page 98, lines -12 to -11

Verification that $d^{A\otimes B}_{n-1} \circ d^{A\otimes B}_{n} = 0$.

Details: Since $(A\otimes B)_n$ is generated by classes $y\otimes z$ where $y \in A_p$ and $z \in B_{q=n-p}$ for some p, it suffices to check that the composite sends these elements to zero.

We have $d(y\otimes z) = dy\otimes z + (-1)^p y\otimes dz$, so $dd(y\otimes z) = d(dy\otimes z + (-1)^p y\otimes dz) =$ $ddy\otimes z + (-1)^{p-1} dy\otimes dz + (-1)^p dy\otimes dz +$ $(-1)^{p-1}(-1)^p y\otimes ddz$. The first and last terms are zero because $dd=0$, and the middle two terms cancel each other because $(-1)^p = -(-1)^{p-1}$. Thus $dd=0$ in $A\otimes B$, which is what we wanted to verify.

page 98, line -9

Proof of Proposition 1.

(i) Suppose $0 \to K_* \to A_* \to C_* \to 0$ is short exact and B_* is free in each dim. [As earlier on p. 99, assume all groups are zero in negative dims]. Note that if F is a free module and $0 \to K \to A \to C \to 0$ is short exact, then $F \cong \bigoplus_{\gamma \in \Gamma} \mathbb{D}_\gamma \Rightarrow$

$$0 \to K \otimes_{\mathbb{D}} F \to A \otimes_{\mathbb{D}} F \to C \otimes_{\mathbb{D}} F \to 0$$

is just a direct sum of the form

$$0 \to \bigoplus_\gamma K_\gamma \to \bigoplus_\gamma A_\gamma \to \bigoplus_\gamma C_\gamma \to 0$$

and is short exact (more generally, a direct sum of short exact sequences is short exact). Thus for each $n \geq 0$ the n-dimensional modules in the tensored chain complexes are given by the short exact sequence page 98, line -8, continued

$$0 \to \bigoplus_{p\geq 0} K_p \otimes B_{n-p} \to \bigoplus_{p \geq 0} A_p \otimes B_{n-p} \to \bigoplus_{p\geq 0} C_p \otimes B_{n-p} \to 0$$

and hence $0 \to K_* \otimes B_* \to K_* \otimes A_* \to K_* \otimes C_* \to 0$

$A_* \otimes B_* \quad C_* \otimes B_*$

is short exact.

(ii) Now assume K_*, A_*, C_* are free in each dimension. Then for all p we have $A_p \cong C_p \oplus K_p$ and the n-dim modules in

$$0 \to K_* \otimes B_* \to A_* \otimes B_* \to C_* \otimes B_* \to 0$$

are given by the short exact sequence

$$0 \to \bigoplus_{p\geq 0} K_p \otimes B_{n-p} \to \bigoplus_{p\geq 0} (K_p \oplus C_p) \otimes B_{n-p} \to \bigoplus_{p\geq 0} C_p \otimes B_{n-p} \to 0$$

which is in fact a _split_ short exact sequence.

Although the chain complex sequence

$$0 \to K_* \otimes B_* \to A_* \otimes B_* \to C_* \otimes B_* \to 0$$

might not be (and usually is not) split, we can still conclude that it is short exact in the chain complex category.

page ~~100~~ 99, lines 1-9

Details for the proof of Proposition 2.

(i) If $x \in A_p$ and $y \in B_q$ are cycles, then $x \otimes y$ is also a cycle.

Proof. $dx = 0$ and $dy = 0 \Longrightarrow$
$$d(x \otimes y) = \underset{0}{\cancel{dx}} \otimes y + (-1)^p x \otimes \underset{0}{\cancel{dy}} = 0 + 0 = 0.$$

(ii) If $x, x' \in A_p$ and $y, y' \in B_q$ are cycles such that $x - x' = dw$ and $y - y' = dz$ for some w and z, then $[x \otimes y] = [x' \otimes y']$ in $H_{p+q}(A \otimes B)$.

Proof. $d(w \otimes y) = (x - x') \otimes y + (-1)^{p+1} w \otimes \underset{0}{\cancel{dy}} =$
$x \otimes y - x' \otimes y$, so $[x \otimes y] = [x' \otimes y]$ in $H_*(A \otimes B)$. Likewise, $d(x' \otimes z) =$
$$\underset{0}{\cancel{dx'}} \otimes z + (-1)^p x' \otimes (y - y') = (-1)^p (x' \otimes y - x' \otimes y'),$$
so $[x' \otimes y] = [x' \otimes y']$ in $H_*(A \otimes B)$.

page ~~100~~ 99, lines 1-9, continued

The preceding shows that the map

$$[x]\otimes[y] \xrightarrow[\text{“}\otimes\text{”}]{\mu_*} [x\otimes y]$$ is well-defined.

(iii) The map ↑ is bilinear.

This is easy given the well-definition of μ_*: $([x_1]+[x_2])\otimes[y] = [x_1+x_2]\otimes[y] =$
$[(x_1+x_2)\otimes y] = [(x_1\otimes y)+(x_2\otimes y)] = ([x_1]\otimes[y])+([x_2]\otimes[y])$,
and similarly we have $[x]\otimes([y_1]+[y_2]) =$
$([x]\otimes[y_1])+([x]\otimes[y_2])$.

Furthermore, if $c\in\mathbb{D}$ then $c\cdot([x]\otimes[y]) =$
$c[x\otimes y] = [c(x\otimes y)] = [(cx)\otimes y] = [cx]\otimes[y]$,
and similarly we have $c([x]\otimes[y]) = [x]\otimes[cy]$
(recall that $c(a\otimes b) = (ca)\otimes b = a\otimes cb$ in
the tensor product $M\otimes_{\mathbb{D}} N$, where $\mathbb{D}$ is a
commutative ring with unit).

page 103, line 5

Verification that the Alexander-Whitney map (on line 4) is a chain map.

As in many other instances, we start with the universal example where $T:\Delta_n\to\Delta_n\times\Delta_n$ is the diagonal map, and we write this in the form $(\underset{\substack{\uparrow\\ \text{1st coord}}}{x_0\dots x_n}; \underset{\substack{\uparrow\\ \text{2nd coord.}}}{y_0\dots y_n})$. Then

$$d_n\psi(x_0\dots x_n; y_0\dots y_n) = d\sum_{p=0}^{n} x_0\dots x_p\otimes y_p\dots y_n =$$

$$\sum_{p=0}^{n} d(x_0\dots x_p\otimes y_p\dots y_n) =$$

$$\sum_{p=0}^{n} d(x_0\dots x_p)\otimes y_p\dots y_n + (-1)^p\sum_{p=0}^{n} x_0\dots x_p\otimes d(y_p\dots y_n)$$

$$= \sum_{p=0}^{n}\sum_{i=0}^{p}(-1)^i\, x_0\dots \overset{\text{OMIT}}{x_i}\dots x_p\otimes y_p\dots y_n +$$

$$\sum_{p=0}^{n}\sum_{i=p}^{n} x_0\dots x_p\otimes y_p\dots \overset{\text{OMIT}}{y_i}\dots y_n\cdot(-1)^i$$

page 10~~4~~3, line 5, continued

On the other hand,

$$\psi d_n\,(x_0\cdots x_n;\, y_0\cdots y_m) = \psi \sum_{j=0}^{n} (-1)^j (x_0\cdots \hat{x}_j \cdots x_n;\, y_0 \cdots \hat{y}_j \cdots y_m)$$

$$= \sum_{j=0}^{n} (-1)^j \Big[\sum_{p\le j} x_0\cdots x_p \otimes y_p \cdots \overset{\text{OMIT}}{\widehat{y_j}} \cdots y_n \;+$$

$$\sum_{p\ge j} x_0 \cdots \overset{\text{OMIT}}{\widehat{x_j}} \cdots x_p \otimes y_p \cdots y_n \Big].\ I_6$$

we subtract the second expression from the first, we are left with

$$\sum_{p=0}^{n} (-1)^p\, x_0 \cdots \overset{\text{OMIT}}{\widehat{x_p}} \otimes y_p \cdots y_n \;+$$

$$\sum_{p=0}^{n} (-1)^{p-1} x_0 \cdots x_{p-1} \otimes \overset{\text{OMIT}}{\widehat{y_{p-1}}}\, y_p \cdots y_n$$

which is zero. Hence $\psi d_n = d_n \psi$ on $(x_0 \cdots x_n;\, y_0 \cdots y_n)$.

page ~~104~~ 103, line 5, continued

General case $T : \Delta_n \to X \times Y$, $T = (T_X, T_Y)$

Note that $T_* \Psi = \Psi T_\#$ by construction, for both evaluated at $\underset{= U}{(x_0 \cdots x_n; y_0 \cdots y_n)}$ yield
$$\sum_{p=0}^{n} \mathrm{Front}_p(T_X) \otimes \mathrm{Back}_{n-p}(T_Y).$$

Then $\Psi dT = \Psi d T_\#(U) = \Psi T_\# d(U) =$

$T_\# \Psi d(U) \underset{\text{PREV}}{=} T_\# d\Psi(U) = dT_\# \Psi(U) =$

$d\Psi T_\#(U) = d\Psi(T)$.

page 105, line 16

$$d\varphi(F_X\otimes B_Y) = d(F_X\times B_Y)_\# \varphi(id_p\otimes id_{n-p}) =$$

$$(F_X\times B_Y)_\# \varphi d(id_p\otimes id_{n-p}) \overset{\text{check}}{\underset{\text{this!}}{=}}$$

$$\varphi\circ(F_X\times B_Y)_\# d(id_p\otimes id_{n-p}) =$$

$$\varphi d(F_X\times B_Y)_\#(id_p\otimes id_{n-p}) = \varphi d(F_X\otimes B_Y)$$

Therefore d is a chain map.

page 108 lines 20-21

Relative cross products for closed subsets.

$A \subseteq U \subseteq X$, $B \subseteq V \subseteq Y$ s.t.
closed open, closed open

A is a def. retract of U & B is a def. retract of B.

Then we have

$$
\begin{tikzcd}
H^*(X,A) \otimes H^*(Y,B) \arrow[ddrr, dashed] & & H^*(X,U) \otimes H^*(Y,B) \arrow[ll, "\cong"'] \arrow[d, "\text{prod}", "\times"'] \\
& & H^*(X \times Y, U \times Y \cup X \times V) \arrow[d, "(\text{inclusion})^*"] \\
& & H^*(X \times Y, A \times Y \cup X \times B)
\end{tikzcd}
$$

Note If (K, L) is a simplicial complex pair with underlying space pair $(|K|, |L|)$, then $|L|$ is a deformation retract of an open neighborhood W in $|K|$

page ~~107~~ lines 12-13

WORK ON THE CHAIN/COCHAIN LEVEL (see notes)

Proof of Proposition 14.

(i) Let $c \in S_q(X)$ be a chain. Then

$$\varepsilon_X \frown c = \underbrace{\varepsilon_X(T|\{e_0\})}_{=1} . \underbrace{\mathrm{Back}_q(c)}_{=c} = c$$

and $c \frown \varepsilon_X = \mathrm{Front}_q(c) \cdot \varepsilon_X(T|\{e_q\}) =$ c likewise.

[$\mathrm{Back}_q = \mathrm{id}$ on q-simplices!]

(ii) Check that both cochains $(f \cup g) \frown c$ and $f \frown (g \frown c)$ are equal to $f(T|e_0 \ldots e_q) \cdot g(T|e_q \ldots e_p) \cdot \mathrm{Back}_r\, c$

where $r = n - p - q$.

(iii) Let $g \in S^p(Y)$ and $c \in S_n(X)$. Then

$g \frown f_\# c = g(\mathrm{Front}_p(f_\# c)) \cdot \mathrm{Back}_{n-p}(f_\# c) =$

$g(f_\# \mathrm{Front}_p\, c) \cdot f_\# \mathrm{Back}_{n-p}(c) =$

$f^\# g(\mathrm{Front}_p\, c) \cdot f_\# (\mathrm{Back}_{n-p}(c))$ $\dfrac{f_\# \text{ is a}}{\text{module hom}}$

page 10~~7~~, lines 12-13 continued

$$f_{\#}\left(f_g^{\#}(\mathrm{Front}_p c)\cdot \mathrm{Back}_{n-p}(c)\right) =$$
$$f_{\#}\left(f_g^{\#} \cap c\right).$$

By the discussion preceding the statement of Prop 14, the corresponding identities in homology and cohomology follow directly from these.

page 108 lines −5 to −4

Proof that
(1) the Alexander Whitney map $S_*(X) \longrightarrow S_*(X) \otimes S_*(X)$

is coassociative,
(2) the Alexander-Whitney map is functorial with respect to cont. maps $f: X \to Y$.

---

(1) Both $(\Psi \otimes id) \circ \Psi$ and $(id \otimes \Psi) \circ \Psi$ send

$T: \Delta_n \to X$ to $\sum_{0 \le s \le t \le n} 2(T|e_0 \ldots e_s) \otimes (T|e_s \ldots e_t) \otimes (T|e_t \ldots e_n)$.

(2) The goal is to verify that the diagram

$$\begin{array}{ccc} S_*(X) & \xrightarrow{\Psi_X} & S_*(X) \otimes S_*(X) \\ f_* \downarrow & & \downarrow f_* \otimes f_* \\ S_*(Y) & \xrightarrow{\Psi_Y} & S_*(Y) \otimes S_*(Y) \end{array}$$

commutes.

If we apply both/either of these composites to $T: \Delta_n \to X$, the result is

$$\sum \mathrm{Front}_p(f \circ T) \otimes \mathrm{Back}_{n-p}(f \circ T).$$

page ~~110~~ 109, line 10

Singular augmentations are co-units

(1) $S_*(X) \xrightarrow{\Psi} S_*(X) \otimes S_*(X) \longrightarrow \mathbb{D} \otimes S_*(X) \cong S_*(X)$.

Evaluate on $T : \Delta_n \to X$:

$$T \mapsto \sum_p \mathrm{Front}_p(T) \otimes \mathrm{Back}_{n-p}(T) \mapsto \varepsilon(\mathrm{Front}_0(T)) \otimes T \approx T$$

$\Rightarrow$ composite is the identity.

(2) $S_*(X) \xrightarrow{\Psi} S_*(X) \otimes S_*(X) \longrightarrow S_*(X) \otimes \mathbb{D} \cong S_*(X)$

Evaluate on $T : \Delta_n \to X$:

$$T \mapsto \sum_p \mathrm{Front}_p(T) \otimes \mathrm{Back}_{n-p}(T) \mapsto T \otimes \varepsilon(\mathrm{Back}_0(T)) \approx T$$

$\Rightarrow$ composite is the identity.