In many contexts it is useful to have a slight refinement of the cup product described above. Specifically, if A and B are both subspaces of X which satisfy some regularity condition — for example, if both are open in X — then we shall define a relative cup product

$$H^p(X, A; \mathbb{D}) \times H^q(X, B; \mathbb{D}) \longrightarrow H^{p+q}(X, A \cup B; \mathbb{D})$$

which is a very slight modification of the definition given above.

Suppose first that A and B are open in X, and let $\mathcal{F}$ be the open covering of X given by $\{A, B\}$. Let $S_*^{\mathcal{F}}(A \cup B)$ be the subcomplex of $\mathcal{F}$-small singular chains, let $S_{\mathcal{F}}^*(A \cup B)$ be the associated cochain complex, and let $S_{\mathcal{F}}^*(X, A \cup B)$ be the kernel of the restriction map from $S^*(X)$ to $S_{\mathcal{F}}^*(A \cup B)$. Equivalently, $S_{\mathcal{F}}^*(X, A \cup B)$ is the cochain complex associated to the quotient

$$S_*(X)/S_*^{\mathcal{F}}(A \cup B) \quad = \quad S_*(X)/\left(S_*(A) + S_*(B) \right)$$

and since A and B are open in X, it follows that $S_{\mathcal{F}}^*(X, A \cup B)$ is a quotient of $S^*(X, A)$ such that the projection from $S^*(X, A \cup B) \to S_{\mathcal{F}}^*(X, A \cup B)$ induces isomorphisms in cohomology.

Suppose now that we are given cochains $f \in S^p(X, A)$ and $g \in S^q(X, B)$; by construction $S^p(X, A)$ and $S^q(X, B)$ are cochain subcomplexes of $S^*(X)$, and therefore the cup product construction defines a cochain $f \cup g : S_{p+q}(X) \to \mathbb{D}$. We need to show that this cochain actually lies inside $S_{\mathcal{F}}^*(A \cup B)$, or equivalently that the restriction of $f \cup g$ to $S_*^{\mathcal{F}}(A \cup B) = S_*(A) + S_*(B)$ is trivial. This will follow if we can show that the restrictions of $f \cup g$ to both $S_*(A)$ and $S_*(B)$ are zero, and thus it suffices to show that $f \cup g(T) = 0$ if T is a singular simplex in A or B.

Let T be a singular simplex in A or B; symmetry considerations show it suffices to consider the first case (reverse the roles of the variables to get the other case). Then $f \cup g(T) = f(T_1) \cdot g(T_2)$, where T_i is obtained by restricting T to a front or back face of Δ_{p+q}. If the restriction of f to $S_*(A)$ is zero, then it follows from the previous formula that $f \cup g(T) = 0$. Similarly, if the restriction of g to $S_*(B)$ is zero, then one obtains the same conclusion. Therefore $f \cup g$ actually lies in $S_{\mathcal{F}}^*(A \cup B)$; the previous arguments show that $f \cup g$ is a cocycle if f and g are cocycles and in this case the cohomology class of $f \cup g$ depends only on the cohomology classes of f and g. This gives us a map from $H^p(X, A) \times H^q(X, B)$ to the cohomology of $S_{\mathcal{F}}^*(X, A \cup B)$, and since the surjection from $S^*(X, A \cup B)$ to this group induces cohomology isomorphisms it follows that we obtain a class in $H^{p+q}(X, A \cup B; \mathbb{D})$. This refined cup product has analogs of all the properties one might expect to generalize from the case $A = B$; for example, it is associative.

As before, let π be an abelian group.

Given a simplicial complex $(P, \mathbf{K})$ and a subcomplex $(Q, \mathbf{L})$, one can define the (unordered) simplicial cochain complex $C^*(\mathbf{K}, \mathbf{L}; \pi)$ to be $\mathrm{Hom}(C_*(\mathbf{K}, \mathbf{L}); \pi)$. These objects are contravariantly functorial with respect to subcomplex inclusions, and as before one obtains long exact cohomology sequences for pairs. Furthermore, if we apply $\mathrm{Hom}(...; \pi)$ to the canonical natural maps $\lambda : C_*(\mathbf{K}, \mathbf{L}) \to S_*(P, Q)$, then we obtain canonical natural cochain complex maps

$$\psi : S^*(P, Q; \pi) \;\longrightarrow\; C^*(\mathbf{K}, \mathbf{L}; \pi)$$

and these in turn yield a commutative ladder diagram relating the long exact cohomology sequences for (P, Q) and $(\mathbf{K}, \mathbf{L})$. Previous experience suggests that the associated cohomology maps ψ^* should be isomorphisms, and we shall prove this below.

PROPOSITION 5. *The maps ψ^* define isomorphisms relating the long exact cohomology sequences for (P, Q) and $(\mathbf{K}, \mathbf{L})$.*

Proof. Consider the functorial chain maps λ as above; we known these maps define isomorphisms in homology. By construction λ maps a free generator $\mathbf{v}_0 \cdots \mathbf{v}_q$ of $C_q(\mathbf{K}, \mathbf{L})$ to an affine singular q-simplex T for (P, Q); therefore, if $\mathbf{V}_*(\mathbf{K}, \mathbf{L})$ is the quotient of $S_*(P, Q)$ by the image of λ, then it follows that the chain group $\mathbf{V}_q(\mathbf{K}, \mathbf{L})$ is free abelian on a subset of free generators for $S_q(P, Q)$, and by the long exact homology sequence for the short exact sequence

$$0 \;\to\; C_* \;\to\; S_* \;\to\; \mathbf{V}_* \;\to\; 0$$

it follows that all homology groups of $\mathbf{V}_*(\mathbf{K}, \mathbf{L})$ are zero. We can now use Proposition VI.0 to conclude that $\mathbf{V}_*(\mathbf{K}, \mathbf{L})$ has a contracting chain homotopy D_*, and we can use the associated maps $\mathrm{Hom}(D_*, \pi)$ to conclude that for each π all the cohomology groups of the cochain complex $\mathrm{Hom}(\mathbf{V}_*, \pi)$ are also zero. If we now apply this observation to the long exact cohomology sequence associated to

$$0 \;\to\; \mathrm{Hom}(\mathbf{V}_*, \pi) \;\to\; \mathrm{Hom}(S_*, \pi) \;\to\; \mathrm{Hom}(C_*, \pi) \;\to\; 0$$

we see that the map $\psi : \mathrm{Hom}(S_*, \pi) \to \mathrm{Hom}(C_*, \pi)$ must also induce isomorphisms in cohomology.■

Given a simplicial complex $(P, \mathbf{K})$ and an ordering of its vertices, one can similarly define an ordered cochain complex $C^*(P, \mathbf{K}^\omega)$ and canonical cochain complex maps

$$\alpha : C^*(P, \mathbf{K}) \;\longrightarrow\; C^*(P, \mathbf{K}^\omega)$$

and an analog of the preceding argument then yields the following result:

COROLLARY 6. *The associated maps in cohomology α^* are isomorphisms.*■

CUP PRODUCTS. If $\mathbb{D}$ is a commuative ring with unit, then one can define cup products on the cochain complexes $C^*(\mathbf{K}, \mathbb{D})$ using the same construction as in the singular case, and it is an elementary exercise to check that (a) this cup product has the previously described properties of the singular cup product, (b) the cochain map ψ preserves cup products at the cochain level (hence also in cohomology)[(⋆)].

Examples of cochains

Formally speaking, cochains are fairly arbitrary objects, so we shall describe some "toy models" which reflect typical and important contexts in which concrete examples arise (also see Exercise VI.2 in `advnotesexercises.pdf`). As usual, let $(P, \mathbf{K})$ be a polyhedron in $\mathbb{R}^n$, and let $f : P \to \mathbb{R}$ be a continuous function. We can then define a (simplicial) **line integral cochain** $\mathbf{L}_f \in C^1(\mathbf{K}; \mathbb{R})$ on free generators $\mathbf{v}_0\mathbf{v}_1$ by the formula

$$\mathbf{L}_f(\mathbf{v}_0\mathbf{v}_1) \quad = \quad \int_0^1 f\big(t\mathbf{v}_1 + (1-t)\mathbf{v}_0\big)\,|\mathbf{v}_1 - \mathbf{v}_0|\,dt \quad \in \quad \mathbb{R}\ .$$

By construction, this is just the scalar line integral of f along the directed straight line curve from $\mathbf{v}_0$ to $\mathbf{v}_1$.

Similarly, if $(P, \mathbf{K})$ is a polyhedron in $\mathbb{R}^3$ and $f : P \to \mathbb{R}$ is continuous, then we can define a surface integral cochain $\mathbf{S}_f \in C^2(\mathbf{K}; \mathbb{R})$ by the standard surface integral formula for scalar functions:

$$\mathbf{S}_f(\mathbf{v}_0\mathbf{v}_1\mathbf{v}_2) \quad = \quad \int_0^1 \int_0^{1-t} f(s\mathbf{v}_1 + t\mathbf{v}_2) \cdot |(\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_2 - \mathbf{v}_0)|\,ds\,dt$$

In this formula "$\times$" denotes the usual vector cross product. There are also versions of this construction in higher dimensions which yield cochains of higher dimension, but we shall not try to discuss them here.

Finally, given a field $\mathbb{F}$ we shall construct an explicit example of a cocycle in $C^1(\partial\Delta_2^\omega; \mathbb{F})$ which is not a coboundary.

By construction $C_1(\partial\Delta_2^\omega)$ is free abelian on free generators $\mathbf{e}_i\mathbf{e}_j$, where $0 \leq i < j \leq 2$. Thus a 1-dimensional cochain f is determined by its three values at $\mathbf{e}_0\mathbf{e}_1$, $\mathbf{e}_0\mathbf{e}_2$, and $\mathbf{e}_1\mathbf{e}_2$, each such cochain must be a cocycle because $C^2(\partial\Delta_2^\omega; \mathbb{F})$ is trivial (hence $\delta^1 = 0$). Also, a cochain f is a coboundary if and only if there is some 0-dimensional cochain g such that

$$f(\mathbf{e}_i\mathbf{e}_j) \quad = \quad g(\mathbf{e}_i) \ - \ g(\mathbf{e}_j)$$

for all i and j such taht $0 \leq i < j \leq 2$.

Now consider the cochain f with $f(\mathbf{e}_0\mathbf{e}_1) = f(\mathbf{e}_0\mathbf{e}_2) = f(\mathbf{e}_1\mathbf{e}_2) = 1$. We claim that f cannot be a coboundary. If it were, then as above we could find integers $x_i = g(\mathbf{v}_i)$ such that

$$x_1 - x_0 \quad = \quad x_2 - x_0 \quad = \quad x_2 - x_1 \quad = \quad 1\ .$$

This is a system of three linear equations in three unknowns, but it has no solutions. The nonexistence of solutions means that f cannot possibly be a coboundary. Similar considerations show that if k is an integer which is prime to the characteristic of $\mathbb{F}$ (in the characteristic zero case this means $k \neq 0$), then $k \cdot f$ is a cocycle which is not a coboundary.

By the previous results on cohomology isomorphisms, it follows that the singular cohomology $H^1(S^1; \mathbb{F})$ and simplicial cohomology $H^1(\partial\Delta_2; \mathbb{F})$ must also be nonzero.■

RELATIVE CUP PRODUCTS. If $(P, \mathbf{K})$ is a simplicial complex and we are given two subcomplexes $(Q_i, \mathbf{L}_i)$ for $i = 1, 2$, then one can define relative cup products on the simplicial cochain level

$$C^i(\mathbf{K}, \mathbf{L}_1; \mathbb{D}) \times C^j(\mathbf{K}, \mathbf{L}_2; \mathbb{D}) \longrightarrow C^{i+j}(\mathbf{K}, \mathbf{L}_1 \cup \mathbf{L}_2; \mathbb{D})$$

in much the same way that one defines such products of singular cochains, and once again these products pass to bilinear maps of cohomology groups

$$H^i(\mathbf{K}, \mathbf{L}_1; \mathbb{D}) \times H^j(\mathbf{K}, \mathbf{L}_2; \mathbb{D}) \longrightarrow H^{i+j}(\mathbf{K}, \mathbf{L}_1 \cup \mathbf{L}_2; \mathbb{D}) .$$

Specifically, if $\mathbf{L}_1$ and $\mathbf{L}_2$ are both subcomplexes of $\mathbf{K}$ and we are given cochains

$$f : C^i(\mathbf{K}, \mathbf{L}_1; \mathbb{D}) \to \mathbb{D} , \quad g : C^j(\mathbf{K}, \mathbf{L}_2; \mathbb{D})$$

then the cochain level cup product

$$C^i(\mathbf{K}, \mathbf{L}_1; \mathbb{D}) \times C^j(\mathbf{K}, \mathbf{L}_2; \mathbb{D}) \longrightarrow C^{i+j}(\mathbf{K}, \mathbf{L}_1 \cup \mathbf{L}_2; \mathbb{D})$$

sends (f, g) to the cochain $f \cup g$ whose value on a simplex generator T of $C_{i+j}(\mathbf{K})$ is the product of f evaluated on the front i-face of T and g evaluated on the back j-face of T. Since f and g are cochains which vanish on $C_i(\mathbf{L}_1)$ and $C_j(\mathbf{L}_2)$ respectively, it follows that $f \cup g$ vanishes on $C_{i+j}(\mathbf{L}_1 \cup \mathbf{L}_2)$ and hence defines a relative cochain. One can then reason exactly in the singular case to show that this cochain level cup product passes to a cup product in simplicial cohomology, and once again this refined cup product has analogs of all the properties one has in the singular case.

COMPATIBILITY OF THE SINGULAR AND SIMPLICIAL CUP PRODUCTS. Clearly it would be very useful to know that the singular and simplicial cup products correspond under the standard isomorphism from singular to simplicial cohomology. This is slightly less trivial than one might initially expect, for the relative cup product in singular cohomology is defined for pairs (X, A) such that A is open in X and the corresponding product in simplicial cohomology is defined for pairs (X, A) such that A is a closed subset of X. We shall need the following result in order to prove compatibility:

> *Suppose that $(P, \mathbf{K})$ is a simplicial complex and $(Q, \mathbf{L})$ is a $(Q_1, \mathbf{L}_1)$ and $(Q_2, \mathbf{L}_2)$ are subcomplex. Then there is an open neighborhood W of Q in P such that Q is a deformation retract of W.*

There is a proof of this result in Section II.9 of Eilenberg and Steenrod (and there are also proofs in many other algebraic topology texts).■

Now suppose that $(P, \mathbf{K})$ is a simplicial complex and that $(Q_1, \mathbf{L}_1)$ and $(Q_2, \mathbf{L}_2)$ are subcomplexes. Let W_1 and W_2 be open subsets of P such that Q_i is a deformation retract of W_i for $i = 1, 2$. Then a Five Lemma argument implies that the restriction mappings $H^*(P, W_i) \to H^*(P, Q_i)$ are isomorphisms, and the following result relates the singular and simplicial cup products:

THEOREM 7. *In the setting of the preceding paragraph, we have the following commutative diagram*

$$\begin{array}{ccc}
H^s(P, W_1; \mathbb{D}) \times H^t(P, W_2; \mathbb{D}) & \stackrel{\cup}{\longrightarrow} & H^{s+t}(P, W_1 \cup W_2; \mathbb{D}) \\
\Big\downarrow j_1^* \times j_2^* & & \Big\downarrow j^* \\
H^s(P, Q_1; \mathbb{D}) \times H^t(P, Q_2; \mathbb{D}) & & H^{s+t}(P, Q_1 \cup Q_2; \mathbb{D}) \\
\Big\downarrow \theta^* \times \theta^* & & \Big\downarrow \theta^* \\
H^s(\mathbf{K}, \mathbf{L}_1; \mathbb{D}) \times H^t(\mathbf{K}, \mathbf{L}_2; \mathbb{D}) & \stackrel{\cup}{\longrightarrow} & H^{s+t}(\mathbf{K}, \mathbf{L}_1 \cup \mathbf{L}_2; \mathbb{D})
\end{array}$$

in which the terms are given as follows and have the specified properties:

(i) *The horizontal arrows in the top and bottom row denote the singular and simplicial cup products.*

(ii) *The the mappings j_1^*, j_2^*, and j^*, are (restriction) maps induced by the appropriate inclusions of pairs, and the first two maps (and hence also their product) are isomorphisms.*

(iii) *The maps θ^* are the usual natural isomorphisms from singular to simplicial cohomology.*

It follows that the horizontal arrow in the first column is an isomorphism, and in fact a more precise application of the results from Eilenberg and Steenrod implies that we can choose the neighborhoods W_i so that the horizontal arrow in the second column is also an isomorphism, although this is not needed for many applications. Frequently we shall abuse language and say that the bottom line is the relative cup product in singular cohomology.

Method of proof. The proof follows immediately from the definitions of the various morphisms in the diagram (verify this!).■