The Lefschetz Fixed Point Theorem

(P, K) polyhedron

$f : P \to P$ cont.

The <u>Lefschetz number</u> $\Lambda(f) =$

$$\sum (-1)^k \operatorname{trace} f_{k*} : H_k(P;\mathbb{Q}) \to H_k(P;\mathbb{Q}).$$

<u>Claim</u> This is an integer.

<u>Idea</u> Look at the diagram

$$\begin{tikzcd}
H_k(P;\mathbb{Z})/\text{torsion} \arrow[r, hook, "\subseteq"] \arrow[d, "f^{\mathbb{Z}}_{k*}"] & H_k(P;\mathbb{Q}) \arrow[d, "f^{\mathbb{Q}}_{k*}"] \\
H_k(P;\mathbb{Z}/\text{torsion}) \arrow[r, hook, "\subseteq"] & H_k(P;\mathbb{Q})
\end{tikzcd}$$

and choose free generators for $H_k(P;\mathbb{Z})/\text{torsion}$ which yield a basis for $H_k(P;\mathbb{Q})$. This shows that a matrix for $f^{\mathbb{Q}}_*$ comes from an integral matrix for $f_*^{\mathbb{Z}}$.

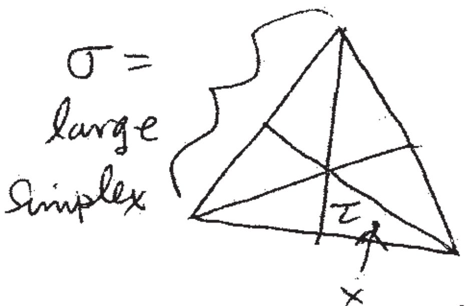
Hence each trace $f_{k*}^{\mathbb{Q}}$ is an integer.

THEOREM $\Lambda(f) \neq 0 \Rightarrow f$ has a fixed point.

PROOF. Suppose not, and let $\delta =$ min distance from $f(x)$ to x, so that $\delta > 0$ by compactness. Subdivide K into simplices of diameter $< \delta/4$. Then $f[\sigma] \cap \sigma$ is empty for all σ, and more generally $f[\mathrm{Star}\,\sigma] \cap \mathrm{Star}\,\sigma = \phi$ for all σ. Choose a simplicial approximation $g : B^r K \to K$ to f as in the Simplicial Approximation Thm.

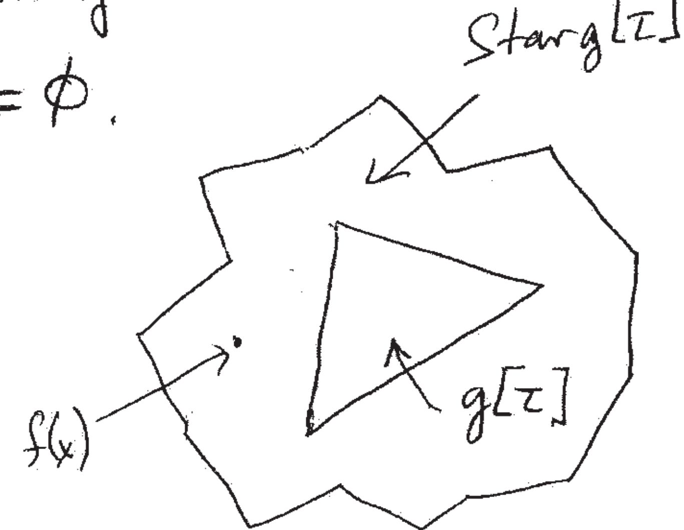
Then if τ is a simplex of $B^r(K)$ we have $f[\tau] \subseteq \mathrm{Star}\, g[\tau]$.

<u>**CLAIM**</u> $g[\tau] \cap \tau = \phi$.



τ in $B^r(K)$

$\tau \subseteq \sigma$ in K.



$g[\tau]$ in K.

$$x \in \tau \Rightarrow f(x) \in \mathrm{Star}\, g[\tau]$$

$$\text{and } d(x, f(x)) \geq \delta$$

$$y \in \tau \Rightarrow d(x,y) < \delta/4$$

$$z \in g[\tau] \Rightarrow f(x), z \in \mathrm{Star}\, g(\tau)$$

$$\Rightarrow d(f(x), z) < \delta/2.$$

It follows that $y \in \tau$, $z \in g[\tau] \Rightarrow$
$d(y,z) > \delta/4$.

In fact, we can say more: Given

$\tau \subseteq \sigma$ we have $\sigma \cap g(\tau) = \phi$.
in $B^r(K)$ in K

Consider what this means for the chain map $C_*(K) \xrightarrow{\beta_r} C_*(B^r(K)) \xrightarrow{g_\#} C_*(K)$:

Let $\sigma = v_0 \dots v_q \in C_q(K)$ be a typical generator, so that $\beta_r(v_0 \dots v_q)$ lies in the chain subcomplex $C_*(B^r(v_0 \dots v_q))$, and consider the effect of $g_\#$ on a typical free generator of $C_q(B^r(v_0 \dots v_q))$.

If σ is the simplex with vertices $v_0 \dots v_q$, then $g_\#$ must take a typical free generator of $C_q(B^r(v_0 \dots v_q))$ into a chain subcomplex $C_*(\sigma') \subseteq C_*(K)$ such that $\sigma' \cap \sigma = \phi$. Therefore the image of $v_0 \ldots v_q$ in $C_q(K)$ actually lies in some $C_q(K')$ where σ and K' are disjoint, and hence $g_\# \beta_r(v_0 \ldots v_q)$ will lie in a subgroup of $C_q(K)$ consisting of chains whose $v_0 \ldots v_q$-coordinates are zero. This means that the trace of $g_\# \beta_r : C_q(K) \longrightarrow C_q(K)$ is zero, and by the trace identity the same is true for $H_q(K;\mathbb{Q}) \xrightarrow{\beta_{r*}} H_q(B^r(K);\mathbb{Q}) \xrightarrow{g_*} H_q(K;\mathbb{Q})$. We know the latter corresponds to the singular homology map

$$g_* : H_q(P;\mathbb{Q}) \longrightarrow H_q(P;\mathbb{Q})$$

and hence its trace is also zero. Taking alternating sums, we see that $\Lambda(g)=0$. Finally, $f \simeq g \Rightarrow \Lambda(f)=\Lambda(g)$, and hence we also have $\Lambda(f)=0$.

To summarize, we have shown that if f has no fixed points, then $\Lambda(f)=0$.