IV.3 : Künneth formulas

(Hatcher, §§ 3.2, 3.B)

One obvious point about the preceding discussion is that we have not yet produced examples for which the cup product of two positive-dimensional cohomology classes is nontrivial. Our next order of business is to find classes of examples with this property. The first step is to prove purely algebraic versions of the results we want.

Algebraic cross products

The proof of the topological result in the preceding paragraph depends on finding a suitable chain complex for computing the homology of a product space $X \times Y$; more precisely, we want this to be an algebraic construction on the singular chain complexes of X and Y which is somehow an algebraic product of $S_*(X)$ and $S_*(Y)$. The correct model is given by a tensor product construction.

References for several of the straightforward verifications are given along the margins (all from notes11a).

Definition. Let (A_*, d^A_*) and (B_*, d^B_*) be chain complexes over a principal ideal domain $\mathbb{D}$ such that the chain groups in negative dimensions are zero. Then the tensor product $(A_*, d^A_*) \otimes_{\mathbb{D}} (B_*, d^B_*)$ has chain groups

$$(A \otimes B)_n \quad = \quad \bigoplus_{p=0}^{n} A_p \otimes B_{n-p}$$

and the differential is given on $A_p \otimes B_q$ by the formula

$$d^{A\otimes B}(x \otimes y) \quad = \quad d^A(x) \otimes y \; + \; (-1)^p x \otimes d^B(y) \; .$$

The sign is needed to ensure that $d^{A\otimes B} \circ d^{A\otimes B} = 0$ so that we actually get a chain complex; proving this algebraic identity is a fairly straightforward (and not too messy) exercise. It is also fairly straightforward to verify that this construction is covariantly functorial in A and B.$^{(\star)}$

See notes11a, p. 1

If we are simply given graded modules A_* and B_* (which may be viewed as chain complexes with zero differentials), then the preceding also yields a definition of the tensor product $A_* \otimes B_*$.

We shall need the following elementary observation, whose proof is left to the reader$^{(\star)}$.

PROPOSITION 1. *If B_* is a graded module which is free in all gradings (each B_k is free) and we are given a short exact sequence $0 \to K_* \to A_* \to C_* \to 0$ of graded modules, then the tensor product*

$$0 \;\longrightarrow\; K_* \otimes_{\mathbb{D}} B_* \;\longrightarrow\; A_* \otimes_{\mathbb{D}} B_* \;\longrightarrow\; C_* \otimes_{\mathbb{D}} B_* \to 0$$

See notes11a, pp. 2 - 3

is also a short exact sequence of graded modules. A similar conclusion holds for all B_* provided each of the graded modules K_*, A_*, C_* is free.■

One important consequence of the definition for tensor products of chain complexes is the following method of constructing classes in $H_*(A \otimes B)$ from classes in $H_*(A)$ and $H_*(B)$.

PROPOSITION 2. *In the setting above, there are bilinear mappings*

$$\times : H_p(A;\mathbb{D}) \;\otimes\; H_q(B;\mathbb{D}) \;\longrightarrow\; H_{p+q}(A \otimes B;\mathbb{D})$$

with the following property: If $x \in A_p$ is a cycle representing u and $y \in B_q$ is a cycle representing v, then $x \otimes y$ is a cycle representing $u \times v$.

This construction is called the *external homology cross product.*

See notes11a. pp. 4 - 5

Proof. We shall only sketch the main steps and leave the details to the reader$^{(\star)}$. First of all, if x and y are cycles, then the definitions imply that $x \otimes y$ is also a cycle, and if $x = dw$ or $y = dz$ then $x \otimes y$ is a boundary. Bilinearity follows from the definition, and this plus the preceding sentence imply that the bilinear map is well defined.■

Our next result states that these products are maximally nontrivial if $\mathbb{D}$ is a field.

THEOREM 3. (The algebraic Künneth Theorem) *If $\mathbb{F}$ is a field, then the external homology cross product defines an isomorphism*

$$\bigoplus_{p=0}^{n} H_p(A;\mathbb{F}) \otimes H_{n-p}(B;\mathbb{F}) \;\longrightarrow\; H_n(A \otimes B;\mathbb{F})$$

for all $n \geq 0$.

Proof. In the argument below, all tensor products are taken over the field $\mathbb{F}$.

For each integer k let $\mathfrak{z}(A_q) \subset A_q$ be the subspace of cycles; if we define a chain complex structure on $\mathfrak{z}(A_*)$ by setting all boundary homomorphisms equal to zero, then $\mathfrak{z}(A_*)$ is a chain subcomplex of A_*, and the quotient complex

$$\mathfrak{q}(A_*) \;=\; A_*/\mathfrak{z}(A_*)$$

also has a trivial differential because $\mathfrak{q}(A_k) \cong d[A_k] \subset A_{k-1}$ and $d \circ d = 0$.

Since we are working over a field $\mathbb{F}$, all modules are free, and hence if we apply Proposition 3 to the previous short exact sequence we obtain a short exact sequence of chain complexes

$$0 \;\longrightarrow\; \mathfrak{z}(A_*) \otimes_{\mathbb{F}} B_* \;\longrightarrow\; A_* \otimes_{\mathbb{F}} B_* \;\longrightarrow\; \mathfrak{q}(A_*) \otimes_{\mathbb{F}} B_* \;\longrightarrow\; 0$$

which of course has an associated long exact homology sequence. Since the differentials in $\mathfrak{z}(A_*)$ and $\mathfrak{q}(A_*)$ are trivial, this long exact sequence has the form

$$\cdots\; [\mathfrak{z}(A)_* \otimes H_*(B)]_k \;\rightarrow\; H_k((A \otimes B) \;\rightarrow\; [\mathfrak{q}(A)_* \otimes H_*(B)]_k \;\rightarrow\; [\mathfrak{z}(A)_* \otimes H_*(B)]_{k-1} \;\cdots$$

and the definitions of the connecting homomorphisms imply that the right hand arrow ∂_k is given by $\widetilde{d}_* \otimes \mathrm{id}\,[H_*(B)]$, where

$$\widetilde{d}_m : \mathfrak{q}(A)_m \quad \longrightarrow \quad \mathfrak{z}(A)_{m-1}$$

is the composite

$$\widetilde{d}_m : \mathfrak{q}(A)_m \quad \cong \quad d_m[A_m] \quad \subset \quad \mathfrak{z}(A)_{m-1} \ .$$

This implies that the map $\widetilde{d}_m$ is injective, and since the tensor product functor over a field preserves short exact sequences it follows that the connecting homomorphisms ∂_k are also injective. Therefore the maps

$$H_k((A \otimes B) \quad \to \quad [\mathfrak{q}(A)_* \otimes H_*(B)]_k$$

are zero, so by exactness it follows that $H_k(A \otimes B)$ is isomorphic to the quotient

$$[\mathfrak{z}(A)_* \otimes H_*(B)]_k \ / \ [\mathfrak{q}(A)_* \otimes H_*(B)]_k \quad \cong \quad [H_*(A)_* \otimes H_*(B)]_k$$

and a check of the definitions shows that the isomorphism is induced by the homology cross product.■

There is also a **dual cross product in cohomology**. If $g : A_p \to \mathbb{F}$ and $h : B_q \to \mathbb{F}$ are cochains, then we define a cross product cochain $g \times h : [A \otimes B]_{p+q} \to \mathbb{F}$ such that the restriction to $A_r \otimes B_{p+q-r}$ is zero if $r \neq p$ and the restriction to $A_p \otimes B_q$ satisfies the identity

$$g \times h(x \otimes y) \quad = \quad g(x) \cdot h(y) \ , \qquad (x \in A_p \ , \quad y \in B_q) \ .$$

The coboundary of $g \times h$ is given by the following identity:

LEMMA 4. *In the setting above we have*

$$\delta(g \times h) \quad = \quad \delta g \times h \ + \ (-1)^p \, g \times \delta h \ .$$

In particular, if g and h are cocycles then so is $g \times h$, and if in addition one of g and h is a coboundary then so is $g \times h$, so that the cross product passes to a bilinear mapping from $H^p(A) \otimes H^q(B) \to H^{p+q}(A \otimes B)$.

Sketch of proof. By definition we have

$$\delta(g \times h) \quad = \quad (g \times h)^\circ d \quad = \quad (g \times h)^\circ \left(d^A \otimes \mathrm{id} \ + \ (-1)^p \mathrm{id} \otimes d^B \right)$$

and if we apply the right hand expression to a typical generator $z \otimes w \in A_m \otimes B_{p+q-m-1}$ we see that the value equals the value of the right hand side in the displayed expression of the lemma. The second sentence in the lemma follows by adding the conditions $\delta g = \delta h = 0$ for the first assertion, and adding one of the additional conditions $g = \delta g'$ or $h = \delta h'$ in the second. The third sentence follows immediately from these and the fact that the cochain cross product is bilinear.■

It is now very easy to show that the cross product of two nontrivial cohomology classes is nonzero.

COROLLARY 5. *If $\alpha \in H^p(A)$ and $\beta \in H^q(B)$ are nonzero, then so is $\alpha \times \beta$.*

Proof. By the Weak Universal Coefficient Theorem there are homology classes $u \in H_p(A)$ and $v \in H_q(B)$ such that the Kronecker indices $\alpha(u)$ and $\beta(v)$ are nonzero elements of $\mathbb{F}$. Since the Kronecker index of the cohomology cross product satisfies

$$\langle \alpha \times \beta, u \times v \rangle \quad = \quad \alpha(u) \cdot \beta(v)$$

and the right hand side is nonzero (it is a product of two nonzero elements in $\mathbb{F}$), it follows that $\alpha \times \beta$ is also nonzero.■

Topological cross products

We can define the cross product of two singular cochains by a variant of the cup product definition. If $\mathbb{D}$ is a commutative ring with unit and we are given two singular cochains $f : S_p(X, \mathbb{D}) \to \mathbb{D}$ and $g : S_q(Y, \mathbb{D}) \to \mathbb{D}$, then their cross product

$$f \times g : S_{p+q}(X \times Y; \mathbb{D}) \longrightarrow \mathbb{D}$$

is defined on a singular simplex $T = (T_X, T_Y) : \Delta_{p+q} \to X \times Y$ by the formula

$$f \times g(T) \quad = \quad f\,(\,\mathbf{Front}_p(T_X)\,) \cdot g\,(\,\mathbf{Back}_q(T_Y)\,)\ .$$

The usual bilinearity and associativity properties follow directly from the definition (details are left to the reader). We also have the following identities showing that each of the cup and cross products can be easily described in terms of the other:

PROPOSITION 6. *In the setting above we have the following identities, whose verifications are left to the reader:*

- (i) *If $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are coordinate projections with associated singular cochain homomorphisms $\pi_X^\#$ and $\pi_Y^\#$, then $f \times g = \pi_X^\#(f) \cup \pi_Y^\#(g)$.*
- (ii) *If $X = Y$ and $\Delta_X : X \to X \times X$ is the diagonal map, then $f \cup g = \Delta_X^\#(f \times g)$.*■

The singular cohomology cross product also satisfies analogs of the basic properties for cohomology products in Lemma 4;

LEMMA 7. *In the setting above we have*

$$\delta(f \times g) \quad = \quad \delta f \times g \ + \ (-1)^p\, f \times \delta g\ .$$

In particular, if f and g are cocycles then so is $f \times g$, and if in addition one of f and g is a coboundary then so is $f \times g$, so that the cross product passes to a bilinear mapping from $H^p(X; \mathbb{D}) \otimes H^q(Y; \mathbb{D}) \to H^{p+q}(X \times Y; \mathbb{D})$.■

At this point we need a result relating the singular homology relating the singular homology of $X \times Y$ to the singular homology of the factors X and Y; to shorten the discussion, we restrict ourselves to field coefficients in these notes. Some of the earliest general results of this type were due to H. Künneth in the early 1920s, and in singular homology this relationship follows from a general method of **acyclic models** due to Eilenberg and J. A. Zilber. We shall not formulate this method abstractly, but the reader may be able to see the general pattern emerge.

THEOREM 8. (Eilenberg-Zilber) *If $\mathbb{D}$ is a principal ideal domain and X and Y are topological spaces, then there are functorial chain homotopy equivalences*

$$\psi_{X,Y} : S_*(X \times Y;\mathbb{D}) \;\rightarrow\; S_*(X;\mathbb{D}) \otimes_{\mathbb{D}} S_*(Y;\mathbb{D})\,,\; \varphi_{X,Y} : S_*(X;\mathbb{D}) \otimes_{\mathbb{D}} S_*(Y;\mathbb{D}) \;\rightarrow\; S_*(X \times Y;\mathbb{D})$$

with the following properties:

(i) *The composites $\varphi \circ \psi$ and $\psi \circ \varphi$ are naturally chain homotopic to the identity.*

(ii) *In degree 0 the map ψ takes a singular 0-simplex $T = (T_X, T_Y)$ to $T_X \otimes T_Y$, and φ is inverse to ψ.*

We shall describe an explicit choice for $\psi_{X,Y}$ known as the *Alexander-Whitney map.*

Before proving Theorem 8, we shall list some of its consequences:

THEOREM 9. (Classical Künneth Formula for field coefficients) *Let X and Y be topological spaces, and let $\mathbb{F}$ be a field. Then the composite of the homology cross product and induced mapping ψ_* in homology defines isomorphisms of singular homology groups*

$$\bigoplus_{p=0}^{n} H_p(X;\mathbb{F}) \otimes_{\mathbb{F}} H_{n-p}(Y;\mathbb{F}) \quad\longrightarrow\quad H_n(X \times Y;\mathbb{F})$$

for all $n \geq 0$.

This result follows directly from Theorem 8 and the Algebraic Künneth Formula (Theorem 3).∎

THEOREM 10. (Cohomological Künneth Formula) *Let X and Y be topological spaces, let $\mathbb{F}$ be a field, and assume that the homology groups $H^p(X;\mathbb{F})$ and $H^q(Y;\mathbb{F})$ are finite for all $p.q \geq 0$. Then the cohomology cross product map defines isomorphisms of singular homology groups*

$$\bigoplus_{p=0}^{n} H^p(X;\mathbb{F}) \otimes_{\mathbb{F}} H^{n-p}(Y;\mathbb{F}) \quad\longrightarrow\quad H^n(X \times Y;\mathbb{F})$$

for all $n \geq 0$.

Sketch of the proof that Theorem 9 implies Theorem 10. This is a consequence of the following observations:

(1) The Universal Coefficient isomorphism from the finite dimensional vector space $H^k(W;\mathbb{F})$ to the dual space of $H_k(W;\mathbb{F})$, where $W = X$ or Y and $k \geq 0$.

(2) The natural isomorphism $(V_1 \oplus V_2)^* \cong V_1^* \oplus V_2^*$, where V^* denotes the dual space and V_1 and V_2 are vector spaces over $\mathbb{F}$.

(3) The natural isomorphism $(V_1 \otimes_{\mathbb{F}} V_2)^* \cong V_1^* \otimes_{\mathbb{F}} V_2^*$, where V^* denotes the dual space and V_1 and V_2 are **finite dimensional** vector spaces over $\mathbb{F}$. In fact, the conclusion of the theorem is generally false if the finite dimensionality conditions do not hold, but there still is a natural monomorphism from $V_1^* \otimes_{\mathbb{F}} V_2^*$ to $(V_1 \otimes_{\mathbb{F}} V_2)^*)$.

Note that under these isomorphisms the homology and cohomology cross products correspond; namely, if $f_i \in V_i^*$ and $x_i \in V_i$ then $f_1 \otimes f_2(x_1 \otimes x_2) = f_1(x_1) \cdot f_2(x_2)$.■

The first step in proving Theorem 7 is to consider the special case where $X = Y = \Delta_n$ for some n.

LEMMA 11. *Let $\mathbb{D}$ be a commutative ring with unit. If $p, q \geq 0$ and an augmentation is defined on $S_*(\Delta_n;\mathbb{D}) \otimes S_*(\Delta_n;\mathbb{D})$ using the multiplication and tensor product maps*

$$S_0(\Delta_n : \mathbb{D}) \otimes_{\mathbb{D}} S_0(\Delta_n : \mathbb{D}) \ \to \ \mathbb{D} \otimes \mathbb{D} \ \to \ \mathbb{D}$$

then $S_*(\Delta_p;\mathbb{D}) \otimes_{\mathbb{D}} S_*(\Delta_q;\mathbb{D})$ is acyclic.

Proof of Lemma 11. Let $C_*(\mathbb{D})$ be the ordered simplicial chain complex $C_*(\{\mathbf{e}\}_0^\omega;\mathbb{D})$, where $\mathbf{e}\}_0$ is a standard vertex of Δ_n, let $\eta : C_*(\mathbb{D}) \to S_*(\Delta_p;\mathbb{D})$ be the augmentation-preserving inclusion determined by viewing the generator of $C_0(\mathbb{D})$ as the singular 0-simplex sending the unique point in Δ_0 to the vertex $\mathbf{e}\}_0 \in \Delta_p$, and note that the augmentation map ε on $S_*(\Delta_p)$ can be viewed as a chain map from the latter to $C_*(\mathbb{D})$. Then the proof of homotopy invariance for singular homology implies that $\eta[C_*(\mathbb{D})]$ is a chain homotopy deformation retract of $S_*(\Delta_p;\mathbb{D})$. We can then construct the tensor product of a contracting chain homotopy with the identity on $S_*(\Delta_q;\mathbb{D})$, and it follows immediately that $S_*(\Delta_q;\mathbb{D}) \cong C_*(\mathbb{D}) \otimes \otimes_{\mathbb{D}} S_*(\Delta_q;\mathbb{D})$ is a chain deformation retract of $S_*(\Delta_p;\mathbb{D}) \otimes_{\mathbb{D}} S_*(\Delta_q;\mathbb{D})$. Since the smaller chain complex is acyclic, it follows that the larger chain complex is also acyclic.■

SIMPLICIAL ANALOGS OF LEMMA 11. Similar results hold for the ordered and unordered simplicial chain complexes of $\Delta - n$. The proofs are straightforward adaptations of the proof in the singular case and are left to the reader.■

Proof of Theorem 8. The Alexander-Whitney map $\psi_{X,Y}$ is just a formalization of earlier constructions, Specifically, if $T = (T_X, T_Y) : \Delta_n : X \times Y$ is a singular simplex given by the coordinate projections T_X and T_Y, then

$$\psi_{X,Y}(T) \quad = \quad \sum_{p=0}^{n} \mathbf{Front}_p(T_X) \otimes \mathbf{Back}_{n-p}(T_Y) \ .$$

See notes11a, pp. 6 - 8

It is a routine exercise to check that this construction defines a natural chain map$^{(\star)}$.

The idea behind constructing φ and the chain homotopies is to look at universal examples and extend to the general case by naturality. The chain groups

$$[S_*(X;\mathbb{D}) \otimes S_*(Y;\mathbb{D})]_n$$

are free modules, and explicit free generators are given by all objects of the form $F_X \otimes B_Y$, where $F_X : \Delta_p \to X$ and $B_Y : \Delta_{n-p} \to Y$ are singular simplices and $0 \leq p \leq n$. We shall define φ on such objects recursively with respect to n. The stated conditions define φ in degree 0. Once we are given φ in degrees $\leq n-1$, we shall define φ first on the universal class

$$\mathrm{id}[\Delta_p] \otimes \mathrm{id}[\Delta_{n-p}] \in S_p(\Delta_p;\mathbb{D}) \otimes_{\mathbb{D}} S_{n-p}(\Delta_{n-p};\mathbb{D})$$

and then we shall define $\varphi(F_X \otimes B_Y)$ by the naturality condition

$$\varphi(F_X \otimes B_Y) \quad = \quad F_{X\#}\,(\mathrm{id}[\Delta_p]) \otimes B_{Y\#}\,(\mathrm{id}[\Delta_{n-p}]) \ \ .$$

It is a straightforward exercise to verify that this construction defines a chain map in degree n, and this completes the inductive step$^{(\star)}$. **See notes11a, p. 9**

By the preceding discussion, the construction of φ reduces to finding a choice $W(p,q)$ for $\varphi(\mathrm{id}[\Delta_p] \otimes \mathrm{id}[\Delta_p])$ which satisfies the chain map condition

$$dW(p,q) \quad = \quad \varphi \circ d\,(\mathrm{id}[\Delta_p] \otimes \mathrm{id}[\Delta_p]) \ \ .$$

Since $S_*(\Delta_p \times \Delta_{n-p};\mathbb{D})$ is acyclic, such a class exists if and only if the right hand side is a cycle, so everything comes down to computing the boundary map on the right hand side. By the induction hypothesis, we know that φ is a chain map in degree $n-1$, and therefore the boundary of the right hand side is given by

$$d\,(\,\varphi \circ d\,(\mathrm{id}[\Delta_p] \otimes \mathrm{id}[\Delta_p])\,) \quad = \quad (\varphi \circ d) \circ d\,(\mathrm{id}[\Delta_p] \otimes \mathrm{id}[\Delta_p])$$

which vanishes because $d \circ d = 0$ as required. This completes the inductive step and the construction of φ.

The chain homotopies from $\varphi \circ \psi$ and $\psi \circ \varphi$ to the respective identity maps are also constructed using universal examples. We shall start by constructing the chain homotopy $\varphi \circ \psi \simeq \mathbf{id}$. Since $\varphi_0 \circ \psi_0$ is the identity map, we can take $D_0 = 0$. Assume that se have defined D_k for $k < n$; as before, we first define D_n on the diagonal map $\mathrm{Diag}_n : \Delta_n \to \Delta_n \times \Delta_n$ and then we extend by naturality. The classes

$$\theta_{n+1} \quad = \quad D_n\,(\mathrm{Diag}_n) \in S_{n+1}(\Delta_n \times \Delta_n)$$

are required to satisfy the identity

$$d\theta_n \;\; = \;\; \mathrm{Diag}_n \; - \; \varphi\psi\,(\mathrm{Diag}_n) \; - \; D_{n-1} \circ d_n\,(\mathrm{Diag}_n)$$

for all $n > 0$. Once again, everything reduces to showing that the right hand side is a cycle because $S_*(\Delta_n \times \Delta_n; \mathbb{D})$ is acyclic. Much as before, one can use the inductive hypothesis

$$\varphi_{n-1} \circ \psi_{n-1} \; - \; \text{identity} \;\; = \;\; D_{n-2} \circ d_{n-1} \; + \; d_n \circ D_{n-1}$$

and $d \circ d = 0$ to prove that $\text{Diag}_n \; - \; \varphi\psi\,(\text{Diag}_n) \; - \; D_{n-1} \circ d_n\,(\text{Diag}_n)$ is a cycle. As before, one extends by naturality, and it is another formal exercise to check that the construction defines a natural chain homotopy from $\varphi \circ \psi$ to the identity.

Similar considerations yield the chain homotopy $E : \psi \circ \varphi \simeq \mathbf{id}$. In this case we must use the free generators of $S_*(X; \mathbb{D}) \otimes S_*(Y; \mathbb{D})$ described above, and we need Lemma 10 for the fact that $S_*(\Delta_p; \mathbb{D}) \otimes_{\mathbb{D}} S_*(\Delta_q; \mathbb{D})$ is acyclic.■

The next result, which yields many examples of nontrivial cross products in singular homology and cohomology, is an immediate consequence of the results in this section.

COROLLARY 12. *Let X and Y be nonempty topological spaces, and let $\mathbb{F}$ be a field.*

(*i*) *If $u \in H_p(X; \mathbb{F})$ and $v \in H_q(Y; \mathbb{F})$ are nonzero, then so is $u \times v$.*

(*ii*) *If $\alpha \in H^p(X; \mathbb{F})$ and $\beta \in H^q(Y; \mathbb{F})$ are nonzero, then so is $\alpha \times \beta$.*■

Products in relative homology groups

We would also like to have a version of Corollary 11 for cross products in relative homology and cohomology. There are a few complications, but one can develop a reasonably good theory in this case. The first step is a generalization of the Eilenberg-Zilber Theorem. For the rest of this section we shall assume that all coefficients lie in some commutative ring with unit $\mathbb{D}$ which is suppressed from the notation.

THEOREM 13. *Suppose that (X, A) and (Y, B) are pairs of spaces such that A and B are open in X and Y respectively. Then there is a relative cross product on the cochain level*

$$S^p(X, A) \otimes S^q(Y, B) \;\longrightarrow\; S^{p+q}(X \times Y, A \times Y \cup X \times B)$$

which is compatible with the absolute cross product defined in this unit. This product satisfies analogs of the coboundary formulas in the absolute case and passes to a cohomology cross product which is also compatible with the previous construction when $A = B = \emptyset$. Furthermore, if the coefficients lie in a field $\mathbb{F}$ and all the cohomology groups $H^p(X, A)$ and $H^q(Y, B)$ are finite dimensional vector spaces, then the cross product defines an isomorphism from $H^*(X, A) \otimes H^*(Y, B)$ to $H^*(X \times Y, A \times Y \cup X \times B)$.

Proof. Let $\mathcal{U}$ be the open covering of $A \times Y \cup X \times B$ given by $\{A \times Y, X \times B\}$. Then one can check directly that the composite of the cochain level cross product from $S^p(X, A) \otimes S^q(Y, B)$ to $S^{p+q}(X \times Y)$ with the restriction mapping

$$S^{p+q}(X \times Y) \;\to\; S^{p+q}_{\mathcal{U}}(A \times Y \cup X \times B) \;\;=\;\; \text{Hom}_{\mathbb{D}}(S_{p+q}(A \times Y) + S_{p+q}(X \times B), \mathbb{D})$$

is trivial; in other words, if f is a cochain on $S_p(X)$ which vanishes on $S_p(A)$ and g is a cochain on $S_q(Y)$ which vanishes on $S_q(B)$, then $f \times g$ vanishes on $S_{p+q}(A \times Y) + S_{p+q}(X \times$

$B) = S^{\mathcal{U}}_{p+q}(A\times Y\cup X\times B)$. The Leibniz Formula for the coboundary of a cross product is an immediate consequence of the construction and known results when $A = B = \emptyset$ (recall that the relative cochain groups $S^*(Z, C)$ are contained in the absolute groups $S^*(Z)$ as the subgroups of cochains whose restrictions to $S_*(C)$ are zero). It follows that the cochain level cross product passes to cohomology mappings

$$H^p(X,A)\otimes H^q(Y,B) \;\to\; H^{p+q}\left(S^*(X\times Y)/S^*_{\mathcal{U}}(A\times Y\cup X\times B)\right)$$

and since the inclusion of the latter in $S_{p+q}(A\times Y\cup X\times B)$ is a chain homotopy equivalence by the proof of excision we get the desired map from $H^p(X,A)\otimes H^q(Y,B)$ to $H^{p+q}(X\times Y, A\times Y\cup X\times B)$. This completes the derivation of the construction, We must now prove the relative Künneth Formula in the final sentence of the theorem.

Consider first the case where $B = \emptyset$. Since the short exact sequences of singular chain complexes for the pair (X,A) is split in each degree (the standard free generators of $S_k(A)$ are a subset of the standard free generators for $S_k(X)$), we have the following commutative diagram in which the vertical maps are Alexander-Whitney maps and the rows are short exact sequences; all tensor products are taken with respect to the field $\mathbb{F}$.

$$\begin{array}{ccccccccc}
0 & \to & S_*(A\times Y) & \to & S_*(X\times Y) & \to & S_*(X\times Y, A\times Y) & \to & 0 \\
 & & \downarrow\psi & & \downarrow\psi & & \downarrow\psi & & \\
0 & \to & S_*(A)\otimes S_*(Y) & \to & S_*(X)\otimes S_*(Y) & \to & S_*(X,A)\otimes S_*(Y) & \to & 0
\end{array}$$

Since the vertical maps on the left and center induce isomorphisms in homology, it follows that the vertical map on the right also induces isomorphisms in homology; in fact, this part of the argument does not require A to be an open subset of X.

Now consider the following commutative diagram, in which the second and third rows are short exact sequences of chain complexes and the maps denoted by ψ are Alexander-Whitney maps:

$$\begin{array}{ccccc}
S_*(X\times B, A\times B) & & & & \\
\cong & & & & \\
S^{\mathcal{U}}_*(A\times Y\cup X\times B, A\times Y) & & & & \\
\downarrow\subset & & & & \\
S_*(A\times Y\cup X\times B, A\times Y) & \to & S_*(X\times Y, A\times Y) & \to & S_*(X\times Y, A\times Y\cup X\times B) \\
\downarrow ? & & \downarrow\psi & & \downarrow\psi \\
S_*(X,A)\otimes S_*(B) & \to & S_*(X,A)\otimes S_*(Y) & \to & S_*(X,A)\otimes S_*(Y,B)
\end{array}$$

The question mark represents the following **claim**: *There is a chain map ψ' from $S_*(A\times Y\cup X\times B, A\times Y)$ to $S_*(X,A)\otimes S_*(B)$ whose restriction to the subcomplex $S_*(x\times$*

$B, A \times B) \cong S_*^{\mathcal{U}}(A \times Y \cup X \times B, A \times Y)$ *is the usual Alexander-Whitney map.* The existence of ψ' follows from the commutativity of the right hand square, for the latter implies that $\psi : S_*(X \times Y, A \times Y) \to S_*(X, A) \otimes S_*(Y)$ maps the kernel of the surjection from $S_*(X \times Y, A \times Y)$ to the kernel of the surjection from $S_*(X, A) \otimes S_*(Y)$. Furthermore, the commutativity of the diagram

$$\begin{array}{ccc} S_*(X \times B, A \times B) & \to & S_*(X \times Y, A \times Y) \\ \downarrow \psi & & \downarrow \psi \\ S_*(X, A) \otimes S_*(B) & \to & S_*(X, A) \otimes S_*(Y) \end{array}$$

implies that ψ' extends the Alexander-Whitney map ψ on $S_*(X, A) \otimes S_*(B)$.

By earlier discussion of special cases, which applies to the pair $(X \times B, A \times B)$ by interchanging the roles of the first and second factors, we know that the Alexander-Whitney map on $S_*(X, A) \otimes S_*(B)$ induces isomorphisms in singular homology, and by the proof of excision we know that the inclusion of $S_*(X \times B, A \times B)$ in $S_*(A \times Y \cup X \times B)$ induces isomorphisms in singular homology, and it follows immediately that the mapping ψ' also induces isomorphisms in singular homology. We have already shown that the Alexander-Whitney map from $S_*(X \times Y, A \times Y)$ induces an isomorphism in homology, and therefore the Five Lemma implies that the Alexander-Whitney map from $S_*(X \times Y, A \times Y \cup X \times B)$ also induces isomorphisms in homology.

If we now take coefficients in a field and assume all homology and cohomology groups are finite dimensional, then the weak Universal Coefficient Theorem implies that the dual cochain complex maps

$$S^*(X, A; \mathbb{F}) \otimes_{\mathbb{F}} S^*(Y, B; \mathbb{F}) \ \to S^*(X \times Y, A \times Y \cup X \times B; \mathbb{F})$$

induce isomorphisms in cohomology from $H^*(X, A; \mathbb{F}) \otimes_{\mathbb{F}} H^*(Y, B; \mathbb{F})$ to $H^*(X \times Y, A \times Y \cup X \times B; \mathbb{F})$.■

In particular, just as before we know that if the homology groups of (X, A) and (Y, B) are finite dimensional over $\mathbb{F}$ in each dimension, then the cross product of a nontrivial cohomology class $\alpha \in H^*(X, A; \mathbb{F})$ and a nontrivial cohomology class $\beta \in H^*(Y, B; \mathbb{F})$ will always be nontrivial.

CORRESPONDING RESULTS FOR CLOSED SUBSETS. Frequently we want versions of the preceding when A and B are closed subsets rather than open subsets. As in earlier discussions, analogous results hold if we assume that A and B are deformation retracts of open neighborhoods $A \subset U \subset X$ and $B \subset V \subset Y$ (details are left to the reader$^{(\star)}$ — the crucial point is that pairs like (X, A) and (X, U) have isomorphic homology), and in many (most?) situations of interest in algebraic and geometric topology this sort of condition is satisfied. For example, this is the case if X is a polyhedron and A is a subpolyhedron.

Although the homology groups of a space do not have a ring structure, it turns out that the graded object $H_*(X, \mathbb{D})$ is a graded module over the cohomology ring if one multiply cohomology degrees by (-1).

Definition. Let X be a space, and let A_1 and A_2 be open subsets of X. The chain/cochain level cap product

$$\cap : S^p(X, A_1; \mathbb{D}) \otimes_{\mathbb{D}} S_n(X; \mathbb{D})/[S_n(A_1; \mathbb{D}) + S_n(A_2; \mathbb{D})] \quad \longrightarrow \quad S_{n-p}(X, A_2; \mathbb{D})$$

is defined as follows: Given $g : S_p(X, A_1) \to \mathbb{D}$ and a singular simplex $T : \Delta_n \to X$, the cochain $g \cap T$ is given by $g\big(\mathbf{Front}_p(T)\big) \cdot \mathbf{Back}_{n-p}(T)$. Strictly speaking this construction is defined on $S^p(X, A_1; \mathbb{D}) \otimes S_n(X; \mathbb{D})$, but it factors through the displayed quotient because $g \cap T$ is trival on all singular simplices in $S_n(A_1; \mathbb{D}) + S_n(A_2; \mathbb{D}) \subset S_n(X; \mathbb{D})$; if the image of T lies in A_2, then the image is trivial by the definition of $S_{n-p}(X, A_2; \mathbb{D})$, and if the image of T lies in A_1 then triviality follows because $f|S_p(X, A_1; \mathbb{D})$ is zero. If c is a p-chain in $S^p(X, A_1; \mathbb{D})$, then one has the usual sort of graded Leibniz rule for computing $d(g \cap c)$, and it follows that (1) $g \cap c$ is a cycle if g is a cocycle and c is a cycle, (2) $g \cap c$ is a boundary if either g is a coboundary or c is a boundary. Since A_1 and A_2 are open subsets of X, the proof of excision implies that the chain complex inclusion

$$S_*(A_1; \mathbb{D}) + S_*(A_2; \mathbb{D}) \quad \subset \quad S_*(A_1 \cup A_2; \mathbb{D})$$

induces isomorphisms in homology, it follows that the chain/cochain level cap product induces a map in homology/cohomology

$$\cap : H^p(X, A_1; \mathbb{D}) \otimes_{\mathbb{D}} H_n(X, A_1 \cup A_2; \mathbb{D}) \longrightarrow S_{n-p}(X, A_2; \mathbb{D}) \ .$$

The cap product map is $\mathbb{D}$-bilinear, and it also has the following formal properties:

PROPOSITION 14. *Let X be a space. Then the cap product has the following properties:*

(i) If $\varepsilon_X : S_0(X) \to \mathbb{D}$ is the augmentation and $[\varepsilon_X] \in H^0(X; \mathbb{D})$ is its cohomology class, then cap product with $[\varepsilon_X]$ induces the identity on $H_*(X; \mathbb{D})$.

(ii) The cap and cup product satisfy a mixed associative law: If $u \in H^q(X; \mathbb{D})$, $v \in H^p(X; \mathbb{D})$, and $z \in H_n(X; \mathbb{D})$, then $(u \cup v) \cap z = u \cap (v \cap z)$.

(iii) If $f : X \to Y$ is continuous with $u \in H^p(Y; \mathbb{D})$ and $z \in H_n(x; \mathbb{D})$, then $f_*(f^*u \cap z) = u \cap f_*(z)$.

In each case, one can verify that the corresponding identities hold at the chain/cochain level; details are left to the reader$^{(\star)}$.■ **See notes11a, pp. 11 - 12**