty$ if $\dim X \leq n$ is false for all n.

Munkres states and proves many fundamental results about the Lebesgue covering dimension, and we shall not try to copy or rework most of his results here. Instead, our emphasis in this section will be on the following key issues:

(1) Describing precise connections between the topological theory of dimension as in Munkres and the algebraic notions of k-dimensional homology groups for various choices of k.

(2) Using the methods of these notes to give an alternate proof of Theorem 50.6 in Munkres; namely, if $A \subset \mathbb{R}^n$ is compact, then the topological dimension of A satisfies $\dim A \leq n$.

(3) Using algebraic topology to prove that the topological dimension of an n-dimensional polyhedron is in fact **equal** to n (the results in Munkres show that this dimension is at most n).

We shall begin by addressing the dimension question in (2); one reason for doing this is that the approach taken here will play a crucial role in our treatment of the subject.

MUNKRES, THEOREM 50.6. *If A is a compact subset of $\mathbb{R}^n$, then $\dim A \leq n$.*

Alternate proof. We know that there is some very large hypercube K of the form $[-M, M]^n$ which contains A, and we also know that A is closed in this hypercube. By Theorem 50.1 on pages 306–307 of Munkres, it is enough to show that the hypercube has dimension at most n. Since every hypercube has a simplicial decomposition with simplices of dimension $\leq n$, it will suffice to prove the following result:

LEMMA 1. *If $P \subset \mathbb{R}^m$ is a polyhedron with an n-dimensional simplicial decomposition, then the topological dimension of P is at most n.*

If we know this, then we know that the hypercube, and hence A, must have topological dimension $\leq n$.∎

Proof of Lemma 1. Let $\mathcal{U}$ be an open covering of the hypercube K, and let $\varepsilon > 0$ be a Lebesgue number for $\mathcal{U}$. Using barycentric subdivisions, we can find an n-dimensional simplicial decomposition of K whose simplices all have diameter less than $\varepsilon/2$. Therefore if $\mathbf{v}$ is a vertex of this simplicial decomposition, then the open set $\mathbf{Openstar}(\mathbf{v})$ is contained in some element of $\mathcal{U}$. Now these sets form an open covering of K (see Section 2.C of Hatcher), and therefore these open stars form a finite open refinement of $\mathcal{U}$. Since an intersection of open stars $\cap_i \mathbf{Openstar}(\mathbf{v}_i)$ is nonempty if and only if the vertices $\mathbf{v}_i$ lie on a simplex from the underlying simplicial decomposition, the n-dimensionality of the decomposition implies that every intersection of $(n+2)$ distinct open stars must be empty. This is exactly the criterion for the covering by open stars to have order at most $(n+1)$. Therefore we have shown that $\mathcal{U}$ has a finite open refinement with at most this order, which means that the topological dimension of K is at most n.∎

The discussions of the first and third issues are closely related, and they use the material on partitions of unity on pags 225–226 of Munkres (see Theorem 36.1 in particular).

Definitions. Let X be a $\mathbf{T_4}$ space, and let $\mathcal{U}$ be a finite open covering of X. Set $\mathbf{Vec}(\mathcal{U})$ equal to the (finite-dimensional) real vector space with basis given by the sets in $\mathcal{U}$, and define the **nerve** of $\mathcal{U}$, written $\mathfrak{N}(\mathcal{U})$, to be the simplicial complex whose simplices are given by all vertex sets of the form $U_{\alpha(0)}, \cdots, U_{\alpha(q)}$ such that

$$U_{\alpha(0)} \cap \cdots \cap U_{\alpha(q)} \neq \emptyset .$$

By construction, the vertices of this simplicial complex are all symbols of the form $[U_\alpha]$, where U_α is nonempty and belongs to $\mathcal{U}$.

If $\{ \varphi_\alpha \}$ is a partition of unity which is subordinate to (= dominated by) $\mathcal{U}$, then there is a *canonical map* k_φ from X to $\mathfrak{N}(\mathcal{U})$ given by the partition of unity:

$$k_\varphi(x) = \sum \varphi_\alpha(x) \cdot [U_\alpha]$$

Different partitions of unity yield different maps, but we have the following:

CLAIM: *For each finite open covering $\mathcal{U}$, all canonical maps from X to $\mathfrak{N}(\mathcal{U})$ are homotopic to each other.*

Proof of the claim. For each choice of x and canonical maps φ_0, φ_1, we know that the points $\varphi_i(x)$ lie on the simplex whose vertices are all $[U_\alpha]$ such that $x \in U_\alpha$. Thus the straight line segment joining $\varphi_0(x)$ to $\varphi_1(x)$ also lies on this simplex, and hence also lies in the nerve of $\mathcal{U}$. In other words, the image of the straight line homotopy from φ_0 to φ_1 is always contained in $\mathfrak{N}(\mathcal{U})$, and therefore the two canonical maps into $\mathfrak{N}(\mathcal{U})$ are homotopic.■

In the special case where $(P, \mathbf{K})$ is a simplicial complex and $\mathcal{U}$ is the open covering given by open stars of vertices (see Hatcher for the definitions), the canonical map(s) from P to the nerve of $\mathcal{U}$ can be described very simply as follows:

PROPOSITION 2. *Let P, $\mathbf{K}$ and $\mathcal{U}$ be as above, and for each vertex $\mathbf{v}$ of $\mathbf{K}$ define the* extended barycentric coordinate function $\mathbf{v}^* : P \to [0,1]$ *as follows: If $\mathbf{x} \in A$ for some simplex A which contains $\mathbf{v}$ as a vertex, let $\mathbf{v}^*(\mathbf{x})$ denote the barycentric coordinate of $\mathbf{x}$ with respect to $\mathbf{v}$, and if $\mathbf{x}$ lies on a simplex A which does not contain $\mathbf{v}$ as a vertex, set $\mathbf{v}^*(\mathbf{x}) = 0$ (it follows immediately that this map is well-defined and continuous). Define a map $\kappa : P \to \mathfrak{N}(\mathcal{U})$ by $\kappa(\mathbf{x}) = \sum \mathbf{v}^*(\mathbf{x}) \cdot \mathbf{v}$. Then κ defines a homeomorphism from P to $\mathfrak{N}(\mathcal{U})$, and every canonical map with respect to the open covering $\mathcal{U}$ is homotopic to κ.*

Sketch of proof. First of all, the barycentric coordinate functions are well-defined, for if $\mathbf{x}$ lies on a simplex A with vertex $\mathbf{v}$ and also on a simplex B for which $\mathbf{v}$ is not a vertex, then it follows that the barycentric coordinate of $\mathbf{x}$ with respect to $\mathbf{v}$ must be zero. The assertion that κ defines a homeomorphism from P to the nerve of $\mathcal{U}$ follows because κ maps the simplices of $\mathbf{K}$ bijectively to the simplices of $\mathfrak{N}(\mathcal{U})$; more precisely, there is a 1–1 correspondence of simplices and each simplex of $\mathbf{K}$ is sent to a simplex of the nerve by a bijective affine map.

Finally, the proof that κ is homeomorphic to a canonical map associated to a partition of unity follows from the same considerations which appear in the proof that two canonical maps are homotopic (for every $\mathbf{x} \in P$, there is a simplex in the nerve containing both $\kappa(\mathbf{x})$ and the value of a canonical map at $\mathbf{x}$).■

Čech homology groups

The idea behind singular homology groups is that one approximates a space by maps **from** simplicial complexes (in particular, simplices) into a space X. Dually, the idea behind Čech homology groups is that one approximates a space by maps **into** simplicial complexes. Constructions of this type play an important role in the theory and applications of machinery from algebraic topology, but we shall only focus on what we need. As is often the case, the first step is to construct some necessary algebraic machinery.

Inverse systems and inverse limits

The definition of Čech homology requires the notion of *inverse limit*; special cases of this concept appear in Hatcher, but since we need the general case we must begin from scratch.

Definition. A *codirected set* is a pair $(A, \prec)$ consisting of a set A and a binary operation $\prec$ such that the following hold:

(a) (Reflexive Property) For all $x \in A$ we have $x \prec x$.

(b) (Transitive Property) If $x, y, z \in A$ are such that $x \prec y$ and $y \prec z$, then $x \prec z$.

(c) (Lower Bound Property) For all $x, y \in A$ there is some $w \in A$ such that $w \prec x$ and $w \prec y$.

These are similar to the defining conditions for a partially ordered set, but we do **not** assume the symmetric property (so $x \prec y$ and $y \prec x$ does not necessarily imply $x = y$), and the Lower Bound Property does not necessarily hold for a partially ordered set which is not linearly ordered. On the other hand, if a partially ordered set is a **lattice** (*i.e.*, finite subsets always have least upper bounds and greatest lower bounds), then it is a codirected set.

The basic example of a codirected set in Hatcher is given by the positive integers $\mathbb{N}^+$ with the **reverse** of the usual partial ordering, so that $a \prec b$ if and only if $b \geq a$.

Given a codirected set $(A, \prec)$, there is an associated category $\mathsf{CAT}\,(A, \prec)$ for which $\mathsf{Morph}\,(x, y)$ is nonempty if and only if $x \prec y$, and in this case $\mathsf{Morph}\,(x, y)$ contains exactly one element.

Definition. Let $(A, \prec)$ be a codirected set, and let $\mathbf{C}$ be a category. An *inverse system* in $\mathbf{C}$ indexed by $(A, \prec)$ is a covariant functor F from $\mathsf{CAT}\,(A, \prec)$ to $\mathbf{C}$. If $a \prec b$, then the value of F on the unique morphism $a \to b$ is frequently denoted by notation like $f_{a,b}$; in other words, $f_{a,b} = F(a \prec b)$.

There is a closely related concept of *inverse limit* for inverse systems. One can do this in purely categorical terms, but we are only interested in working with inverse limits over categories of modules. For inverse systems $F = \{F(a)\}$ of modules, the inverse limit

$$\lim_{\leftarrow} \quad = \quad \text{inv} \lim_{A} \; F(a) \quad = \quad \text{proj} \lim_{A} \; F(a)$$

is defined to be the set of all $x = (x_a)$ in $\prod_A \; F(a)$ such that for each $a \prec b$ we have $f_{a,b}(x_a) = x_b$. For each $a \in A$ the map p_a denotes projection onto the a-coordinate.

Inverse limits have the following universal mapping property, which in fact characterizes the construction.

PROPOSITION 3. *Suppose that F is an inverse system as above, and suppose that we are given a module L with maps $q_a : L \to F(a)$ such that $f_{a,b} \circ q_a = q_b$ whenever $a \prec b$. Then there is a unique homomorphism $h : L \to \lim_{\leftarrow} F(a)$ such that $g_a = f_a \circ h$ for all a.*

This is an immediate consequence of the definitions.∎

There are straightforward analogs of the inverse limit construction for may categories (sets, compact Hausdorff spaces, groups, ...), and we shall leave the details of setting up such objects to the reader as an exercise.

Frequently it is important to recognize that inverse limits of directed systems can be given by inverse limits over "good" subobjects. We shall say that $B \subset A$ is a *codirected subobject* if B is a subset, the binary relation is the restriction of the binary relation on A, and the Lower Bound Property still holds on B (however, if $w \in A$ is such that $w \prec b, a$ we do not necessarily assume that $w \in B$; we only assume that there is some $w' \in B$ with $w' \prec a, b$). We shall say that such a object is *cofinal* if for each $x \in A$ there is some $y \in B$ such that $y \prec x$.

Example. Let γ be a cardinal number, and let $\mathsf{Cov}_\gamma(X)$ be the family of indexed open coverings of X such that the cardinality of the indexing set is at most γ. We shall say that an indexed open covering $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ is an *indexed refinement* of $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ if there is a map of indexing spaces $j : B \to A$ such that $V_\beta \subset U_{j(\beta)}$ for all β; note that if $\mathcal{V}$ is a refinement of $\mathcal{U}$ in the usual sense then by the Axiom of Choice we can always find a function j with the required properties. — Suppose now that X is a compact metric space and $\mathsf{FinCov}\,(X)$ is a set of all finite indexed open coverings whose indexing sets are subsets of the set $\mathbb{N}$ of nonnegative integers. If $\mathbb{A}$ is a subset of $\mathsf{FinCov}\,(X)$ such that for each $k > 0$ there is an open covering $\mathcal{A}_k \in \mathbb{A}$ whose (open) subsets all have diameter less than $1/k$, then a Lebesgue number argument implies that $\mathbb{A}$ is cofinal in $\mathsf{FinCov}\,(X)$.

Given a cofinal subobject B and an inverse system F on A, then there is an associated inverse system $F|B$. The following crucial observation suggests the importance an usefulness of such restricted inverse systems.

PROPOSITION 4. *Suppose that we are given the setting above, and let B be a cofinal subobject. Then there is a canonical isomorphism from $\lim_{\leftarrow} F$ to $\lim_{\leftarrow} F|B$.*

Proof. By definition, the inverse limit L_A over all of A is a submodule of $P_A = \prod_{a\in A} F(a)$ and the inverse L_B limit over B is a submodule of $P_B = \prod_{b\in B} F(b)$. Let $\varphi_0 : P_A \to P_B$ be given by the projections onto the factors $F(b)$; since the operations in the product are defined coordinatewise, it follows immediately that φ_0 is a module homomorphism.

By construction it follows that φ_0 maps L_A to L_B. If $\varphi : L_A \to L_B$ be the homomorphism defined by φ_0, the objective is to prove that φ is an isomorphism. It is straightforward to verify that φ is onto. Suppose now that we are given $x = (x_a)$ and $y = (y_a)$ such that $\varphi(x) = \varphi(y)$. Then $x_b = y_b$ for all $b \in B$, and we need to show that this implies $x_a = y_a$ for all a. Let $\alpha \in A$ be arbitrary, and choose $\beta \in B$ such that $\beta \prec \alpha$. Then we have $x_\alpha = f_{\beta,\alpha}(x_\beta)$ and $y_\alpha = f_{\beta,\alpha}(y_\beta)$. Since we are assuming that $y_\beta = x_\beta$, it follows that $y_\alpha = x_\alpha$.■

Definition and properties of Čech homology

Suppose that X is a compact Hausdorff space, let $A \subset X$ be a closed subspace, and let $\mathsf{FinCov}\,(X, A)$ denote the codirected set of all pairs $(\mathcal{U}, \mathcal{U}|A)$, where $\mathcal{U}$ is a finite open covering of X and $\mathcal{U}|A$ denotes its restriction to A with all empty intersections deleted; the binary relation

$$\beta \quad = \quad (\mathcal{V}, \mathcal{V}|A) \quad \prec \quad (\mathcal{U}, \mathcal{U}|A) \quad = \quad \alpha$$

is taken to mean that $(\mathcal{V}, \mathcal{V}|A)$ is an indexed refinement of $(\mathcal{U}, \mathcal{U}|A)$. Since we are working with indexed refinements, it follows that the map of indexing sets will define a simplicial mapping of nerve pairs

$$j_{\beta,\alpha} : (N_\beta, N'_\beta) \quad = \quad \Big(\mathfrak{N}(\mathcal{V}), \mathfrak{N}(\mathcal{V}|A) \Big) \quad \longrightarrow \quad \Big(\mathfrak{N}(\mathcal{U}), \mathfrak{N}(\mathcal{U}|A) \Big) \quad = \quad (N_\alpha, N'_\alpha)$$

and therefore we obtain an inverse system of simplicial complex pairs and simplicial mappings. If we take the simplicial or singular chain complexes associated to such a system we obtain inverse systems of chain complexes, and if we pass to homology we obtain inverse systems of homology groups; at the chain complex level the inverse systems are different, but their homology groups are the same.

Definition. If X is a compact Hausdorff space and $A \subset X$ is a closed subspace, then the **Čech homology groups** $\check{H}_q(X, A)$ are the inverse limits of the inverse systems $H_q(N_\alpha, N'_\alpha)$, where α runs through all pairs $(\mathcal{U}, \mathcal{U}|A)$.

Presumably we have introduced these groups because they have implications for dimension theory, and one can also ask if these groups can be computed for finite simplicial complexes. The next two results confirm these expectations.

THEOREM 5. *If X is a compact Hausdorff space whose Lebesgue covering dimension is $\leq n$ and A is a closed subset of X, then $\check{H}_q(X, A) = 0$ for all $q > n$.*

Proof. The condition on the Lebesgue covering dimension implies that every finite open covering $\mathcal{U}$ of X has a (finite) refinement such that each subcollection of $n + 2$ open subsets from $\mathcal{U}$ has an empty intersection. This condition means that the nerve of $\mathcal{U}$ has no simplices with $n+2$ vertices and hence no simplices of dimension $\geq n+1$; in other words, the (geometric) dimension of the nerve is at most n. By Proposition 4 and the assumption on the Lebesgue covering dimension, we know that the Čech homology of (X, A) can be computed using open coverings for which each subcollection of $n+2$ open subsets from $\mathcal{U}$ has an empty intersection, and hence the Čech homology is an inverse limit of homology groups of simplicial complexes with dimension $\leq n$. Since the q-dimensional homology of such complexes vanishes if $q > n$, it follows that the same is true for the inverse limit groups when $q > n$, and therefore we must have $\check{H}_q(X, A) = 0$ for all $q > n$.■

The next main result states that the Čech homology for a simplicial complex pair is the same as the homology we have already defined. a more general result:

THEOREM 6. *If X is a compact Hausdorff space and $A \subset X$ is a closed subspace, then there is a canonical mapping φ_∞ from $H_*(X, A)$ to $\check{H}_*(X, A)$ (the* singular-Čech comparison map*), where the groups on the left are singular homology groups. If X is a polyhedron with some simplicial $\mathbf{K}$ such that A is a subcomplex with respect to this decomposition, then the singular-Čech comparison map is an isomorphism.*

Before proving this result, we shall use the conclusion to derive the main implications for dimension theory.

THEOREM 7. (i) *For all $n \geq 0$, the Lebesgue covering dimension of the disk D^n is equal to n.*

(ii) *If $(P, \mathbf{K})$ is a simplicial complex whose geometric definition is equal to n, then the Lebesgue covering dimension of P is also equal to n.*

(iii) *If $A \subset \mathbb{R}^n$ is a compact subset with a nonempty interior, then the Lebesgue covering dimension of A is equal to n.*

(iv) *If $\mathbf{Q} = [0,1]^\infty$ is the Cartesian product of countably infinitely many copies of the unit interval (the so-called* Hilbert cube*), then the Lebesgue covering dimension of $\mathbf{Q}$ is equal to ∞.*

Proof. We shall take these in order.

Proof of (i). By the discussion at the beginning of this section (or the corresponding discussion in Munkres), we know that the Lebesgue covering dimension of D^n is at most n, so we need to show that it cannot be $\leq (n-1)$. We shall exclude this by deriving a contradiction from it. If the Lebesgue covering dimension was strictly less than n, then it would follow that $\check{H}_n(D^n, A)$ would vanish for all closed subsets $A \subset D^n$. By Theorem 6 we know that $\check{H}_n(D^n, S^{n-1}) \cong H_n(D^n, S^{n-1})$, and since the latter is isomorphic to $\mathbb{Z}$ it follows from Theorem 5 that the Lebesgue covering dimension cannot be $\leq n-1$. Therefore this dimension must be equal to n.■

Proof of (ii). This follows immediately from (i) and Theorem 50.2 of Munkres (see page 307 for details).■

Proof of (iii). By the discussion at the beginning of this section we know that the Lebesgue covering dimension of A is $\leq n$. Since A has a nonempty interior, it follows that

A contains a closed subset which is homeomorphic to D^n. This means that the Lebesgue covering dimension of A must be at least as large as the Lebesgue covering dimension of D^n, which is n. Combining these observations, we conclude that the Lebesgue covering dimension of A is equal to n.■

Proof of (iv). Let $H\langle n\rangle \subset \mathbf{Q}$ be the subset of all points whose coordinates satisfy $x_k = 0$ for $k \geq n+1$. Then it follows that $H\langle n\rangle$ is a closed subset of $\mathbf{Q}$ which is homeomorphic to D^n, and therefore we have $n = \dim H\langle n\rangle \leq \dim \mathbf{Q}$ for all n.■

Remark. The preceding result implies that the Lebesgue covering dimension does not behave well with respect to quotients, even if the space and its quotient are polyhedra. In particular, if $f : X \to Y$ is a continuous and onto mapping of compact Hausdorff spaces, then in general we cannot say anything about the relation between the Lebesgue covering dimensions of X and Y even if we know that both numbers are finite. The simplest counterexamples are given by the continuous surjection from $[0,1]$ to $[0,1]^2$ given by the Peano curve (described in Section 44 of Munkres) and the usual first coordinate projection from $[0,1]^2$ to $[0,1]$; in the first case the dimension increases when one passes to the quotient, and in the second case the dimension decreases (which is what one reasonably expects). Of course, if we take f as above to be an identity map, then the dimension does not change.

We shall discuss the behavior of dimensions under taking products after proving Theorem 6.

Proof of Theorem 6. We begin by proving the general statement. If $\mathcal{U}$ is an open covering of X and A is a closed subset of X, then we have seen that a partition of unity subordinate to $\mathcal{U}$ defines a canonical map from X into the nerve $\mathfrak{N}(\mathcal{U})$, and by construction this map sends A into $\mathfrak{N}(\mathcal{U}|A)$. We have also seen that the homotopy class of this map is well defined (at least when $A = \emptyset$, but the same argument implies that the canonical maps of pairs associated to different partitions of unity will be homotopic as maps of pairs). Therefore we have homomorphisms

$$(k_\alpha)_* : H_*(X,A) \longrightarrow H_*\big(\mathfrak{N}(\mathcal{U}), \mathfrak{N}(\mathcal{U}|A)\big)$$

and we need to show that these yield a map into the inverse limit of the groups on the right hand side, which is true if and only if

$$(k_\alpha)_* = (j_{\beta\alpha})_* \circ (k_\beta)_*$$

for all α and β such that $\beta \prec \alpha$. But if the latter holds, then it follows that the composite $j_{\beta\alpha} \circ (k_\beta)$ defines a canonical map into the nerve pair (N_α, N'_α), and therefore this composite is homotopic to k_α; therefore the associated maps in homology are equal, and this implies that we have the desired homomorphism φ_∞ into the inverse limit $\check{H}_*(X,A)$.

We must now show that the singular-Čech comparison map φ_∞ is an isomorphism if X is a polyhedron with simplicial decomposition $\mathbf{K}$ and A corresponds to a subcomplex of $(X,\mathbf{K})$. Let $r > 0$, and let $\mathcal{W}_r$ be the open covering by open stars of vertices in the

r^{th} barycentric subdivision $B^r(\mathbf{K})$. Then by construction we have $\mathcal{W}_{r+1} \prec \mathcal{W}_r$ for all r, and a Lebesgue number argument shows that the set of all open coverings $\mathcal{W}_r$ determines a cofinal subset of $\mathsf{FinCov}\,(X)$. If (N_r, N_r') denotes the nerve pair associated to $\mathcal{W}_r$, then it follows that $\check{H}_*(X,A)$ is isomorphic to the inverse limit of the groups $H_*(N_r, N_r')$.

If we can show that the canonical maps k_r into N_r all define isomorphisms from $H_*(X,A)$ to $H_*(N_r, N_r')$, then the map into the inverse limit will be an isomorphism for the following reasons:

(1) If $\varphi_\infty(u) = 0$, then $(k_r)_*\,(u) = 0$ for all r, and since each of these maps is an isomorphism it follows that $u = 0$.

(2) If v lies in the inverse limit, then v has the form $(v_1, v_2, \cdots)$ where $v_r = (j_{r,r+1})_*\,(v_{r+1})$ for all r. Since k_r defines an isomorphism, it follows that $v_r = (k_r)_*\,(u_r)$ for some unique $u_r \in \check{H}(X,A)$, and if we can show that $u_r = u_{r+1}$ for all r then it will follow that $v = \varphi_\infty(u)$. But the previous equations imply that

$$(k_r)_*\,(u_{r+1}) \;=\; (j_{r,r+1})_* \circ (k_{r+1})_*\,(u_{r+1}) \;=\; (j_{r,r+1})_*\,(v_{r+1})\;\; v_r \;=\; (k_r)_*\,(u_r)$$

and since $(k_r)_*$ is injective it follows that $u_{r+1} = u_r$.

To conclude the proof, we note that the relative version of Proposition 2 implies that the map of pairs determined by each k_r is homotopic to a homeomorphism of pairs.∎

As noted before, this concludes the proof that the Lebesgue covering dimension of D^n is equal to n. It is also possible to prove the following result:

THEOREM 8. *For every $n \geq 0$ the Lebesgue covering dimension of $\mathbb{R}^n$ is equal to n.*

Sketch of proof. The exercises at the end of Section 50 in Munkres (see pages 315–316) provide machinery for extending results on covering dimensions to "reasonable" noncompact spaces. In particular, Exercise 8 shows that the Lebesgue covering dimension of $\mathbb{R}^n$ is at most n. Since the dimension of the closed subspace D^n is equal to n, it follows that the Lebesgue covering dimension of $\mathbb{R}^n$ is at least n, and therefore it must be exactly n.∎

One cap proceed similarly to extend the conclusions for Exercises 9 and 10 on page 316 of Munkres. Specifically, *every (second countable) topological n-manifold has Lebesgue covering dimension equal to n, and if $A \subset \mathbb{R}^n$ is a close subset with nonempty interior, then the Lebesgue covering dimension of A is also equal to n.*∎

Note. For topological n-manifolds, second countability is equivalent to the σ-compactness condition which appears on page 316 of Munkres (proof?).

Dimensions of products

The standard homeomorphism $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{m+n}$ strongly suggests the following question:

QUESTION. *If we know that the Lebesgue covering dimensions of the nonempty compact Hausdorff spaces X and Y are m and n respectively, does it follow that the Lebesgue covering dimension of the product $X \times Y$ is equal to $m+n$?*

In the next subheading we shall prove the following result:

PROPOSITION 9. *If X and Y are compact Hausdorff spaces whose Lebesgue covering dimensions are m and n respectively, then the Lebesgue covering dimension of the product $X \times Y$ is less than or equal to $m + n$.*

We shall derive this result as an immediate consequence of Proposition 18 below.

If we assume that our spaces are somewhat reasonable, then we can prove a stronger and more satisfying result:

PROPOSITION 10. *In the setting of Proposition 9, suppose that $X = \cup_i A_i$ and $Y = \cup_j B_j$ where the sets A_i and B_j are all homeomorphic to k-disks for suitable values of k. Then the Lebesgue covering dimension of $X \times Y$ is equal to $m + n$.*

Proof of Proposition 10. By Theorem 50.2 of Munkres and finite induction, it follows that the dimension of $X \times Y$ is equal to the maximum of the dimensions of the closed subsets $A_i \times B_j$. On the other hand, the same result implies that there are some indices p and q such that A_p is homeomorphic to D^m and B_q is homeomorphic to D^n (otherwise the dimensions of X and Y would be strictly less than m and n). Since $D^m \times D^n$ is homeomorphic to D^{m+n} it follows that $X \times Y$ has a closed subset with Lebesgue covering dimension equal to $m + n$. On the other hand, we also know that the dimension of each disk A_i is at most m and the dimension of each disk B_j is at most n, so the dimension of $X \times Y$ is at most $m + n$. If we combine these, we find that the dimension of $X \times Y$ is equal to $m + n$.■

Counterexamples to the general question

Although Propositions 9 and 10 may suggest that the formula $\dim(X \times Y) = \dim X + \dim Y$ holds more generally, it is possible to construct examples where the left hand side is less than the right. The first examples of this sort were discovered by L. S. Pontryagin; here is a reference to the original paper:

> **L. S. Pontryagin.** *Sur une hypothèse fondementale de la théorie de la dimension.* Comptes Rendus Acad. Sci. (Paris) **190** (1930), 1105–1107.

In Pontryagin's example one has $X = Y$ and $\dim X = 2$ but $\dim(X \times X) = 3$. By the following result, these are the lowest dimensions in which one can have $\dim(X \times Y) < \dim X + \dim Y$.

DIMENSION ESTIMATES FOR PRODUCTS. *Let X and Y be nonempty compact metric spaces. Then the following hold:*

(a) *If* $\dim Y = 0$, *then* $\dim X \times Y = \dim X$.

(b) *If* $\dim Y = 1$, *then* $\dim X \times Y = \dim X + 1$.

(c) *If* $\dim Y \geq 2$, *then* $\dim X \times Y \geq \dim X + 1$.

Proofs of these results are beyond the scope of this course, so we shall limit ourselves to mentioning some key points which arise in the proofs.

The proof of the first statement is actually fairly direct, and it only requires a small amount of additional machinery. Proofs of the second and third statements using an alternate approach to defining topological dimensions (the *weak inductive* or *Menger-Urysohn dimension*) are due to Hurewicz (we should note that the Menger-Urysohn definition is the one which appears in Hurewicz and Wallman). Here is a reference to the original paper.

> **W. Hurewicz.** *Sur la dimension des produits cartésiens.* Annals of Mathematics **36** (1935), 194–197.

There is a brief indication of another way to retrieve (b) at the top of page 241 in the book by Nagami (however, this requires a substantial amount of input from algebraic topology). One proof of (c) can be obtained by combining (b) with the following existence theorem: *If* Y *is a compact metric space such that* $n = \dim Y$ *is finite and* $0 < k < n$*, then there is a closed subset* $B \subset Y$ *such that* $\dim B = k$. — This result and the equivalence of the Lebesgue and Menger-Urysohn dimensions for compact metric spaces are discussed in an appendix to this section.

Spaces for which $\dim(X \times Y) < \dim X + \dim Y$ are generally far removed from the sorts of objects studied in most of topology, but it is important to recognize their existence. On the other hand, even though there is no general product formula for the dimensions of compact metric spaces, the validity of the formula for many well-behaved examples (see Proposition 9) leads one naturally to look for necessary and sufficient conditions under which one has $\dim(X \times Y) = \dim X + \dim Y$. Here is one reference which answers the question:

> **Y. Kodama.** *A necessary and sufficient condition under which* $\dim(X \times Y) = \dim X + \dim Y$. Proc. Japan. Acad. **36** (1960), 400–404.

As in several previously cited cases, the proofs of the main results in this paper rely heavily on input from algebraic topology.

Further results

We shall consider two issues related to the discussion of dimension theory:

1. Giving an example of a compact subset of $\mathbb{R}^2$ for which the singular and Čech homology groups are not isomorphic.

2. Showing that a compact subset of $\mathbb{R}^n$ has Lebesgue covering dimension n if and only if it has a nonempty interior (one can then use the previously cited exercises in Munkres to show that the same conclusion holds for arbitrary closed subsets). The machinery developed for this question will also yield a proof of Proposition 9 on the Lebesgue covering dimensions of cartesian products.

The example for the first problem will be the *Polish circle*, and our discussion will be based upon the following online reference:

`http://math.ucr.edu/~res/math205B/polishcircle.pdf`

The key to studying the Čech homology of arbitrary compact subsets in $\mathbb{R}^n$ is a fundamental *continuity property* which does not hold in singular homology.

Continuity in Čech homology

The results in Chapter IX of Eilenberg and Steenrod show that Čech homology is functorial with respect to continuous maps of compact Hausdorff spaces. Given this, we can the basic result very simply.

THEOREM 11. (Continuity Property) *Suppose that X is a subspace of some Hausdorff topological space E, and suppose further that there are compact subsets $X_\alpha \subset E$ such that $X = \cap_\alpha X_\alpha$ for all α and the family X_α is closed under taking finite intersections. Then we have*

$$\check{H}_*(X) \ \cong \ \lim_{\leftarrow} \ \check{H}_*(X_\alpha) \ .$$

If $E = \mathbb{R}^n$ for some n, then it is always possible to find such a family of compact subsets X_n such that $X_{n+1} \subset X_n$ for all n and X_n is a finite union of hypercubes of the form

$$\prod_{i=1}^{n} \left(x_i,\, x_i + \frac{1}{2^n} \right)$$

where each x_i is a rational number expressible in the form $p_i/2^n$ for some integer p_i. For example, one can take X_n to be the union of all such cubes which have a nonempty intersection with X.

Reference for the proof of Theorem 11. A proof is given on page 261 of Eilenberg and Steenrod (specifically, see theorem X.3.1).■

Remark. One can also make the singular-Čech comparison map into a natural transformation of covariant functors, but we shall not do this here because it is not needed for our purposes except for a remark following the proof of Theorem 15 (as before, details may be found in Chapters IX and X of Eilenberg and Steenrod).

Singular and Čech homology of the Polish circle

As in the previously cited document

`http://math.ucr.edu/∼res/math205B/polishcircle.pdf`

the Polish circle P is defined to be the union of the following curves:

(1) The graph of $y = \sin(1/x)$ over the interval $0 \leq x \leq 1$.

(2) The vertical line segment $\{1\} \times [-2, 1]$.

(3) The horizontal line segment $[0, 1] \times \{-2\}$.

(4) The vertical line segment $\{0\} \times [-2, 1]$.

One important fact about the Polish circle is that it is arcwise connected but not locally arcwise connected. The proof of this is analogous to the discussion on page 66 of the online notes

`http://math.ucr.edu/~res/math205A/gentopnotes2008.pdf`

which shows that the space B, which is given by closure (in $\mathbb{R}^2$) of the graph of $\sin(1/x)$ for $x > 0$, is connected but not arcwise connected. For the sake of completeness, we shall indicate how one modifies the argument to show the properties of P stated above. First of all, since P is the union of four arcwise connected subspaces $A \cup B \cup C \cup D$ such that $A \cap B$, $B \cap C$ and $C \cup D$ are all nonempty, the arcwise connectedness of P follows immediately. To prove that P is not arcwise connected, we need the following result, whose proof is similar to the previously cited argument which shows that B is not arcwise connected:

LEMMA 12. *Let Y be a compact, arcwise connected, locally arcwise connected topological space, let $f : Y \to P$ be continuous, and suppose that $a_0 \in Y$ is such that the first coordinate of $f(a_0)$ is zero and $f(a_0) \neq (0, -2)$. Then there is an arcwise connected open neighborhood V of a_0 in Y such that $f[V]$ is contained in the intersection of Y with the $y-$axis.*$\blacksquare$

This observation has far-reaching consequences for the fundamental group and singular homology of P, all of which come from the following:

PROPOSITION 13. *Let Y and f be as in the preceding lemma. Then there is some $\varepsilon > 0$ such that $f[Y]$ is disjoint from the open rectangular region $(0, \varepsilon) \times (-2, 2)$.*

In terms of the presentation of P given above, this means that $f[Y]$ is contained in the union of $B \cup C \cup D$ with the graph of $\sin(1/x)$ over the interval $[\varepsilon, 1]$. This subspace M_ε is homeomorphic to a closed interval and as such is contractible. Therefore Proposition 13 has the following application to the algebraic-topological invariants of the Polish circle:

THEOREM 14. *If P is the Polish circle, then $\pi_1(P, p)$ is trivial for all $p \in P$, and the inclusion of $\{p\}$ in P induces an isomorphism of singular homology groups.*

Proof of Theorem 14, assuming Proposition 13. We begin with the result on the fundamental group. Suppose that γ is a closed curve in P based at p. By Proposition 13 we know that the image of γ lies in M_ε for some $\varepsilon > 0$, so that the class of γ in $\pi_1(P, p)$ lies in the image of $\pi_1(M_\varepsilon, p)$. Since M_ε is contractible, it follows that the image of $\pi_1(M_\varepsilon, p)$ in $\pi_1(P, p)$ is trivial, and therefore the latter must also be trivial.

The proof for singular homology is similar. If $z \in S_q(P)$ is a cycle, then there is some M_ε such that $p \in M_\varepsilon$ and z lies in the image of $S_q(M_\varepsilon)$. Of course, this means that the class u represented by z lies in the image of the homomorphism $H_q(M_\varepsilon) \to H_q(P)$, and since M_ε is contractible it follows that this image is trivial if $q > 0$. On the other hand, if $q = 0$, then the arcwise connectedness of all the spaces implies that the various inclusion maps all induce isomorphisms in 0-dimensional singular homology.$\blacksquare$

Proof of Proposition 13. Let E denote the inverse image of the intersection of P with $\{0\} \times [-\frac{3}{2}, 1]$. Then for each $c \in E$ there is an arcwise connected open neighborhood V_c of c in Y such that $f[V_c]$ is contained in the intersection of Y with the $y-$axis. Let W_c be an open neighborhood of c whose closure is contained in V_c. By continuity E is closed in Y and hence E is a compact subset, so there is a finite subcollection of the sets W_c, say $\{W_1, \cdots, W_n\}$, which covers E.

Define $G \subset Y$ to be the closed subset

$$Y \ - \ \cup_{i=1}^n \ W_i$$

so that $f[G]$ is compact and disjoint from $P \cap \{0\} \times [-\frac{3}{2}, 1]$. If $A \subset P$ is the piece of the graph of $\sin(1x)$ described above, then it follows that the second coordinates of all points in $f[G] \cap A$ are positive and by compactness must be bounded away from zero; in other words, there is some $\varepsilon > 0$ such that $f[G] \cap A$ is disjoint from $(0, \varepsilon) \times \mathbb{R}$. But this means that

$$f[Y] \quad = \quad f[G] \ \cup \ \left(\cup_{i=1}^n \ f[W_i] \right)$$

must be disjoint from $(0, \varepsilon) \times (-2, 2)$.■

In contrast to the preceding, we have the following result:

THEOREM 15. *The Čech homology groups of the Polish circle P are given by $\check{H}_q(P) = \mathbb{Z}$ if $q = 0, 1$ and zero otherwise.*

The results on Čech homology groups in Eilenberg and Steenrod show that these groups are functorial for continuous mappings and that homotopic mappings induce the same algebraic homomorphisms in Čech homology. If we combine this with Theorem 15 and the results on singular homology, we see that *the Polish circle P is a space which is simply connected and has the singular homology of a point, but P is* `not` *a contractible space*. A self-contained proof of the preceding statement is given in `polishcircle.pdf`.

Proof. We shall prove this using the continuity property of Čech homology as stated above, and we shall use the presentation of P as an intersection of the decreasing closed subsets B_n in the previously cited `polishcircle.pdf`. Since $P \ = \ \cap_n \ B_n$ it follows that

$$\check{H}_* \quad \cong \quad \lim_{\leftarrow} \ \check{H}_*(B_n)$$

and since each B_n is homeomorphic to a finite simplicial complex (describe this explicitly — it is fairly straightforward), we can replace Čech homology with singular homology on the right hand side. It will suffice to prove that each B_n is homotopic to a circle and the inclusion mappings $B_{n+1} \subset B_n$ are all homotopy equivalences. We shall do this using the subspaces C_n from the `polishcircle` document.

By construction, C_n is a subset of B_n, and we claim that C_n is a deformation retract of B_n. Let X_n be the closed rectangular box

$$\left[\frac{2}{(4n+3)\pi} \right] \ \times \ [-1, 1]$$

(the piece shaded in blue in the third figure of `polishcircleA.pdf`), and let Q_n denote the bottom edge of X_n defined by the equation $y = -1$. It follows immediately that Q_n is a strong deformation retract of X_n; since the closure of $B_n - X_n$ intersects X_n in the two endpoints of Y_n, we can extend the retract $X_n \to Y_n$ and homotopy $X_n \times [0,1] \to X_n$ by taking the identity on $\overline{B_n - X_n}$ to extend the retraction and the trivial homotopy from the identity to itself on $\overline{B_n - X_n}$. This completes the proof that C_n is a strong deformation retract of B_n.

By construction the space C_n is homeomorphic to the standard unit circle, and furthermore it is straightforward to check that the composite

$$C_{n+1} \quad \subset \quad B_{n+1} \quad \subset \quad B_n \quad \longrightarrow \quad C_n$$

(where the last map is the previously described homotopy inverse) must be a homeomorphism which is the identity off the points which lie in the vertical strip

$$\left(\frac{2}{4n+7}, \frac{2}{4n+3} \right) \times \mathbb{R}$$

and on this strip it is the flattening map which sends a point $(x,y) \in C_{n+1}$ to $(x,-1) \in C_n$. Therefore the map in homology from $H_q(C_{n+1})$ to $H_q(C_n)$ is an isomorphism of infinite cyclic groups in dimensions 0 and 1 and of trivial groups otherwise, and it follows that the map from $H_q(B_{n+1})$ to $H_q(B_n)$ is also an isomorphism of of infinite cyclic groups in dimensions 0 and 1 and of trivial groups otherwise. As in the proof of the second half of Theorem 6, it follows that $\check{H}_q(X)$ must be infinite cyclic if $q = 0$ or 1 and trivial otherwise.∎

In fact, as noted before the proof of Theorem 15 one can show that a standard map from P to S^1 induces isomorphisms in Čech homology. This requires the naturality property of the comparison map from singular to Čech homology.

Dimensions of nowhere dense subsets

We have seen that if A is a compact subset of $\mathbb{R}^n$ with a nonempty interior, then the Lebesgue covering dimension of A is equal to n; we shall conclude this section with a converse to this result. In order to prove the converse we shall need some refinements of the ideas which arise in the proof of the embedding theorem stated as Theorem 50.5 in Munkres (see pages 311–313).

Definition. Let $(X, \mathbf{d})$ be a metric space, let $f : X \to Y$ be a continuous map of topological spaces, and let $\varepsilon > 0$. We shall say that f is an *ε−map* if for all $u, v \in X$ the equation $f(u) = f(v)$ implies that $\mathbf{d}(u,v) \le \varepsilon$; an equivalent formulation is that for all $y \in Y$ the diameter of the level set $f^{-1}[\{y\}]$ is less than or equal to ε.

Clearly a continuous map f is 1–1 if and only if it is an ε−map for all $\varepsilon > 0$ (equivalently, it suffices to have this condition for all numbers of the form $1/k$ where k is a positive integer or all numbers of the form 2^{-k} where k is a positive integer).

We shall need the following result, which is entirely point set-theoretic.

LEMMA 17. *Let $(X, \mathbf{d}_X)$ and $(Y, \mathbf{d}_Y)$ be compact metric spaces, let $\varepsilon > \varepsilon' > 0$, and let $f : X \to Y$ be a continuous ε'−map. Then there is a $\delta > 0$ such that if $A \subset Y$ has diameter less than or equal to δ, then $f^{-1}[A]$ has diameter less than ε.*

Proof. Let $\eta = \frac{1}{2}(\varepsilon + \varepsilon')$ and let $K_\eta \subset X \times X$ be the set of all (x_1, x_2) such that $\mathbf{d}_X(x_1, x_2) \geq \eta$. Then K_η is a closed (hence compact) subset of $X \times X$ and $f \times f[K_\eta]$ is a compact subset of $Y \times Y$ which is disjoint from the diagonal Δ_Y because f is an ε'−map. It follows that the restriction of the distance function $\mathbf{d}_Y$ to $f \times f[K_\eta]$ is bounded away from zero by a positive constant h; in other words, if $U_h \subset Y \times Y$ is the set of all $(y_1, y_2) \in Y \times Y$ such that $\mathbf{d}_Y(y_1, y_2) \leq h/2$, then $(y_1, y_2) \notin f \times f[K_\eta]$.

Suppose now that the diameter of A is less than $\delta = h/2$; then we have $A \times A \subset U_h$, and it follows that if $(p, q) \in f^{-1}[A]$, then $\mathbf{d}_Y\big(\, f(p),\ f(q)\,\big) < \delta$, and this means that (p, q) cannot lie in K_η because the image of the latter under $f \times f$ is disjoint from U_h, which contains $A \times A$. In other words, if the diameter of A is less than δ, then the diameter of $f^{-1}[A]$ must be less than or equal to η, which is less than ε.∎

The next result gives a method for approximating n-dimensional compact metric spaces by n-dimensional simplicial complexes.

PROPOSITION 18. *Let X be a compact metric space, and let n be a nonnegative integer. Then the Lebesgue covering dimension of X is $\leq n$ if and only if for every $\varepsilon > 0$ there is an ε−map from X into some n-dimensional polyhedron P.*

Proof. Suppose first that the Lebesgue covering dimension of X is $\leq n$. Take the open covering of X by open disks of radius $\varepsilon/2$ about the points of X, and extract a finite subcovering

$$\mathcal{U} \quad = \quad \left\{ N_{\varepsilon/2}(x_1),\ \cdots\ ,\ N_{\varepsilon/2}(x_m) \right\} .$$

Let $\{ \varphi_j \}$ be a partition of unity subordinate to this finite covering, and consider the canonical map k from X to $\mathfrak{N}(\mathcal{U})$. If $k(u) = k(v)$, then $\varphi_j(u) = \varphi_j(v)$ for all j; at least one of these values must be positive, and therefore we can find some j such that $u, v \in N_{\varepsilon/2}(x_j)$. Since the latter implies $\mathbf{d}(u, v) \leq \mathrm{diameter} N_{\varepsilon/2}(x_j) \leq \varepsilon$, it follows that k is an ε−map.

As usual, with respect to this metric there is a Lebesgue number $\eta > 0$ for this open covering. Let $0 < \varepsilon' < \varepsilon < \eta$, and let $f : X \to P$ be an ε'−map from X to some polyhedron P of dimension $\leq n$. By the preceding lemma there is some $\delta > 0$ such that if $A \subset Y$ has diameter less than δ then $f^{-1}[A]$ has diameter less than ε.

Take a sufficiently large barycentric subdivision of P such that all simplices have diameter at most $\delta/2$, and let $\mathcal{V}$ be the open covering given by the inverse images (under f) of open stars of the vertices in P. Then the intersection of any $n+2$ open subsets in $\mathcal{V}$ is empty; if we can show that $\mathcal{V}$ is a refinement of $\mathcal{U}$, then we are done. But the open stars of vertices in P all have diameter at most δ, and thus by Lemma 17 their inverse images have diameters which are at most ε. Since ε is less than a Lebesgue number for $\mathcal{U}$, it follows that each of the open subsets in $\mathcal{V}$ must be contained in some open set from $\mathcal{U}$, and thus $\mathcal{V}$ is an open refinement of $\mathcal{U}$ such that every subcollection $n+2$ subsets in $\mathcal{V}$ has an empty intersection.∎

Before proceeding, we shall show that Proposition 18 yields the previously stated result about the dimensions of Cartesian products (namely, $\dim(X \times Y) \leq \dim X + \dim Y$). In this argument we assume that $\dim X$ and $\dim Y$ are both finite; it is straightforward to verify that if X and Y are $\mathbf{T_1}$ spaces and either $\dim X = \infty$ or $\dim Y = \infty$, then $\dim(X \times Y) = \infty$ (look at the contrapositive statement).

Proof of Proposition 9. Suppose that $\dim X \leq m$ and $\dim Y \leq n$, and let $\varepsilon > 0$. By Proposition 18, it will suffice to construct an ε-map from $X \times Y$ to some polyhedron T of dimension at most $m + n$. For the sake of definiteness, in this argument the metrics on products are given by the $\mathbf{d}_2$ metrics associated to metrics on the factors (using the notation of the 205A notes).

The construction is fairly straightforward. By the dimension hypotheses and Proposition 18 we know there are $(\varepsilon/\sqrt{2})$-maps $f : X \to P$ and $g : Y \to Q$, where P and Q are polyhedra of dimension at most m and n respectively. It follows that the product map $f \times g : X \times Y \to P \times Q$ is an ε-map into a polyhedron whose dimension is at most $m + n$.■

Using Proposition 18, we can prove the result on the dimensions of nowhere dense subsets mentioned above.

THEOREM 19. *Suppose that $A \subset \mathbb{R}^n$ is compact and nowhere dense. Then the Lebesgue covering dimension of A is at most $n - 1$.*

The estimate in the theorem is the best possible estimate because we know that the Lebesgue covering dimension of the nowhere dense subset S^{n-1} is equal to $n - 1$.

Proof. We shall prove that A satisfies the criterion in Proposition 18. One step in the proof involves the following result:

CLAIM. *If $\mathbf{v}$ is an interior point of the disk D^n where $n > 0$, then S^{n-1} is a retract of $D^n - \{\mathbf{v}\}$.*

The quickest way to prove this is to take the map $\rho : D^n \times D^n - \text{diagonal} \to S^{n-1}$ constructed in the file `brouwer.pdf` and restrict it to $(D^n - \{\mathbf{v}\}) \times \{\mathbf{v}\}$.

The first steps in the proof are to let $\varepsilon > 0$ and to take a large hypercube Q containing A. We know that Q has a simplicial decomposition, and if we take repeated barycentric subdivisions we can construct a decomposition whose simplices all have diameter less than $\varepsilon/2$. Let σ be an n-simplex in this decomposition. Since σ has a nonempty interior (in the sense of point set topology) and A is nowhere dense in $\mathbb{R}^n$, it follows that there is some interior point $w(\sigma)$ in σ such that $w(\sigma) \notin A$. By the claim above, we know that the boundary $\partial\sigma$ is a retract of $\sigma - w(\sigma)$, and we can piece the associated retractions together to obtain a retraction

$$r : Q \ - \left(\bigcup_{\dim \sigma = n} \{w(\sigma)\} \right) \ \longrightarrow \ Q^{[n-1]}$$

where $Q^{[n-1]}$ (the *n-skeleton*) is the union of all simplices in Q with dimension strictly less than n. By construction the set A is contained in the domain of r, and therefore we also obtain a retraction $r : A \to Q^{[n-1]}$. The inverse image of a point z in the codomain is contained in all simplices which contain z, and since these simplices all have diameter less than $\varepsilon/2$, it follows that each set $r^{-1}[\{z\}]$ has diameter less than ε. Therefore we have shown that $r|A$ is an ε−map into the $(n-1)$-dimensional polyhedron $Q^{[n-1]}$. By Proposition 18, it follows that the Lebesgue covering dimension of A is at most $n-1$.■

Using results from Section 50 of Munkres (including the exercises), it is a straightforward exercise to prove the following generalization of Theorem 19:

COROLLARY 20. *Let M^n be a second countable topological n-manifold, and suppose that $A \subset M$ is a closed nowhere dense subset of M^n. Then the Lebesgue covering dimension of A is strictly less than n.*■

Appendix : The Flag Property

Default hypothesis. Unless stated otherwise, all spaces discussed in this Appendix are compact metric spaces with finite Lebesgue covering dimensions.

In our discussion of product formulas for the Lebesgue covering dimension, we noted that $\dim X \times Y > \dim X$ if $\dim Y > 0$, and we gave references for the proof when $\dim Y = 1$. We also asserted that the general case followed quickly from this special case because $\dim Y > 0$ implies the existence of a closed subset $A \subset Y$ with $\dim A = 1$. In fact, we have the following:

PROPOSITION A1. (Flag Property) *Suppose that X satisfies the Default Hypothesis and $\dim X = n > 0$. Then there is a chain of closed subsets*

$$\{y\} \quad = \quad A_0 \quad \subset \quad A_1 \quad \subset \quad \cdots \quad \subset A_n \quad = \quad X$$

such that $\dim A_k = k$ for all k.

Note. The name for this result is motivatived by a standard geometrical concept of a *flag* of subspaces in $\mathbb{R}^n$, which is a sequence of vector subspaces

$$\{\mathbf{0}\} \quad = \quad V_0 \quad \subset \quad V_1 \quad \subset \cdots \subset \quad V_n \quad = \quad \mathbb{R}^n$$

such that $\dim V_k = k$ for all k; of course, there is a similar concept if $\mathbb{R}$ is replaced by an arbitrary field.

The proof of the Flag Property is a fairly direct consequence of equality of the Lebesgue covering dimension and the previously cited Menger-Urysohn or weak inductive dimension for compact metric spaces. Here is a summary of what we need in order to prove the Flag Property:

THEOREM A2. *Let X be a compact metric space such that $\dim X \leq n$, and let $x \in X$. Then there is a countable neighborhood base at x of the form*

$$\mathfrak{B} \quad = \quad \{\, W_1 \quad \supset \quad W_2 \quad \cdots \qquad \}$$

such that for each k the set $\mathsf{Bdy}_x(W_k)$ has dimension at most $n-1$. Conversely, if such neighborhood bases exist for each point of X, then $\dim X \leq n$.

As in Munkres, the boundary (or frontier) $\mathsf{Bdy}_X(E)$ of $E \subset X$ (in X) is the intersection of the limit point sets $\mathbf{L}_X(E) \cap \mathbf{L}_X(X - E)$; since we are working with metric spaces, this is a closed subset of X.

Idea of proof for Theorem A2. The statement in the conclusion is essentially the same as the condition for the Menger-Urysohn dimension of X to be at most n (this is given on page 24 of Hurewicz and Wallman). Therefore the conclusion will follow if we know that the Lebesgue covering dimension and the Menger-Urysohn dimension are equal for compact metric spaces. Virtually every book on dimension theory from the past 50 years contains some abstract version of this equality. More directly, one can use Theorem V.8 on page 67 of Hurewicz and Wallman (in which "dimension" means the Menger-Urysohn dimension) to show that the two definitions are the same for compact metric spaces.

One reason that the standard references for dimension theory phrase things in more abstract terms is that the Lebesgue covering dimension and Menger-Urysohn dimension are not necessarily equal for more general topological spaces (usually it is easy to find examples; see also the Wikipedia article on inductive dimension mentioned earlier).

Proof of Proposition A1. (Compare Hurewicz and Wallman, Proposition III.1.D, pp. 24–25.) If $\dim X = 1$ then X is nonempty and the conclusion follows immediately. Proceeding by induction on the dimension, we shall assume the result is true for compact metric spaces of dimension $\leq n-1$. Suppose that X is an n-dimensional compact metric space. Since $\dim X$ is not less than or equal to $n-1$, Theorem A2 implies the existence of some point $z \in X$ such that for all countable neighborhood bases at z of the form

$$\mathfrak{A} \quad = \quad \{\, V_1 \quad \supset \quad V_2 \quad \cdots \qquad \}$$

we have $\dim(\mathsf{Bdy}_X(V_k)) > n-2$ for infinitely many k (why?). In particular, this holds for the neighborhood base $\mathfrak{B}$ for z described in the statement of Theorem A2 (we know such a neighborhood base exists because $\dim X = n$). It follows that $\dim(\mathsf{Bdy}_X(W_k)) = n-1$ for all such k. Choose a specific m such that $\dim(\mathsf{Bdy}_X(W_m)) = n-1$. By the induction hypothesis, there is a chain of closed subspaces

$$\{y\} \quad = \quad A_0 \quad \subset \quad A_1 \quad \subset \quad \cdots \quad \subset \quad A_{n-1} \quad = \quad \mathsf{Bdy}_X(W_m)$$

and we may extend this to a chain of subspaces as in the conclusion of the proposition by taking $A_n = X$.■

III.6 : Homology and line integrals

(Lee, *Introduction to Smooth Manifolds*, Chs. 6, 12, 14)

See Section VIII.6 in the course directory file `fundgp-notes.pdf`, the exercises for this section in `fundgpexercises2014.pdf`, and the file `disks-with-holes.pdf`. Among other things, these documents discuss some basic questions involving analytic functions of one complex variable. Far-reaching generalizations of the results in these documents are discussed at the end of Unit V.