IV.4 : Grade-commutativity and examples

(Hatcher, §§ 3.2, 3.B)

DEFAULT HYPOTHESES. Unless explicitly stated otherwise, througout this section $\mathbb{D}$ will denote a commutative ring with unit, all chain complexes will be assumed to be modules over $\mathbb{D}$, and tensor products will be assumed to be given over $\mathbb{D}$.

It was fairly easy to prove that the cup product on cohomology is associative, and in fact this is also true on the cochain level. Furthermore, it was even easier to prove that the augmentation cocycle $\varepsilon : S_*(X) \to \mathbb{D}$ is a two sided identity in the absolute case (pairs where the subspace is empty). We shall now consider commutativity properties of cup products, both at the cohomology level and at the cochain level.

One can do direct calculations to show that the cup product is usually not commutative in the standard sense. For example, one can check this in the simplicial cohomology of complexes homeomorphic to $T^2 = S^1 \times S^1$. Our results contain both good news and bad news:

> Good news. On the cohomology level the cup product is **grade-commutative** in the sense that if $\alpha \in H^p(X, \mathbb{D})$ and $\beta \in H^q(Y, \mathbb{D})$, then $\beta \cup \alpha = (-1)^{pq} \alpha \cup \beta$.
>
> Bad news. On the cochain level the cup product is usually not even grade-commutative, although it is so up to a system of *higher chain homotopies* (however, we shall only show commutativity up to an ordinary chain homotopy).

In particular, it turns out that the Steenrod squares and reduced powers in Section 4.L of Hatcher are defined using such higher chain homotopies and in fact imply the impossibility of constructing a grade-commutative cup product on the cochain level for coefficients in a field $\mathbb{F}$ of finite characteristic. On a more positive note, relatively recent results of M. Mandell show that if X is reasonably nice — for example, if X is a polyhedron — then the homotopy type of X is determined by the singular chain complex together with the a suitably defined structure of higher chain homotopies for cup product commutativity. Here is the reference:

> **M. A. Mandell.** *Cochains and homotopy type*, Publ. Math. Inst. Hautes Études Sci. **103** (2006), 213–246.

In contrast, if we work over a field $\mathbb{F}$ of characteristic zero, then it is possible to define cohomology groups using cochain constructions that are grade commutative (on the cochain level). There is a more extensive discussion of commutative cochains on pages 110–111 of the following book:

> **P. A. Griffiths and J. W. Morgan.** Rational homotopy theory and differential forms, Progress in Mathematics Vol. 16. Birkhäuser, Boston, MA, 1981.

In the next unit we shall discuss some fundamental constructions which are closely related to the topics covered in this book.

Since homology and cohomology are essentially dual to each other, the existence of a product structure in the latter but not the former may seem puzzling. However, it turns out that one can resolve this by a more systematic approach to dualization. One can view a multiplicative structure on a (graded) algebraic object as a (grade-preserving0 homomorphism $\mu : A \otimes A \to A$; dually, one can define a COMULTIPLICATIVE structure as a homomorphism $\psi : A \to A \otimes A$; such a structure can also be called a **coproduct**. Every concept that is meaningful for an algebra or product has a natural dual concept which is meaningful for a coalgebra or coproduct. For example, just as an algebra is associative if and only if the diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu\otimes 1} & A \otimes A \\ \big\downarrow 1 \otimes \mu & & \big\downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

commutes, we can say that an algebra is *coassociative* if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A \otimes A \\ \big\downarrow \psi & & \big\downarrow 1 \otimes \psi \\ A \otimes A & \xrightarrow{\psi\otimes 1} & A \otimes A \otimes A \end{array}$$

commutes. All of this is intrinsically formal, but the next result shows that such structures actually arise in concrete situations.

PROPOSITION 1. *If X is a topological space, let Δ_X be the diagonal and let $\psi_{X,X}$ be the Alexander-Whitney map for $X \times X$. Then the chain map $\Psi_X = \psi_{X,X} \circ \Delta_{X\#}$ defines a coassociative comultiplication on $S_*(X)$. Furthermore, if $f : X \to Y$ is a continuous mapping, then the induced map of singular chain complexes is a morphism of coalgebras.*

This follows directly from the definition of the Alexander-Whitney map (details are left to the reader$^{(\star)}$).∎

COROLLARY 2. *If our underlying commutative ring with unit is a field $\mathbb{F}$, then the chain level comultiplication induces a comultiplication in homology, and this comultiplication is functorial and coassociative.*∎

The conceptual point of the proposition and corollary is that one can view the multiplicative structure in cohomology as the dual of the given comultiplicative structure in homology.

We can view a *two-sided identity* in an algebra as a homomorphism from $\mathbb{D} \to A$ such that the composites

$$A \cong \mathbb{D} \otimes A \to A \otimes A \to A\ , \qquad A \cong A \otimes \mathbb{D} \to A \otimes A \to A$$

are the identity mapping (in the graded case, we assume $\mathbb{D}$ is contained in degree zero). The dual notion is basically just an augmentation $A \to \mathbb{D}$, and of course it is supposed to satisfy the dual conditions that the composite mappings

$$A \to A \otimes A \to \mathbb{D} \otimes A \cong A \ , \qquad A \to A \otimes A \to A \otimes \mathbb{D} \cong A$$

are the identity. It is routine to verify that the standard augmentation maps on singular chain complexes have this property, so in fact the singular chain complex may be viewed as a functor from spaces to coassociative coalgebra chain complexes with augmentations.

Algebraic and topological twist maps

The preceding discussion indicates that grade-commutativity properties of cup products should be dual to *grade-cocommutativity* properties of the functorial comultiplication on singular chain complexes. At this point we need to introduce some algebra.

Definition. Suppose that A_* and B_* are chain complexes over $\mathbb{D}$. The transposition or twist isomorphism

$$\tau_{A,B} : A_* \otimes B_* \longrightarrow B_* \otimes A_*$$

is determined by the identity $\tau_{A,B}(a_p \otimes b_q) = (-1)^{pq}(b_q \otimes a_p)$, where $a_p \in A_p$ and $b_q \in B_q$.

It follows immediately that τ is a functorial chain map and $\tau_{B,A} \circ \tau_{A,B}$ is the identity$^{(\star)}$.

One motivation for this construction is the following result:

THEOREM 3. *Let X and Y be topological spaces, and let $T : X \times Y \to Y \times X$ be the map $T(x,y) = (y,x)$ which transposes coordinates. Then there is a commutative diagram up to natural chain homotopy*

$$\begin{array}{ccc} S_*(X \times Y) & \xrightarrow{\psi} & S_*(X) \otimes S_*(Y) \\ \downarrow T_\# & & \downarrow \tau \\ S_*(X \times Y) & \xrightarrow{\psi} & S_*(X) \otimes S_*(Y) \end{array}$$

in which the horizontal maps are Alexander-Whitney maps and τ is the algebraic transposition map on $S_*(X) \otimes S_*(Y)$.

Proof. The first things to observe is that all maps of chain complexes are augmentation preserving and the diagram commutes in degree zero. Assume inductively that the diagram commutes up to natural chain homotopy through dimension $n - 1 \geq 0$. We shall use the method of acyclic models to construct the chain homotopy in degree n for the universal example of the diagonal singular simplex $\mathrm{Diag}_n : \Delta_n \to \Delta_n \times \Delta_n$ and extend it to all singular simplices by naturality.

To construct the chain homotopy on the universal example in degree n, we need to find some $\varphi_n \in [S_*(\Delta_n) \otimes S_*(\Delta_n)]_{n+1}$ such that

$$d\varphi_n \quad = \quad \psi \circ T_\# (\mathrm{Diag}_n) \ - \ \tau \circ \psi (\mathrm{Diag}_n) \ - \ D \circ d (\mathrm{Diag}_n) \ .$$

Since $S_*(\Delta_n) \otimes S_*(\Delta_n)$ is acyclic, we can find a suitable element φ_n if and only if the right hand side is a cycle. As in previous examples, this can be shown using the facts that ψ, T and τ are chain maps, the chain complex identity $d \circ d = 0$, and the fact that D is a chain homotopy through degree $n-1$; details are left to the reader.■

We are now ready to state the grade-commutativity properties of the cross product and their implications for grade-commutativity of the cup product.

THEOREM 4. *Let X and Y be topological spaces, let $u \in H^p(X)$ and $v \in H^q(Y)$ be cohomology classes, and let $T : X \times Y \to Y \times X$ denote the transposition homeomorphism. Then the cohomology cross product satisfies $u \times v = (-1)^{pq} T^*(v \times u)$.*

COROLLARY 5. *If X is a space with $u \in H^p(X)$ and $v \in H^q(X)$, then $u \cup v = (-1)^{pq} v \cup u$.*

Proof of Corollary 5. If $\Delta_X : X \to X \times X$ is the diagonal, then $u \cup v = \Delta_X^*(u \times v)$, and if $T : X \times X \to X \times X$ transposes coordinates, then $T \circ \Delta_X = \Delta_X$. Therefore by Theorem 5 we have

$$u \cup v \;=\; \Delta_X^*(u \times v) \;=\; (T \circ \Delta_X)^*(u \times v) \;=\; \Delta_X^* \circ T^*(u \times v) \;=\; \Delta_X^*\left((-1)^{pq} v \times u\right) \;=\; (-1)^{pq} v \cup u$$

which is what we wanted to prove.■

Proof of Theorem 4. Choose cocycles f and g representing u and v respectively, and let $\psi_{X,Y}$ and $\psi_{Y,X}$ be the Alexander-Whitney maps for $X \times Y$ and $Y \times X$. Then we have the following diagram in which the right hand square commutes and the left hand square commutes up to chain homotopy:

$$\begin{array}{ccccc} S_{p+q}(X \times Y) & \xrightarrow{\text{proj}(p,q)\psi} & S_p(X) \otimes S_q(Y) & \xrightarrow{f \otimes g} & \mathbb{D} \otimes \mathbb{D} \cong \mathbb{D} \\ \Big\downarrow T_\# & & \Big\downarrow \tau & & \Big\downarrow = \\ S_{p+q}(Y \times X) & \xrightarrow{\text{proj}(p,q)\psi} & S_p(X) \otimes S_q(Y) & \xrightarrow{(-1)^{pq} g \otimes f} & \mathbb{D} \otimes \mathbb{D} \cong \mathbb{D} \end{array}$$

The map $\text{proj}(p,q)$ is projection onto the direct summand $S_p(X) \otimes S_q(Y)$ in $[S_*(X) \otimes S_*(Y)]_{p+q}$, while the top row is a cochain level representative for $u \times v$ and the bottom row is a cochain level representative for $(-1)^{pq} u \times v$.

Let E be the chain homotopy relating $\psi_{Y,X} \circ T_\#$ and $\tau \circ \psi_{X,Y}$. Then we have

$$f \times g \;=\; (f \otimes g) \circ \psi_{X,Y} \;=\; (-1)^{pq} (g \otimes f) \circ \tau \circ \psi_{X,Y} \;=\; (-1)^{pq} (g \otimes f) \circ \left(\psi_{Y,X} \circ T_\# + dE + Ed\right) \;=$$

$$T^\# \left((-1)^{pq} (g \otimes f) \circ \psi_{Y,X} \right) \;+\; (-1)^{pq} \delta(g \times f) \circ E \;+\; \delta U$$

where δ is the coboundary map and U is some cochain whose precise value is unimportant because it disappears when we take cohomology classes. The term $\delta(g \times f)$ vanishes because f and g are cocycles, and therefore the displayed identities show that the cohomology classes represented by $f \times g$ and $(-1)^{pq}(g \times f)$ — namely, $u \times v$ and $(-1)^{pq} T^*(v \times u)$ — must be equal.■

The results of this and the previous section yield complete information on the cup product structure for a product of spheres with coefficients in a field. This can be done inductively using the theorem stated below. Before stating this result, we need the following construction:

Definition. Let A_* and B_* be graded algebras over $\mathbb{D}$ with multiplication maps μ_A and μ_B respectively. Then the tensor product $A_* \otimes B_*$ has a multiplication given by

$$(A_* \otimes B_*) \otimes (A_* \otimes B_*) \to A_* \otimes A_* \otimes B_* \otimes B_* \to A_* \otimes B_*$$

where the first map is the middle four interchange

$$x_p \otimes y_q \otimes z_r \otimes w_s \quad \longrightarrow \quad (-1)^{qr}\, x_p \otimes z_r \otimes y_q \otimes w_s$$

and the second map is $\mu_A \otimes \mu_B$.

THEOREM 6. *Let n be a positive integer, let $\mathbb{F}$ be a field, and let X be a space such that $H^k(X;\mathbb{F})$ is finite dimensional for all k. Then the cohomology algebra $H^*(S^n \times X;\mathbb{F})$ is isomorphic to the tensor product algebra $H^*(S^n) \otimes_{\mathbb{F}} H^*(X)$.*

This result is an immediate consequence of the cohomological Künneth Theorem; details are again left to the reader.∎

The preceding theorem and induction yield the computation for the cohomology of a product

$$\prod_{k=1}^{r} S^{n(k)}$$

where each $n(k)$ is positive.

COROLLARY 7. *In the setting above we have*

$$H^*\left(\textstyle\prod_{k=1}^{r} S^{n(k)};\mathbb{F}\right) \quad \cong \quad \bigotimes_{k=1}^{r} H^*\left(S^{n(k)};\mathbb{F}\right) \ .∎$$

The following results are immediate consequences of this corollary:

COROLLARY 8. *In the setting above, assume that for each k the cohomology class u_k is the image of a generator for $H^{n(k)}(S^{n(k)};\mathbb{F})$ under the map induced by the coordinate projection p_k onto the k^{th} factor. Then*

$$\prod_{k} u_k \quad \neq \quad 0 \ .$$

This is merely an iteration of the fact that the cross product of two nontrivial cohomology classes is always nontrivial.∎

COROLLARY 9. *In the setting above, assume that the dimensions $n(k)$ are all even and equal to some fixed integer n (hence the cup product is commutative in the usual sense), and assume further that for each k the cohomology class u_k is the image of a generator for $H^(S^n;\mathbb{F})$ under the map induced by the coordinate projection p_k onto the $k^{\rm th}$ factor. Then*

$$\left(\sum_k \; u_k\right)^r \quad \neq \quad 0 \; .$$

This reduces to a purely algebraic computation, which shows that the class in question is equal to $n! \cdot \prod_k \; u_k$ (the details are again left as an exercise$^{(\star)}$).■

IV.5 : Two applications

(Hatcher, §§ 3.2, 4.2)

Although we have only obtained relatively weak versions of the basic results on products in singular homology and cohomology theory, they suffice to yield two fairly significant results. One is a restriction on the maps in homology associated to a homotopy self-equivalence from $S^{2m} \times S^{2m}$ to itself, and the other is a proof that for all $m > 1$ there is a continuous mapping from S^{4m-1} to S^{2m} which is not homotopic to a constant. The existence of such maps reflects several of the fundamental difficulties one encounters when trying to study homotopy theory.

Coefficient homomorphisms in singular homology and cohomology

We would like to have some way of extracting information about homology and cohomology groups with integer coefficients from computations of homology and cohomology with coefficients in various fields (not surprisingly, the usual examples are the rationals $\mathbb{Q}$ and the prime fields $\mathbb{Z}_p$ where p is prime). There are two or three main ideas.

(1) If $\varphi : \mathbb{D} \to \mathbb{E}$ is a homomorphism of commutative rings with unit, then there are associated natural *coefficient homomorphisms* $\varphi_\#$ and $\varphi^\#$ of singular chain and cochain groups. These are compatible with the cup and cap product structures, and they induce corresponding natural transformations in homology and cohomology which commute with the connecting homomorphisms in the long exact sequences of pairs and Mayer-Vietoris exact sequences.

(2) Let $A \to A_{(0)}$ be the rationalization functor on abelian groups which is defined in Section VII.5 of `algtop-notes.pdf`. Then there is a natural isomorphism of ∂-functors from $H_*(X, A; \mathbb{Z})_{(0)}$ to $H_*(X, A; \mathbb{Q})$ which commutes with the connecting homomorphisms in the long exact sequences of pairs and Mayer-Vietoris exact sequences.

(3) The Universal Coefficient Theorems in Hatcher provide the "right" way of extracting information about homology and cohomology groups with integer coefficients from computations involving $\mathbb{Z}_p$ coefficients. — We are avoiding this to minimize the amount of algebraic machinery developed in the course.

The first of these is easy to show; given a pair of spaces (X, A), the natural map $S_*(X, A) \otimes \mathbb{D} \to S_*(X, A) \otimes \mathbb{E}$ is just the tensor product of the identity on $S_*(X, A)$ with φ, and the map on cochains takes $f : S_*(X, A) \otimes \mathbb{D}$ to $f \circ \varphi$. All of the assertions about these maps then follow by purely formal considerations. The second principle follows immediately from Corollary VII.5.3 in `algtop-notes.pdf`.■

Similar considerations hold for simplicial chain and cochain groups, and this is true for groups defined with respect to orderings of vertices and groups defined without such orderings. Furthermore, the coefficient homomorphisms commute with the maps defined by passage from ordered to unordered simplicial chains, and from unordered simplicial chains to singular chains.■

Cellular homology and cohomology with coefficients. Let $(X, \mathcal{E})$ be a finite cell complex, and let X_k denote the k-skeleton of this complex. Then one can define cellular chain groups with coefficients in an arbitrary commutative ring with unit $\mathbb{D}$, and the proof given in the integer case extends directly to show that the groups $H_*(X; \mathbb{D})$ are isomorphic to the homology groups of the complex $C_*(X, \mathcal{E}; \mathbb{D})$. One can also define cellular cochain complexes $C^*(X, \mathcal{E}; \mathbb{D})$ such that $C^k(X, \mathcal{E}; \mathbb{D}) = H^k(X_k, X_{k-1}; \mathbb{D})$ — which will be a free abelian group whose rank is the number of k-cells — and a dualization of the earlier arguments shows that the cohomology of X with coefficients in $\mathbb{D}$ is isomorphic to the cohomology of the cellular cochain complex (the details are left to the reader).

It is not difficult to guess how cellular chain and cochain complexes behave under the coefficient homomorphism associated to a ring homomorphism $\varphi : \mathbb{D} \to \mathbb{E}$:

PROPOSITION 0. *In the setting above, let $F(\mathbb{D})$ denote the chain or cochain group $C_k(X, \mathcal{E}; \mathbb{D})$ or $C^k(X, \mathcal{E}; \mathbb{D})$, and let $\varphi_* : F(\mathbb{D}) \to F(\mathbb{E})$ be the coefficient map induced by φ. Use the standard free generators for the chain or cochain group (corresponding to the k-cells in X) to identify $F(\mathbb{D})$ and $F(\mathbb{E})$ with $F(\mathbb{Z}) \otimes \mathbb{D}$ and $F(\mathbb{Z}) \otimes \mathbb{E}$ respectively. Then the coefficient homomorphism φ_* corresponds to $\mathbf{id}[F(\mathbb{Z})] \otimes \varphi$.*

Sketch of proof. By naturality considerations it suffices to prove the analogous result for the homology of (D^k, S^{k-1}). This case can be treated explicitly using the ordered simplicial chain complex for the pair $(\Delta_k, \partial\delta_k)$.■

Cell decompositions for products of spheres

Let n be a positive integer, and let $\mathbb{D}$ be a commutative ring with unit.

If we take the simplest cell decomposition for S^n with a 0-cell and an n-cell, then the product construction yields a cell decomposition of $S^n \times S^n$ with one 0-cell, two n-cells and one $2n$-cell. If $n \geq 2$ then there are no possible nonzero differentials in the cellular chain complex for computing $H_*(S^n \times S^n; \mathbb{D})$ and hence one can read off the homology immediately. If $\sigma \in H_n(S^n; \mathbb{D}) \cong \mathbb{D}$ is a generator and i_1, i_2 are the usual slice inclusions, then the classes $i_{1*}\sigma$ and $i_{2*}\sigma$ form a free basis for $H_n(S^n \times S^n; \mathbb{D})$. The top cell of this complex is attached to the n-skeleton, which is a wedge of two copies of S^n by a continuous map

$$P : S^{2n-1} \longrightarrow S^n \vee S^n$$

that we shall call the *universal Whitehead product.*

Let n be as in the preceding paragraph, and let $PT^n \subset T^n$ denote the $(n-1)$-skeleton with respect to the standard cell decomposition of T^n described earlier. Then the quotient space T^n/PT^n is homeomorphic to S^n; let $\kappa : T^n \to S^n$ denote the associated collapsing map. It follows that κ_* and κ^* induce isomorphisms in n-dimensional homology and cohomology (say with field coefficients in the second case). Furthermore, it follows that $\kappa \times \kappa : T^n \times T^n \to S^n \times S^n$ induces a monomorphism in cohomology; verifying this is a fairly straightforward exercise using the corresponding property of κ^*, the known structure of $H^*(S^n \times S^n)$, and the known structure of $H^*(T^n \times F^n)$.

The preceding discussion reduces the computation of the cohomology cup product for $S^n \times S^n$ to questions about the corresponding structure for $T^{2n} = T^n \times T^n$. Here is a formal statement of the conclusions:

PROPOSITION 1. *Let $\Omega \in H^n(S^n)$ be such that the Kronecker index $\langle \Omega, \sigma \rangle = 1$, and let π_1, π_2 denote the projections of $S^n \times S^n$ onto the factors. Then the cohomology classes $\pi_j^*\Omega$ are dual to the homology classes $i_{j*}\sigma$ with respect to the Kronecker index pairing, and these classes satisfy the following conditions:*

(i) *Their cup squares are zero.*

(ii) *The class $\pi_1^*\Omega \cup \pi_2^*\Omega$ generates $H^{2n}(S^n \times S^n)$.*

Proof. In the cellular decomposition for $S^n \times S^n$ described above, there are no cells in adjacent dimensions, and hence the cellular chain and cochain complexes have trivial differentials. Thus the cellular decomposition and the discussion of cellular cohomology with coefficients imply that $H^k(S^n \times S^n; \mathbb{D})$ is isomorphic to $\mathbb{D} \oplus \mathbb{D}$ if $k = n$, $\mathbb{D}$ if $k = 0$ or $2n$, and zero otherwise. Furthermore, by construction $H^n(S^n \times S^n; \mathbb{D})$ is freely generated by the classes $\pi_1^*\Omega_{\mathbb{D}}$ and $\pi_2^*\Omega_{\mathbb{D}}$, where $\Omega_{\mathbb{D}}$ is the image of Ω under the coefficient homomorphism $\varphi : \mathbb{Z} \to \mathbb{D}$ sending 1 to the identity in $\mathbb{D}$.

The first conclusion holds because Ω^2 in the cohomology of S^n and the maps π_t^* are multiplicative. To prove the second statment, let p be a prime and take $\mathbb{D} = \mathbb{Z}_p$. Then the Künneth Theorem for cohomology implies that $\pi_1^*\Omega_{\mathbb{D}} \cup \pi_2^*\Omega_{\mathbb{D}}$ generates $H^{2n}(S^n \times S^n; \mathbb{Z}_p) \cong \mathbb{Z}_p$. Therefore by Proposition 0 the image of $\pi_1^*\Omega \cup \pi_2^*\Omega$ in $\mathbb{Z}_p$ is a generator for all primes p, and it follows that $\pi_1^*\Omega \cup \pi_2^*\Omega$ must be a generator for $H^{2n}(S^n \times S^n) \cong \mathbb{Z}$ (otherwise, its image in some $\mathbb{Z}_p$ would be trivial).

Finally, we also note that if n is even then grade-commutativity of cup products implies that $\pi_1^*\Omega \cup \pi_2^*\Omega = \pi_2^*\Omega \cup \pi_1^*\Omega$.■

These computations lead directly to our first application.

THEOREM 2. *Suppose that $m \geq 1$ and f is a homotopy self-equivalence of $S^{2m} \times S^{2m}$. Let σ_1 and σ_2 denote the free basis for $H_{2m}(S^{2m} \times S^{2m}; \mathbb{Z})$ described earlier. Then either the associated map in homology f_* sends the σ_j to $\varepsilon_j \sigma_j$, where $\varepsilon_j = \pm 1$, or else f_* sends σ_1 to $\varepsilon_1 \sigma_2$ and sends σ_2 to $\varepsilon_1 \sigma_1$ where again $\varepsilon_j = \pm 1$.*

All of the possibilities in the theorem can be realized. For the first alternatives this can be done by considering various product of the form 1, $1 \times \rho$, $\rho \times 1$ and $\rho \times \rho$, where ρ is the reflection involution on a sphere, and the second alternatives can be realized by composing the first alternatives with the transposition map τ on $S^{2m} \times S^{2m}$.

Suppose now that n is an arbitrary positive integer. Since $H_n(S^n \times S^n; \mathbb{Z}) \cong \mathbb{Z}^2$, the only general algebraic restriction one can get on a map f_* induced by a homotopy self-equivalence is that it must correspond to a 2×2 matrix over the integers with determinant equal to ± 1. It is fairly simple to construct examples of homotopy self equivalences of

T^2 which realize every such matrix (the associated linear transformations of $\mathbb{R}^2$ pass to homeomorphisms of T^2). If n is odd, then the possible 2×2 matrices are also understood, but this is a deeper result; the conclusion is that one can realize every matrix if $n = 1, 3, 7$, while for the remaining odd values of n it is possible to realize every integral 2×2 matrix with determinant ± 1 whose reduction mod 2 is a permutation matrix. For the exceptional odd values of n, one can show this using standard "multiplications" on S^n (given by restricting complex, quaternionic, and Cayley number multiplication to the unit sphere in $\mathbb{R}^{n+1}$ where $n + 1 = 2, 4, 8$). For the remaining odd values of n, this fact is due to J. F. Adams and was proved in the nineteen fifties. Here are (very terse) references for the latter:

`http://mathworld.wolfram.com/H-Space.html`

`http://mathworld.wolfram.com/HopfInvariantOneTheorem.html`

Proof of Theorem 2. As noted in the preceding paragraph, if σ_1 and σ_2 are the given standard free basis for $H_{2m}(S^{2m} \times S^{2m}; \mathbb{Z}) \cong \mathbb{Z}^2$, then there are integers a, b, c, d such that $ad - bc = \pm 1$ and $f_*(\sigma_1) = a\sigma_1 + b\sigma_2$, $f_*(\sigma_2) = c\sigma_1 + d\sigma_2$. By the naturality of homology with respect to coefficient homomorphisms, it follows that one has a similar description of f_* with rational coefficients. If we take the dual basis ξ_1, ξ_2 of $H^{2m}(S^{2m} \times S^{2m}; \mathbb{Q})$, then it follows that $f^*\xi_1 = a\xi_1 + c\xi_2$ and $f^*\xi_2 = b\xi_1 + d\xi_2$. Since f preserves cup products and $\xi_j^2 = 0$, the same is true for $f^*(\xi_j)$. But Proposition 1 implies that

$$f^*(\xi_1)^2 \quad = \quad 2ac\, \xi_1 \cup \xi_2 \ , \qquad f^*(\xi_2)^2 \quad = \quad 2bd\, \xi_1 \cup \xi_2$$

and since $\xi_1 \cup \xi_2$ is nonzero it follows that $ac = bd = 0$, so that either $a = 0$ or $c = 0$ and also either $b = 0$ and $d = 0$. The cases $a = b = 0$ and $c = d = 0$ both imply that $ad - bc = 0$, so neither can hold, and therefore the only possibilities are $a = d = 0$ or $c = b = 0$. In the first case the condition $ad - bc$ implies that $b, c \in \{\pm 1\}$, while in the second case we must have $a, d \in \{\pm 1\}$. These are precisely the options listed in the theorem.■

Homotopically nontrivial mappings of spheres

If $m < n$ then simplicial approximation implies that every continuous mapping from S^m to S^n is homotopically trivial, and if $m = n$ we know that there are infinitely many homotopy classes of maps $S^n \to S^n$ which can be distinguished homotopically by their degrees; we have not proved that two maps of the same degree are homotopic, but it would not be exceedingly difficult for us to do so at this point (for example, see the argument in Maunder, *Algebraic Topology*, pages 288–291; the statement of this result in Hatcher is Corollary 4.25 on page 361). The important point is that if $m \leq n$, then homotopy classes of maps from S^m to S^n can be distinguished using homology theory. Given that every map from S^m to S^1 is nullhomotopic if $m > 1$, it was natural to hope that all maps $S^m \to S^n$ would be homotopic to constant maps. However, counterexamples began to surface during the nineteen thirties, and describing the homotopy classes of mappings from S^{n+k} to S^n where $k > 0$ turns out to be an exceedingly difficult problem, although it is known that the answer for any specific choice of n and k is finitely computable (although the basic algorithm seems unlikely to be implemented in the foreseeable future). We shall limit ourselves to a single class of important examples:

THEOREM 3. *Suppose that m is a positive integer. Then there is a continuous mapping $f : S^{4m-1} \to S^{2m}$ which is not homotopic to a constant.*

In fact, refinements of our methods show that there are infinitely many distinct homotopy classes of such maps. There is actually a very striking converse to this result discovered by J.-P. Serre in the nineteen fifties:

> *For all $m, n > 0$, there are only finitely many homotopy classes of continuous mappings from S^n to S^m* **unless** *$m = n$ or m is even and $n = 2m - 1$.*

One reference for this result is Section 9.7 of Spanier. The basic reference for the finite computability statement is the following paper;

> **E.** (= *Edgar*) **H. Brown.** *Finite computability of Postnikov complexes.* Annals of Mathematics **65** (1957), 1–20.

Proof. Throughout this discussion the coefficient field will be the rational numbers $\mathbb{Q}$.

The examples will be composites of the form $\nabla \circ P$, where $P : S^{4m-1} \to S^{2m} \vee S^{2m}$ is the universal Whitehead product described earlier and $\nabla : S^{2m} \vee S^{2m} \to S^{2m}$ folds the two wedge summands together (so its restriction to each summand is the identity). This class is generally known as the *Whitehead product* of the identity map on S^{2m} with itself and denoted by $[\iota_{2m}, \iota_{2m}]$ (compare Hatcher, Example 4.52, page 381). The argument wll require the following relatively elementary observation:

LEMMA 4. *Suppose that $f : S^{p-1} \to A$ is a continuous map into a compact metric space and X is the space obtained by attaching a p-cell to A along f. If f is homotopic to a constant map, then the inclusion of A in X is a retract.*

Proof of Lemma 4. If f is homotopic to a constant, then f extends to a mapping $g : D^p \to A$. Write $X = A \cup E$, where E is the p-cell. Then the retraction from X to A is defined by taking the identity on A and using g to define the mapping on E. By construction, it follows that these definitions fit together to yield a well-defined continuous retraction from X to A.■

Returning to the proof of Theorem 3, let $K(f)$ be the space obtained by adjoining a $4m$-cell to S^{2m} along the mapping $\nabla \circ P$. We then have the following commutative diagram, in which the two horizontal arrows on the left are attaching maps, the middle horizontal arrows are inclusions, and the horizontal arrows on the right are maps which collapse the codomains of the attaching maps to points.

$$\begin{array}{ccccccc}
S^{4m-1} & \xrightarrow{P} & S^{2m} \vee S^{2m} & \longrightarrow & S^{2m} \times S^{2m} & \longrightarrow & S^{4m} \\
\big\downarrow = & & \big\downarrow \nabla & & \big\downarrow h & & \big\downarrow = \\
S^{4m-1} & \xrightarrow{\nabla P} & S^{2m} & \longrightarrow & K(f) & \longrightarrow & S^{4m}
\end{array}$$

This diagram yields the following commutative diagrams in cohomology for each $q > 0$; the rows of these diagrams are short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^q(S^{4m}) & \longrightarrow & H^q(K(f)) & \longrightarrow & H^q(S^{2m}) & \longrightarrow & 0 \\ & & \big\downarrow = & & \big\downarrow h^* & & \big\downarrow \nabla^* & & \\ 0 & \longrightarrow & H^q(S^{4m}) & \longrightarrow & H^q(S^{2m} \times S^{2m}) & \longrightarrow & H^q(S^{2m} \vee S^{2m}) & \longrightarrow & 0 \end{array}$$

It follows that $H^*(K(f))$ is isomorphic to $\mathbb{Q}$ in dimensions $0, 2m, 4m$ and is trivial otherwise. Let θ denote a generator for $H^{2m}\big(K(f)\big)$. It follows that $h^*(\theta)$ is a nonzero multiple of $\xi_1 + \xi_2$, and we might as well choose θ so that it maps to this class in $H^{2m}(S^{2m} \times S^{2m})$. Furthermore, we have

$$h^*(\theta)^2 \quad = \quad 2\,\xi_1 \cup \xi_2 \quad \neq 0$$

so that θ^2 must also be nonzero in $H^{4m}\big(K(f)\big)$.

We claim that the statement in the preceding sentence implies that f cannot be nullhomotopic. If it were, then there would be a retraction $\rho : K(f) \to S^{2m}$, and θ would have to be in the image of ρ^*. But if $\theta = \rho^*\theta_0$ for some $\theta_0 \in H^*(S^{2m})$, then $\theta_0^2 = 0$ and hence $\theta^2 = 0$, contradicting the conclusions in the preceding paragraph. Hence the only possibility consistent with the latter is that f is not nullhomotopic.∎

IV.6 : Open disk coverings of manifolds

(Hatcher, § 3.2)

Every compact topological n-manifold is a union of fintely many open subsets U_i such each U_i is homeomorphic to $\mathbb{R}^n$. Since each such open subset is noncompact, it is clear that one needs at least two such open subsets, and of course S^n is an example where the minimum number is exactly two. More generally, one can ask the following question:

> *Suppose that X is a compact Hausdorff space which has at least one open covering consisting entirely of contractible sets. What is the* MINIMUM *number of such sets that are needed to form an open covering of X?*

If X is a topological n-manifold, then the following basic result gives an upper estimate:

THEOREM 1. *If M is a (second countable) arcwise connected topological n-manifold, then M has an open covering by $n + 1$ sets which are homeomorphic to open subsets of $\mathbb{R}^n$.*∎

Here is the standard reference for a proof:

> **E. Luft.** *Covering manifolds with open cells.* Illinois Journal of Mathematics **13** (1969), 321–326.

In this section we shall use cup products to prove that in general the best possible upper bound is $n + 1$.

Lusternik-Schnirelmann category and cup products

There are numerous variants of the contractible open covering question, and we shall be particularly interested in a version where "contractible open sets" is replaced by "open subsets for which the inclusion maps into X are nullhomotopic." In particular, the following homotopy-theoretic concept is closely related to these questions:

Definition. Let X be a second countable, locally compact, Hausdorff space. Then X is said to have *Lusternik-Schnirelmann* or **LS** *category* $\leq m$ if X is a union of m subsets U_i such that the inclusions $U_i \subset M$ are nullhomotopic.

Note. Frequently one finds slightly different spellings of the names "Lusternik" and "Schnirelmann" based upon different conventions for transliterating the Cyrillic spellings Люстерник and Шнирельман into their Latin counterparts.

Definition. We shall say that X has Lusternik-Schnirelmann or **LS** category equal to k if it has **LS** category $\leq k$ but does not have **LS** category $\leq k - 1$. Similarly, we shall say that X has **LS** category $\geq k$ if X does not have **LS** category $\leq k - 1$.

If X is a compact topological n-manifold which has a covering by k open subsets, each homeomorphic to $\mathbb{R}^n$, then it follows immediately that X has **LS** category $\leq k$, and Theorem 1 implies that the **LS** category is always $\leq n+1$. The main result of this section gives an example where equality holds.

THEOREM 2. *The n-torus T^n has* **LS** *category equal to $n + 1$.*

The proof that T^n has **LS** category $\geq n + 1$ will be a consequence of the following general observation.

THEOREM 3. *Suppose that X is an arcwise connected, second countable, locally compact, Hausdorff space with* **LS** *category $\leq m$, and let $u_1 \in H^{d(1)}(X, \mathbb{F})$, $\cdots$, $u_m \in H^{d(m)}(X; \mathbb{F})$ with $d(i) > 0$ for all i. Then $u_1 \ \cdots \ u_m = 0$.*

If the conclusion of the theorem holds for an arcwise connected space X, we shall say that X *has* cuplength $\leq m$ because every product of m positive-dimensional cohomology classes in X is equal to zero.

Proof of Theorem 3. Let $W_1, \cdot, W_m$ be a covering of X such that each inclusion $W_i \to X$ is nullhomotopic. Since each cohomology restriction map $H^{m(i)}(X; \mathbb{F}) \to H^{m(i)}(W_i; \mathbb{F})$ is trivial, the classes u_i lift to classes v_i in the relative cohomology groups $H^{m(i)}(X, W_i; \mathbb{F})$. It follows that $u_1 \ \cdots \ u_m$ is the image of $v_1 \ \cdots \ v_m$ is the group

$$H^*(X, \cup_i U_i; \mathbb{F}) \quad = \quad H^*(X, X; \mathbb{F}) \quad = \quad 0$$

and hence this product equals zero.■

Proof of Theorem 2. Since there are n classes in $H^1(T^n;\mathbb{F})$ whose cup product is nonzero, Theorem 3 implies that T^n has **LS** category $\geq n+1$, and hence every open covering of T^n by sets homeomorphic to $\mathbb{R}^n$ consists of at least $n+1$ such regions.

One can construct an explicit open covering of T^n with $n+1$ open sets as follows: Let $p : \mathbb{R}^n \to T^n$ be the usual universal covering projection sending $(t_1, \cdots, t_n)$ to $(\exp 2\pi\, i\, t_1, \cdots, \exp 2\pi\, i\, t_n)$, and let $a_0, \cdots, a_n$ be distinct points in the half-open interval $[0,1)$, so that the points $z_k = \exp 2\pi\, i\, a_k \in S^1$ are distinct. Now let $W_k \subset \mathbb{R}^n$ be the set of all points such that $a_k < t_k < a_k + 1$ for all k, and take $V_k \subset T^n$ to be the image of W_k under p. By construction each set V_k is contractible. A point of T^n will lie in $T^n - V_k$ if and only if at least one of its coordinates is equal to z_k. The intersection of the sets $T^n - V_k$ will consist of all points $(b_1, \cdots, b_n)$ such that for each k, there is some j for which $b_j = z_k$. Since there are $n+1$ values of z_k and only n coordinates b_j, this is impossible. Therefore $\cap_k\ (T^n - V_k) = \emptyset$, so that $T^n = \cup_k V_k$.

References for further information

The *Wikipedia* article

`http://en.wikipedia.org/wiki/Lusternik%E2%80%Schnirelmann_category`

is a good starting point for learning more about the concept of Lusternik-Schnirelmann category, and it gives several good references for further information on the topic. The book by Cornea, Lupton, Oprea and Tanré (cited in that article) contains a very thorough treatment of this subject.

IV.7 : Real and complex projective spaces

(Hatcher, Ch. 0 and §§ 1.2–1.3, 2.2, 2.C, 3.2)

See the files `projspaces*.pdf` where $* = 1$ or 2.