IV.2 : A weak Universal Coefficient Theorem

(Hatcher, § 3.1)

We have already asserted the q-dimensional cohomology of a space is the dual space of the q-dimensional homology if we take coefficients in a field. However, our basic definition is somewhat different from this, so the next step is to verify the assertion at the beginning of this unit. Hatcher formulates and proves more general results (for example, see Theorem 3.2 on page 195). In this course we do not have enough time to develop the homological algebra necessary to prove such a result, and in any case the results for fields are strong enough to yield some important insights; one slogan might be that our setting only requires linear algebra and not the full force of homological algebra. However, if one goes deeper into the subject then it is necessary to work in the category of modules over arbitrary principal ideal domains.

The Kronecker Index

As usual let $\mathbb{D}$ be a commutative ring with unit, let C_* be a chain complex of $\mathbb{D}$-modules, and define an associated cochain complex by $C^q = \mathrm{Hom}_{\mathbb{D}}(C_q, \mathbb{D})$, with a coboundary map $d^q = \mathrm{Hom}(d_{q+1}, \mathbb{D})$ analogous to the construction for singular cochains. Then evaluation defines a bilinear map $C^q \times C_q \to \mathbb{D}$ which is called the *Kronecker index pairing* and its value at $f \in C^q$ and $x \in C_q$ is usually written as $\langle f, x\rangle$.

LEMMA 1. *Suppose that $f, f' \in C^q$ are cocycles and $x, x' \in C_q$ are cycles such that $f - f' = \delta a$ and $x - x' = db$. Then $\langle f, x\rangle = \langle f', x'\rangle$.*

Proof. For an arbitrary cochain g and chain y it follows immediately that $\langle \delta g, y\rangle = \langle g, dy\rangle$. Therefore we have

$$\langle f, x - x'\rangle \quad = \quad \langle f, db\rangle \quad = \quad \langle \delta f, b\rangle \quad = \quad \langle 0, b\rangle \quad = \quad 0$$

and similarly

$$\langle f - f', x'\rangle \quad = \quad \langle \delta a, x'\rangle \quad = \quad \langle a, dx'\rangle \quad = \quad \langle a, 0\rangle \quad = \quad 0$$

which combine to show that $\langle f, x\rangle = \langle f', x'\rangle$.■

COROLLARY 2. *The chain/cochain level Kronecker index pairing passes to a well-defined bilinear pairing from $H^q(C) \times H_q(C)$ to $\mathbb{D}$.*■

Manipulations with dual vector spaces

We now assume that $\mathbb{F}$ is a field. If V is a vector space over $\mathbb{F}$ and U is a subspace of V, then we have a short exact sequence of vector spaces

$$0 \;\to\; U \;\to\; V \;\to\; V/U \;\to\; 0$$

and applying the dual space functor we obtain the following short exact sequence of dual spaces because V is isomorphic to the direct sum $U \oplus (V/U)$:

$$0 \;\to\; (V/U)^* \;\to\; V^* \;\to\; U^* \;\to\; 0$$

The image of the map from $(V/U)^*$ to V^* is the **annihilator** of U, which consists of all linear functionals which vanish on U and will be denoted by $U^{\dagger}$.

Suppose now that V_1 and V_2 are vector spaces over $\mathbb{F}$ and $T : V_1 \to V_2$ is a linear transformation. Then we can factor T into a composite

$$V_1 \to J_1 \cong J_2 \subset V_2$$

where J_1 is the quotient of V_1 by the kernel of T, the map from J_1 to J_2 is an isomorphism, and J_2 is the image of T. There is also a corresponding factorization for the induced map of dual spaces

$$V_2^* \to J_2^* \cong J_1^* \subset V_1^*$$

These factorizations will be useful in proving the following abstract version of a key result in linear algebra:

PROPOSITION 3. *In the notation above, let $T^* : V_2^* \to V_1^*$ be the associated map of dual spaces. Then we have* $(\text{Kernel } T)^{\dagger} = \text{Image T}^* \subset V_1^*$ *and* $(\text{Image } T)^{\dagger} = \text{Kernel } T^* \subset V_2^*$.

Proof. By our previous observations we know that $(\text{Kernel } T)^{\dagger}$ corresponds to $J_1^* = J_2^*$, and since J_2 is the image of T, we have the asserted relationship. Similarly, we know that $(\text{Image}^T)^{\dagger}$ corresponds to $(V_2/J_2)^*$, and one can check directly that this corresponds to all linear functionals f on V_2 such that $0 = f \circ T = T^*(f)$.■

We now have enough machinery to derive the relationship between homology and cohomology over a field.

PROPOSITION 4. *Let C_* be a chain complex over a field $\mathbb{F}$, and let C^* be the dual cochain complex. Then for each q there is a natural isomorphism from $H^q(C)$ to $H_q(C)^*$.*

Proof. We shall focus on verifying the assertion about the isomorphism first. By definition we know that

$$H^q \;\;\cong\;\; (\text{Kernel } \delta^q)/(\text{Image } \delta^{q-1})\ .$$

Using the relationship $\delta = d^*$ we may rewrite the right hand side in the form

$$(\text{Image } d_{q+1})^{\dagger}/(\text{Kernel } d_q)^{\dagger}$$

and conclude by noting that the latter subquotient of C_q^* corresponds to

$$H_q^* \;\;\cong\;\; \Big((\text{Kernel } d_q)/(\text{Image } d_{q+1})\Big)^*\ .$$

Under these correspondences and the defining isomorphism

$$H_q \quad \cong \quad \Big((\text{Kernel } d_q)/(\text{Image } d_{q+1}) \Big)$$

all the standard pairings which evaluate linear functionals at vectors are preserved. In particular, this means that the isomorphism is given by the pairing described in Corollary 2. Now this pairing is natural by construction, and therefore our isomorphism is also natural.∎

Only a little more work is needed to derive the description of singular cohomology that we want.

COROLLARY 5. *If (X, A) is a topological space and $\mathbb{F}$ is a field, then for each q there is a natural isomorphism from $H^q(X, A; \mathbb{F})$ to the dual space $H_q(X, A; \mathbb{F})^*$.*

Proof. At this point all we need to do is describe a natural isomorphism

$$S^*(X, A; \mathbb{F}) \quad \cong \quad \text{Hom}(S_*(X, A), \mathbb{F}) \quad \longrightarrow \quad \text{Hom}_{\mathbb{F}}\left(S_*(X, A) \otimes \mathbb{F}, \mathbb{F}\right)$$

because the latter is the cochain complex to which Proposition 4 applies. However, the isomorphism in question is given directly by the universal properties of the tensor product construction sending the chain groups $S_q(X, A)$ to $S_q(X, A) \otimes \mathbb{F}$; in other words, there is a 1–1 correspondence between abelian group homomorphisms from $S_q(X, A)$ to $\mathbb{F}$ and $\mathbb{F}$-linear maps from $S_q(X, A) \otimes \mathbb{F}$ to $\mathbb{F}$.∎

If (X, A) is a pair of topological spaces, then similar considerations show that under this isomorphism the connecting morphism in cohomology

$$\delta^* : H^p(A; \mathbb{F}) \;\longrightarrow\; H^{p+1}(X, A; \mathbb{F})$$

corresponds to the map $\text{Hom}_{\mathbb{F}}(\partial, \mathbb{F})$, where $\partial : H_{p+1}(X, ; \mathbb{F}) \to H_p(A; \mathbb{F})$ is the connecting morphism in homology. This reflects the fact that chain complex boundaries and cochain complex coboundaries are adjoint to each other with respect to the Kronecker index pairing; details of the verification are left to the reader$^{(\star)}$.