V. Cohomology and Differential Forms

Courses in multivariable calculus generally end with proofs of fundamental results in vector analysis such as Green's Theorem in the plane, path independence criteria for line integrals, Stokes' Theorem for oriented surfaces with boundaries, and the Divergence Theorem in 3-space. Differential forms provide the standard framework for stating and proving the corresponding results in higher dimensions. We have already seen that algebraic topology also provides a setting in which various global versions of these results can be formulated. These included comprehensive generalizations of Green's Theorem and the Divergence Theorem to regions which have nice decompositions. The purpose of this unit is to describe far-reaching extensions of such relationships to arbitrary finite dimensions. In particular, we shall see that the answer to the question

> *Are there k-dimensional differential forms on an open subset U of $\mathbb{R}^n$ which are closed ($d\omega = 0$) but not exact ($\omega = d\theta$ for some θ)?*

depends only whether or not the singular homology group $H_k(U;\mathbb{R})$ is trivial (in which case the answer is no) or nontrivial (in which case the answer is yes). This even yields new information in the setting of classical vector analysis; specifically, if U is an open subset in $\mathbb{R}^3$, then every smooth vector field $\mathbf{F}$ whose curl satisfies $\nabla \times \mathbf{F} = \mathbf{0}$ is a gradient vector field if and only if $H_1(U;\mathbb{R}) = 0$. This is one of many corollaries of a fundamental result known as **de Rham's Theorem**.

Section 0 is a summary of the main things we need to know about differential k-forms on an open subset of $\mathbb{R}^n$. Roughly speaking, these are formal integrands of line integrals, surface integrals, multiple integrals, and their generalizations to integrals over suitably defined k-dimensional analogs of surfaces in $\mathbb{R}^n$. In Section 1 we make the latter notion precise by defining a variant of singular homology in which the singular simplices are smooth mappings, and in Section 2 we state an analog of Stokes' Theorem for the integral of a k-form over a k-dimensional smooth singular chain. Differentiation of differential forms induces maps d^k from k-forms to $(k+1)$-forms which satisfy $d^{k+1} \circ d^k = 0$, and thus the differential forms on an open subset in $\mathbb{R}^n$ form a cochain complex often called the *de Rham complex* of an open set. The cohomology groups of this cochain complex are called the *de Rham cohomology groups* of the open set, and Section 3 shows that these groups have several formal properties which resemble those of singular cohomology groups with real coefficients. The main result of Section 4 is de Rham's theorem, which states that the two types of cohomology groups are isomorphic. Finally, in Section 5 we shall prove that under this isomorphism the cup product in singular cohomology corresponds to a construction on differential forms known as the wedge product.

Throughout the rest of this section we shall refer to the following textbook for the details of various constructions and proofs:

> **L. Conlon.** Differentiable Manifolds. (Second Edition), *Birkhäuser-Boston, Boston MA*, 2001. ISBN: 0–8176–4134–3.

There will also be references to Lee's book on smooth manifolds; however, in many instances the discussion in Lee is at a more abstract and general level than these notes (in particular, it gets into some complicated issues that we are trying to avoid).

V.0 : Review of differential forms

(Conlon, §§ 6.2, 6.4, 7.1–7.2, 8.1; Lee, Chs. 6, 11-12)

We have already noted that differential forms provide a convenient and powerful setting for generalizing classical vector analysis to higher dimensions, but they also have numerous other uses in both mathematics and physics. Setting up the theory requires some time and effort, but differential forms can be used very effectively to unify and simplify some fundamentally important concepts and results. They have become the standard framework for analyzing an extremely wide range of topics and problems. For the most part, we shall restrict attention to differential forms on open subsets of $\mathbb{R}^n$ where n is allowed to be a more or less arbitrary positive integer.

This is only a summary of the main points of the theory. Additional details can be found on pages 245–288 of Rudin (*Principles of Mathematical Analysis*, Third Edition).

Covariant tensors and differential forms

Let U be an open subset of $\mathbb{R}^n$, and let p be a nonnegative integer. A **covariant tensor field of rank** p is defined to be an expression of the form

$$\sum_{i_1, i_2, \ (etc.)} g_{i_1\, i_2\, \cdots\, i_p}\, dx^{i_1} \otimes \ \cdots \ \otimes dx^{i_p}$$

where

(1) each $g_{i_1\, i_2\, \cdots\, i_p}$ is a smooth real valued function on U,

(2) each i_j ranges from 1 to n,

(3) two expressions are equal if and only if the functional coefficients of each $dx^{i_1} \otimes \cdots \otimes dx^{i_p}$ are equal.

We shall call denote this object by $\mathbf{Cov}_p(U)$. It will be understood that $\mathbf{Cov}_0(U) = \mathcal{C}^\infty(U)$; note also that there is a natural identification of $\mathbf{Cov}^1(U)$ with the space of differential 1-forms we considered in Section V.3 of the lecture notes.

The space of **exterior** or **differential** p**-forms** on U is defined to be the quotient of $\mathbf{Cov}_p(U)$ obtained by the identification

$$dx^{i_1} \otimes \ \cdots \ \otimes dx^{i_p} \quad \approx \quad -\, dx^{j_1} \otimes \ \cdots \ \otimes dx^{j_p}$$

if $[\, j_1\, j_2 \ \cdots \ j_p\,]$ is obtained from $[\, i_1\, i_2 \ \cdots \ i_p\,]$ by switching exactly two of the terms, say i_s and i_t where $s \neq t$. If $i_s = i_t$ for some $s \neq t$ then this is understood to imply that $dx^{i_1} \otimes \ \cdots \ \otimes dx^{i_p}$ is equal to its own negative, and since we are working with real vector spaces this means that the expression in question is identified with zero. The set of all differential p-forms on an open subset $U \subset \mathbb{R}^n$ is denoted by $\wedge^p(U)$, and the images of the basic objects in if $dx^{i_1} \otimes \cdots \otimes dx^{i_p}$ is one of the basic objects in $\mathbf{Cov}_p(U)$ as above, then its image in $\wedge^p(U)$ is denoted by

$$dx^{i_1} \wedge \cdots \wedge dx^{i_p} .$$

By convention we also set $\wedge^0(U)$ equal to $\mathcal{C}^\infty(U)$.

PROPOSITION 1. *If $p > n$ then $\wedge^p(U) = 0$, and if $0 < p \leq n$ then every element of $\wedge^p(U)$ can be written uniquely as a linear combination of the basic forms*

$$dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

with coefficients in $\mathcal{C}^\infty(U)$, where the indexing sequences $\{ i_j \}$ satisfy $i_1 < \cdots < i_p$.

This is an immediate consequence of the construction.■

If $p = 1$ then the definition of $\wedge^1(U)$ is equivalent to the previous one involving sections of the cotangent bundle.

Integrals defined by differential forms The motivation for the definition comes from the use of differential 1-forms as the integrands of line integrals. In particular, we would like 2-forms to represent the integrands of surface integrals and n-forms to represent the integrands of ordinary (Riemann or Lebesgue) integrals over appropriate subsets of U. Note in particular that if U is open in $\mathbb{R}^n$, then every element of $\wedge^n(U)$ is uniquely expressible as

$$h(x) \cdot dx^1 \wedge \cdots \wedge dx^n$$

for some $h \in \mathcal{C}^\infty(U)$.

So how do we form integrals such that the integrand is a p-form and the construction reduces to the usual ones for line and surface integrals if $p = 1$ or 2? The key is to notice that such integrals are first defined using parametric equations for a curve or surface defined for all values of the variable(s) in some open subset of $\mathbb{R}$ or $\mathbb{R}^2$.

Following Rudin, we do so by defining a *smooth singular p-surface piece* in U to be a continuous map $\sigma : \Delta \to U$ such that Δ is compact in $\mathbb{R}^p$ and σ extends to a smooth function on an open neighborhood of Δ in $\mathbb{R}^p$. In multivariable calculus one generally assumes also that the extension of σ to an open set is a *smooth immersion*, or at least this is true if one subdivides the domain of definition into suitable pieces and permits bad behavior at boundary points of such pieces (normally the boundary has measure zero and hence doesn't matter for integration purposes), but we shall not make any such assumptions on the rank of $D\sigma$ in these notes.

For each object σ as in the previous paragraph and each tensor $\Lambda \in \mathbf{Cov}_p(U)$ we can define an integral by the following formula:

$$\int_\sigma \Lambda \quad = \quad \int_\sigma \sum_{i_1, i_2,\ etc.} g_{i_1\, i_2\, \cdots\, i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p} \quad =$$

$$\sum_{i_1, i_2,\ etc.} \int_\Delta g_{i_1\, i_2\, \cdots\, i_p} \circ \sigma(u) \frac{\partial(x^{i_1},\ \cdots\ , x^{i_p})}{\partial(u^1,\ \cdots\ , u^p)}$$

As usual, expressions of the form

$$\frac{\partial(x_a,\ \cdots\ )}{\partial(u_1,\ \cdots\ )}$$

represent Jacobian determinants. We then have the following key observation which allows us to work with forms rather than tensors:

PROPOSITION 2. *In the integral above, the value only depends upon the image λ of Λ in $\wedge^p(U)$.*

Proof. It suffices to consider simple integrands consisting of only one summand. For each sequence

$$x^{i_1},\ \cdots\ , x^{i_p}$$

we need to show that if we switch two terms x^a and x^b then the sign of the integral changes if dx^a and dx^b are both factors of the integrand. The effect of making such a change on the integrand is to switch two columns in the $p \times p$ matrix of functions whose determinant is the Jacobian

$$\frac{\partial(x^{i_1},\ \cdots\ , x^{i_p})}{\partial(u^1,\ \cdots\ , u^p)}$$

and we know this operation changes signs; this proves the point that we need to reach the conclusion of the proposition.■

Because of the preceding result we shall assume henceforth that integrands are differential p-forms.

Operations on differential forms

There are several fundamental constructions on differential forms that are used extensively.

Exterior products. It follows immediately from the definitions that each $\wedge^p(U)$ is a real vector space and in fact is a module over $\mathcal{C}^\infty(U)$ However, there is also an important multiplicative structure that we shall now describe. We shall begin by defining a version of this structure for covariant tensors. Specifically, there are $\mathcal{C}^\infty(U)$-bilinear maps

$$\otimes : \mathbf{Cov}_p(U) \times \mathbf{Cov}_q(U) \longrightarrow \mathbf{Cov}_{p+q}(U)$$

sending a pair of monomials

$$\left(\ g_{i_1\, i_2\, \cdots\, i_p}\, dx^{i_1} \otimes\ \cdots\ \otimes dx^{i_p}\ ,\ h_{j_1\, j_2\, \cdots\, j_q}\, dx^{j_1} \otimes\ \cdots\ \otimes dx^{j_q}\ \right)$$

to the monomial

$$g_{i_1\, i_2\, \cdots\, i_p} h_{j_1\, j_2\, \cdots\, j_q} \cdot dx^{i_1} \otimes\ \cdots\ \otimes dx^{i_p} \otimes dx^{j_1} \otimes\ \cdots\ \otimes dx^{j_q}\ .$$

In order to show this passes to a $\mathcal{C}^\infty(U)$-bilinear map

$$\wedge_{p,q} : \wedge^p(U) \times \wedge^q(U) \longrightarrow \wedge^{p+q}(U)$$

we need to show that if $\xi \in \mathbf{Cov}_p(U)$ and $\eta \in \mathbf{Cov}_q(U)$ are monomials as above and ξ' and η' are related to ξ and η as in the definition of differential forms, then the images of $\otimes(\xi, \eta)$ and $\otimes(\xi', \eta')$ are equal. As above we are assuming

$$\xi \;=\; g_{i_1\, i_2\, \cdots\, i_p}\, dx^{i_1} \otimes \cdots \otimes dx^{i_p} \quad , \quad \eta \;=\; h_{j_1\, j_2\, \cdots\, j_q}\, dx^{j_1} \otimes \cdots \otimes dx^{j_q} \ .$$

Since two covariant monomial tensors determine the same differential form if they are related by a finite sequence of elementary moves (permuting the dx^q's or replacement by zero if there is a repeated such factor), it is enough to show that one obtains the same differential form provided ξ' and η' are related to ξ and η by a single elementary move (which affects one form but not the other).

Suppose the elementary move switches two variables; then we may write

$$\xi' \;=\; \alpha \cdot g_{k_1\, k_2\, \cdots\, k_p}\, dx^{k_1} \otimes \cdots \otimes dx^{k_p} \quad , \quad \eta' \;=\; \beta \cdot h_{\ell_1\, \ell_2\, \cdots\, \ell_q}\, dx^{\ell_1} \otimes \cdots \otimes dx^{\ell_q}$$

where $\{k_1\, k_2\, \cdots\, k_p\}$ and $\{\ell_1\, \ell_2\, \cdots\, \ell_q\}$ are obtained from $\{i_1\, i_2\, \cdots\, i_p\}$ and $\{j_1\, j_2\, \cdots\, j_q\}$ either by doing nothing or by switching two of the variables and the coefficients α and β are ± 1 depending upon whether or not variables were switched in each case. From this description one can check directly (with some tedious computations) that the images of $\otimes(\xi, \eta)$ and $\otimes(\xi', \eta')$ in $\wedge^{p+q}(U)$ are equal. On the other hand, if one has repeated factors in either ξ or η and the corresponding object ξ' or η' is zero, then it is immediately clear that $\otimes(\xi, \eta)$ and $\otimes(\xi', \eta')$ in $\wedge^{p+q}(U)$ both zero and hence are equal.

PROPOSITION 3. *If $\theta \in \wedge^p(U)$ and $\omega \in \wedge^q(U)$, then we have $\theta \wedge \omega = (-1)^{pq}\, \omega \wedge \theta$.*

Proof. Using bilinearity we may immediately reduce this to the special case where

$$\theta \;=\; dx^{i_1} \wedge \cdots \wedge dx^{i_p} \quad , \quad \omega \;=\; dx^{j_1} \wedge \cdots \wedge dx^{j_q} \ .$$

In this case we have

$$\theta\wedge\omega \;=\; dx^{i_1}\wedge \cdots \wedge dx^{i_p}\wedge dx^{j_1}\wedge \cdots \wedge dx^{j_q} \ \ , \ \ \omega\wedge\theta \;=\; dx^{j_1}\wedge \cdots \wedge dx^{j_q}\wedge dx^{i_1}\wedge \cdots \wedge dx^{i_p} \ .$$

Therefore we need to investigate what happens if one rearranges the variables using some permutation.

If γ is an arbitrary permutation then γ is a product of transpositions, and therefore it follows that if one permutes variables by γ the effect on a basic monomial form is multiplication by $\mathrm{sgn}(\gamma)$. Therefore the proof of the formula in the proposition reduces to computing the sign of the permutation which takes the first p numbers in $\{1, \cdot, p+q\}$ to the last p numbers in order and takes the last q numbers to the first q numbers in order.

It is an elementary combinatorial exercise to verify that the sign of this permutation is pq (*e.g.*, fix one of p or q and proceed by induction on the other$^{(\star)}$).■

The following property is also straightforward to verify$^{(\star)}$, and in fact it is a consequence of the analogous property for covariant tensors:

PROPOSITION 4. *If θ and ω are as above and $\lambda \in \wedge^r(U)$, then one has the associativity property $(\theta \wedge \omega) \wedge \lambda = \theta \wedge (\omega \wedge \lambda)$.*■

Exterior derivatives. We have already seen that there is a well-defined map $d : \wedge^0(U) \to \wedge^1(U)$ defined by taking exterior derivatives, and in fact for each p one can define an exterior derivative

$$d^p : \wedge^p(U) \longrightarrow \wedge^{p+1}(U) .$$

These maps are linear transformations of real vector spaces and are defined on monomials by the formula

$$d\big(\, g\, dx^{i_1} \wedge \; \cdots \; \wedge dx^{i_p} \,\big) \;\; = \;\; dg \wedge \; dx^{i_1} \wedge \; \cdots \; \wedge dx^{i_p} \; .$$

If we take $g = 1$ the preceding definition implies

$$d\big(\, dx^{i_1} \wedge \; \cdots \; \wedge dx^{i_p} \,\big) \;\; = \;\; 0 \; .$$

One then has the following basic consequences of the definitions.

THEOREM 5. *The exterior derivative satisfies the following identities:*

(*i*) *If θ is a p-form then $d(\theta \wedge \omega) \; = \; (d\theta) \wedge \omega \; + \; (-1)^p \theta \wedge (d\omega)$.*

(*ii*) *For all λ we have $d(d\lambda) = 0$.*

Sketch of proof. In each case one can use linearity or bilinearity to reduce everything to the special case of forms that are monomials. For examples of this type it is a routine computational exercise to verify the identities described above$^{(\star)}$.■

Definition. A differential form ω is said to be *closed* if $d\omega = 0$ and exact if $\omega = d\lambda$ for some λ. The second part of the theorem implies that exact forms are closed. On the other hand, the 1-form

$$\frac{y\, dx - x\, dy}{x^2 + y^2}$$

on $\mathbb{R}^2 - \{0\}$ is closed but not exact.

Change of variables (pullbacks). The pullback construction on 1-forms extends naturally to forms of higher degree. Specifically, if V is open in $\mathbb{R}^m$ and $f : V \to U$ is smooth then there are real vector space homomorphisms $f^* : \wedge^p(U) \to \wedge^p(U)$ that are defined on monomials by the formula

$$f^*\big(\, g\, dx^{i_1} \wedge \; \cdots \; \wedge dx^{i_p} \,\big) \;\; = \;\; (g \circ f)\, df^{i_1} \wedge \; \cdots \; \wedge df^{i_p}$$

where f^i denotes the i^{th} coordinate function of f. If $p = 1$ this coincides with the previous definition.

The next result implies that the pullback construction preserves all the basic structure on exterior forms that we defined above and it has good naturality properties:

THEOREM 6. *(i) In the notation above we have $f^*(\theta \wedge \omega) = f^*\theta \wedge f^*\omega$ and $f^* \circ d\lambda = d \circ f^*\lambda$.*

(ii) The pullback map for id_U is the identity on $\wedge^p(U)$, and if $h : W \to V$ is another smooth map, then $(f \circ h)^* = h^* \circ f^*$.

(iii) The pullback maps and exterior derivatives satisfy the compatibility relations $d \circ f^* = f^* \circ d$.

Complete derivations of these results appear on pages 263–264 of Rudin$^{(\star)}$.■

The pullback also has the following basic compatibility property with respect to integrals:

CHANGE OF VARIABLES FOR INTEGRALS. *Let $\omega \in \wedge^p(U)$, let $f : V \to U$ be smooth, and let $\sigma : \Delta \to V$ be a smooth p-surface. Then integration of differential forms satisfies the following change of variables formula:*

$$\int_\Delta f^*\omega \quad = \quad \int_{f \circ \sigma} \omega$$

A derivation of this formula appears on pages 264–266 of Rudin$^{(\star)}$.■

Relation to classical vector analysis

We shall now explain how the basic constructions and main theorems of vector analysis can be expressed in terms of differential forms. For most of this section U will denote an open subset of $\mathbb{R}^3$.

Let $\mathbf{X}(U)$ be the Lie algebra of smooth vector fields on U. As a module over $\mathcal{C}^\infty(U)$ the space of vector fields is isomorphic to each of $\wedge^1(U)$ and $\wedge^2(U)$, and $\mathcal{C}^\infty(U)$ is isomorphic to $\wedge^3(U)$; recall that $\mathcal{C}^\infty(U) = \wedge^0(U)$ by definition. For our purposes it is important to give specific isomorphisms $\Phi^1 : \mathbf{X}(U) \to \wedge^1(U)$, $\Phi^2 : \mathbf{X}(U) \to \wedge^2(U)$, $\Phi^3 : \mathcal{C}^\infty(U) \to \wedge^3(U)$. A vector field will be viewed as a vector valued function $\mathbf{V} = (F, G, H)$ where each of F, G, H is a smooth real valued function on U.

$$\Phi_1(F, G, H) \quad = \quad F\,dx + G\,dy + H\,dz$$

$$\Phi_2(F, G, H) \quad = \quad F\,dy \wedge dx + \ G\,dz \wedge dx + H\,dx \wedge dy$$

$$\Phi_3(f) \quad = \quad f\,dx \wedge dy \wedge dx$$

We then have the following basic relationships:

$$(i) \quad \nabla f \;=\; \Phi_1^{-1}(df)$$

$$(ii) \quad \mathbf{curl}(\mathbf{V}) \;=\; \Phi_2^{-1}\circ d\circ \Phi_1(\mathbf{V})$$

$$(iii) \quad \mathbf{div}(\mathbf{V}) \;=\; \Phi_3^{-1}\circ d\circ \Phi_2(\mathbf{V})$$

Each of these is a routine computation$^{(\star)}$.

From this perspective the vector analysis identities

$$\mathbf{curl}(\nabla f) = 0 \quad , \quad \mathbf{div}\,\mathbf{curl}(\mathbf{V}) \;=\; 0$$

are equivalent to special cases of the more general relationship $d\circ d = 0$.

V.1 : Smooth singular chains

(Hatcher, §§ 2.1, 2.3; Conlon, § 8.2; Lee, Ch. 16)

We now need to introduce yet another way of computing the homology groups of an open subset of $\mathbb{R}^n$ for some n.

Let q be a nonnegative integer. In Unit II we defined a *singular q-simplex* in a topological space X to be a continuous mapping $T : \Delta_q \to X$, where Δ_q is the simplex in $\mathbb{R}^{q+1}$ whose vertices are the standard unit vectors; the group of singular q-chains $\mathbf{S}_q(X)$ was then defined to be the free abelian group on the set of singular q-simplices. The first step in this section is to is to define an analog of these groups involving *smooth* mappings if X is an open subset of $\mathbb{R}^n$ for some n.

Definition. Let q be a nonnegative integer, and let $\Lambda_q \subset \mathbb{R}^q$ be the q-simplex whose vertices are $\mathbf{0}$ and the standard unit vectors. Also, let U be an open subset of $\mathbb{R}^n$ for some $n \geq 0$. A *smooth singular q-simplex* in U is a continuous map $T : \Lambda_q \to U$ which is *smooth* — in other words, there is some open neighborhood W_T of Λ_q in $\mathbb{R}^q$ such that T extends to a map $W_T \to U$ which is smooth in the usual sense (the coordinate functions have continuous partial derivatives of all orders). The group of *smooth singular q-chains* $\mathbf{S}_q^{\mathrm{smooth}}(U)$ is the free abelian group on all smooth singular q-simplices in U.

There is an obvious natural relationship between the smooth and ordinary singular chain groups which is given by the standard affine isomorphism φ from Δ_q to Λ_q defined on vertices by $\varphi(\mathbf{e}_1) \;=\; \mathbf{0}$ and $\varphi(\mathbf{e}_i) \;=\; \mathbf{e}_{i-1}$ for all $i > 1$. Specifically, each smooth singular q-simplex $T : \Lambda_q \to U$ determines the continuous singular q-simplex $T\circ\varphi : \Delta_q \to U$. The resulting map of singular chain groups will be denoted by

$$\varphi^{\#} : \mathbf{S}_q^{\mathrm{smooth}}(U) \;\longrightarrow\; \mathbf{S}_q(U)$$