IV. Singular Cohomology

Suppose that $\mathbb{F}$ is a field and (X, A) is a pair of topological spaces. One can then define the q-dimensional cohomology $H^q(X, A; \mathbb{F})$ to be the vector space dual $\mathrm{Hom}_{\mathbb{F}}\big(H_q(X, A; \mathbb{F}),\ \mathbb{F}\,\big)$, and this construction extends to a contravariant functor on the category of pairs of spaces and continuous maps. Similarly, by taking adjoint maps of dual spaces we obtain natural coboundary morphisms $\delta : H^q(A; \mathbb{F}) \to H^{q+1}(X, A; \mathbb{F})$ and long exact cohomology sequences for pairs.

One natural question is why one would bother to do this, especially since it follows that $H_q(X; \mathbb{F}) \cong H^q(X; \mathbb{F})$ if X has the homotopy type of a finite cell complex (because the homology is finite dimensional and is isomorphic to its vector space dual). There are two related answers:

(1) Even when mathematical objects and their duals are equivalent, in many cases it is more convenient to work with the dual object rather than the original one, and vice versa. — For example, vector fields and differential 1-forms on a smooth manifold are dual to each other, but they play markedly different roles in the theory of smooth manifolds. In particular, vector fields are better for working with differential equations, while differential forms provide a more convenient way for manipulating expressions like line integrals.

(2) Frequently the dual objects have some extremely useful extra structure which is not easily studied in the original objects. — To continue with our example of vector fields and differential 1-forms, the latter have better functoriality properties, and the exterior derivative construction on differential forms does not have a functorial counterpart for vector fields unless one adds some further structure like a riemannian metric. On the other hand, there can also be some nice structure on the original objects which is not on their duals; for example, the Lie bracket construction on vector fields has no obvious counterpart on 1-forms unless one adds some further structure.

In fact, it turns out that cohomology groups have a useful additional structure; namely, there are natural bilinear **cup product** mappings

$$\cup\ :\ H^p(X, A; \mathbb{F}) \times H^q(X, A; \mathbb{F}) \ \longrightarrow\ H^{p+q}(X, A; \mathbb{F})$$

which do not have comparably simple counterparts in homology. This illustrates the second point about objects and their duals. Later in these notes we shall illustrate how the first point manifests itself in homology and cohomology.

A useful result

At several points in this unit we shall need the following result on **acyclic** (no homology) chain complexes.

THEOREM 0. *Let C_* be a chain complex such that $C_k = 0$ for $k < M$ for some integer M and each C_k is free abelian on some set of generators G_k. Then $H_*(C) = 0$ in all dimensions if and only if there is a contracting chain homotopy $D_q : C_q \to C_{q+1}$ (for all q) such that $Dd + dD = \mathbf{id}_C$.*

Proof. If D exists then clearly the homology is zero by the usual sort of argument. Conversely, suppose that $H_*(C) = 0$, and let m be the first degree in which C is nonzero. We may construct D_m as follows: If $T \in G_m$, then $dT = 0$ and hence $T = du$ for some $u \in C_{m+1}$. Define $D_m(T) = u$ and extend the map using the freeness property. Now suppose by induction that we have defined D_k for $k \leq N - 1$.

Let T be an element of the free generating set G_N. Then we need to find an element $u_T \in C_{n+1}$ so that $du_T = T - DdT$. Since the complex has no homology, such a class exists if and only if the right hand side is a boundary. But now we have a familiar sort of inductive calculation:

$$d(T - DdT) \;=\; dT - dDdT \;=\; dT - (1 - Dd)dT \;=\; dT - dT - DddT \;=\; -DddT \;=\; D0T \;=\; 0$$

Hence we can define $D(T) = u_T$ and extend by freeness. This completes the inductive step and proves the existence of the contracting chain homotopy.■

REMARK ON THE EXPOSITION. Many of the arguments and constructions in the remaining units of these notes are variations of ideas that were introduced earlier. Partly for this reason, the proofs will often be less detailed than in previous units, with the details left to the reader (also see the earlier quotation from Davis and Kirk on pages 18–19 of these notes and the following quotation from page *vii* of Spanier: The reader is expected to develop facility for the subject as he [or she] progresses, and accordingly, the further he [or she] is in the book, the more he [or she] is called upon to fill in details of proofs).

IV.1 : The basic definitions

(Hatcher, §§ 3.1–3.2)

We begin by defining the singular cohomology of a space with coefficients in an arbitrary $\mathbb{D}$-module, where $\mathbb{D}$ is a commutative ring with unit (a setting broad enough to contain coefficients in fields, the integers, and quotients of the latter). However, we shall quickly specialize to the case of fields in order to minimize the amount of algebraic machinery that is needed.

Definition. Let (X, A) be a pair of topological spaces, and let π be a module over the ring $\mathbb{D}$ as above. The **singular cochain complex** $\big(S^*(X, A; \pi), \delta\big)$ of (X, A) **with coefficients in** π is defined with $S^q(X, A) = \mathrm{Hom}\big(S_q(X, A), \pi\big)$ and the *coboundary mapping*

$$\delta^{q-1} : S^{q-1}(X, A; \pi) \;\longrightarrow\; S^q(X, A; \pi)$$

given by the adjoint map $\mathrm{Hom}(d_q, \pi)$.

Many basic properties of singular cochain complexes follow immediately from the definitions, including the following:

PROPOSITION 1. *(i) We have* $\delta^q \circ \delta^{q-1} = 0$.

(ii) The singular cochain complex is contravariantly functorial with respect to continuous mappings on pairs of topological spaces.

The first of these follows because $d_q \circ d_{q+1} = 0$ and the functor $\mathrm{Hom}(-,-)$ is additive, while the second is basically just a consequence of the definition and the covariant functoriality of the singular chain complex.■

Before going further, we shall define the q-dimensional **singular cohomology** $H^q(X, A; \pi)$ of (X, A) **with coefficients in** π to be the kernel of δ^q modulo the image of δ^{q-1}. Elements of the kernel are usually called **cocycles**, and elements of the image are usually called **coboundaries**. As in the case of singular chain complexes, it follows that the map of singular cochains

$$f^{\#} : S^*(Y, B; \pi) \longrightarrow S^*(X, A; \pi)$$

associated to a continuous map $f : (X, A) \to (Y, B)$ will pass to a homomorphism

$$f^* : H^*(Y, B; \pi) \longrightarrow H^*(X, A; \pi)$$

and this makes singular cohomology into a contravariant functor on pairs of spaces and continuous maps.

If (X, A) is a pair of topological spaces, then for each q we know that $S_q(X) \cong S_q(A) \oplus S_q(X, A)$ **as free abelian groups** (but this is **NOT** an isomorphism of chain complexes!), and from this it follows that for each q we have a split short exact sequence of modules

$$0 \longrightarrow S^*(X, A; \pi) \xrightarrow{\ j^{\#}\ } S^*(X; \pi) \xrightarrow{\ i^{\#}\ } S^*(A; \pi) \longrightarrow 0$$

where $j : X \to (X, A)$ and $i : A \to X$ are the usual inclusions. As in the case of singular chains, this leads to a natural **long exact cohomology sequence**; to simplify the notation we shall omit the coefficient module π in the display below:

$$\cdots \quad H^{k-1}(A) \xrightarrow{\delta} H^k(X, A) \xrightarrow{j^*} H^k(X) \xrightarrow{i^*} H^k(A) \xrightarrow{\delta} H^{k+1}(X, A) \quad \cdots$$

As in the case of homology, this sequence extends indefinitely to the left and right.

Notational convention. The contravariant algebraic maps induced by inclusions are often called **restriction** maps; one motivation for this terminology is that a map like $i^{\#}$ restricts attention from objects defined for X to objects defined only for the subspace A (for example, consider the restriction map from continuous real valued functions on X to those defined on A, which is defined by composing a function $f : X \to \mathbb{R}$ with the inclusion mapping i).

We can now proceed as in the study of singular homology to prove homotopy invariance, excision, and Mayer-Vietoris theorems for singular cohomology; informally speaking, one need only apply the functor $\mathrm{Hom}(-,\pi)$ to everything in sight, including chain homotopies. At some points one needs Theorem 0 to conclude that if C_* is an acyclic, free abelian chain complex, then it has a contracting chain homotopy and the latter implies that $\mathrm{Hom}(C_*,\pi)$ has no nonzero cohomology (verify this!).

Cup products

We shall now assume that our coefficients π are a commutative ring with unit, which we shall call $\mathbb{D}$.

Definition. Let X be a space; then the *augmentation mapping* $\varepsilon_X(\mathbb{D}) = \varepsilon_X \in S^0(X;\mathbb{D})$ is the homomorphism from $S_0(X)$ to $\mathbb{D}$ which sends each singular 0-simplex $T : \Delta_0 \to X$ to the unit element of $\mathbb{D}$.

The following is an immediate consequence of the definitions.

PROPOSITION 2. *If $f : X \to Y$ is continuous, then $f^{\#}(\varepsilon_Y) = \varepsilon_X$. Furthermore, $\delta^0(\varepsilon_X) = 0$.*■

The augmentation plays a key role in the multiplicative structure mentioned earlier. Before proceeding, we need some geometric definitions.

Definition. Let p and q be nonnegative integers, and as usual let Δ_{p+q} denote the standard simplex. Then the *front and back faces* $\mathbf{Front}_p(\Delta_{p+q})$ and $\mathbf{Back}_q(\Delta_{p+q})$ are the p- and q-dimensional faces whose vertices are respectively the first $(p+1)$ and last $(q+1)$ vertices of the original simplex. Note that these intersect in the p^{th} vertex of Δ_{p+q}.

Definition. Given two cochains $f \in S^p(X,A;\mathbb{D})$ and $g \in S^q(X,A;\mathbb{D})$, their **cup product** $f \cup g \in S^{p+q}(X,A;\mathbb{D})$ is given as follows: For each standard free generator of $S_{p+q}(X,A)$ — in other words, each singular simplex of X whose image is not entirely contained in A — we define

$$f \cup g(T) \quad = \quad (f|\mathbf{Front}_p) \cdot (g|\mathbf{Back}_q) \ .$$

We then have the following:

PROPOSITION 3. *The cup product is functorial for continuous maps of pairs. Furthermore, it is bilinear and associative, and if $A = \emptyset$ then ε_X is a two sided multiplicative identity.*$^{(\star)}$■

At this point we do not want to address questions about the possible commutativity properties of the cup product. This is a decidedly nonelementary issue, and in several respects it is a fundamental difficulty which has an enormous impact across most if not all of algebraic topology.

Clearly one would hope the cup product will pass to cohomology, and the following result guarantees this: