page 92, lines 21-22

Proof that $\delta(f\cup g) = (\delta f)\cup g + (-1)^p f\cup(\delta g)$
where $f: S_p(X) \to \mathbb{D}$, $g: S_q(X) \to \mathbb{D}$.

Idea Prove they have the same value at a free generator $T: \Delta_{p+q+1} \to X$ of $S_{p+q+1}(X)$.

$\boxed{\text{Given } i_0 < \dots < i_r \text{, let } v_{i_0} \dots v_{i_r} = \text{restriction of } T \text{ to simplex w/ vertices } e_{i_j}}$

LHS $\delta(f\cup g)(T) =$

$$\sum_i (-1)^i f\cup g(v_0 \dots \widehat{v_i} \dots v_{p+q+1}) =$$

$$\sum_{i\le p} (-1)^i f(v_0 \dots \widehat{v_i} \dots v_{p+1})\, g(v_{p+1} \dots v_{p+q+1}) +$$

$$\sum_{i\ge p+1} (-1)^i f(v_0 \dots v_p)\, g(v_p \dots \widehat{v_i} \dots v_{p+q+1})$$

RHS Do pieces separately:

$(\delta f)\cup g(T) =$

$$\sum_{i=0}^{p+1} (-1)^i f(v_0 \dots \widehat{v_i} \dots v_{p+1})\, g(v_{p+1} \dots v_{p+q+1})$$

<u>page 92, lines 21-22 continued</u>

$$(-1)^p f \cup \delta g(T) =$$

$$(-1)^p \sum_{i=p}^{p+q+1} (-1)^{i-p} f(v_0 \ldots v_p)\, g(v_p \ldots \hat{v_i} \ldots v_{p+q+1})$$

If we subtract $\delta(f \cup g)(T)$ from $\delta f \cup g(T) + (-1)^p f \cup \delta g(T)$ we are left with the $p+1$ term in the sum for $\delta f \cup g(T)$ and the p term in the sum for $(-1)^p f \cup \delta g(T)$, which is

$$(-1)^{p+1} f(v_0 \ldots v_p)\, g(v_{p+1} \ldots v_{p+q+1}) +$$
$$(-1)^p f(v_0 \ldots v_p)\, g(v_{p+1} \ldots v_{p+q+1})$$

and since $(-1)^{p+1} + (-1)^p = 0$ these cancel each other. Hence the two expressions have the same value at T, which was what we wanted to prove.