PROPOSITION 4. *In the notation of the cup product definition, we have*

$$\delta(f \cup g) \;\; = \;\; (\delta f) \cup g \; + \; (-1)^p \, f \cup (\delta g) \; .$$

In particular, it follows that $f \cup g$ *is a cocycle if both* f *and* g *are, and if we are given equivalent representatives* f' *and* g' *for the same cohomology classes, then* $f \cup g - f' \cup g'$ *is a coboundary.*

Proof. The identity involving the coboundary of $f \cup g$ is derived in Lemma 3.6 on page 206 of Hatcher$^{(\star)}$. If f and g are both coboundaries, the formula immediately implies that $f \cup g$ is also a coboundary. Suppose now that we also have $f - f' = \delta v$ and $g - g' = \delta w$. It then follows that

$$\delta(v \cup g) \;\; = \;\; (f - f') \cup g \; , \qquad \delta(f' \cup w) \;\; = \;\; \pm \, f' \cup (g - g') \; .$$

The first of these implies that $f \cup g$ and $f' \cup g$ determine the same cohomology class, while the second implies that $f' \cup g$ and $f' \cup g'$ also determine the same cohomology class.■