$$(i) \quad \nabla f \quad = \quad \Phi_1^{-1}(df)$$

$$(ii) \quad \mathbf{curl}(\mathbf{V}) \quad = \quad \Phi_2^{-1} \circ d \circ \Phi_1(\mathbf{V})$$

$$(iii) \quad \mathbf{div}(\mathbf{V}) \quad = \quad \Phi_3^{-1} \circ d \circ \Phi_2(\mathbf{V})$$

Each of these is a routine computation$^{(\star)}$.

From this perspective the vector analysis identities

$$\mathbf{curl}(\nabla f) = 0 \quad , \quad \mathbf{div}\,\mathbf{curl}(\mathbf{V}) \quad = \quad 0$$

are equivalent to special cases of the more general relationship $d \circ d = 0$.

V.1 : Smooth singular chains

(Hatcher, §§ 2.1, 2.3; Conlon, § 8.2; Lee, Ch. 16)

We now need to introduce yet another way of computing the homology groups of an open subset of $\mathbb{R}^n$ for some n.

Let q be a nonnegative integer. In Unit II we defined a *singular q-simplex* in a topological space X to be a continuous mapping $T : \Delta_q \to X$, where Δ_q is the simplex in $\mathbb{R}^{q+1}$ whose vertices are the standard unit vectors; the group of singular q-chains $\mathbf{S}_q(X)$ was then defined to be the free abelian group on the set of singular q-simplices. The first step in this section is to is to define an analog of these groups involving *smooth* mappings if X is an open subset of $\mathbb{R}^n$ for some n.

Definition. Let q be a nonnegative integer, and let $\Lambda_q \subset \mathbb{R}^q$ be the q-simplex whose vertices are $\mathbf{0}$ and the standard unit vectors. Also, let U be an open subset of $\mathbb{R}^n$ for some $n \geq 0$. A *smooth singular q-simplex* in U is a continuous map $T : \Lambda_q \to U$ which is *smooth* — in other words, there is some open neighborhood W_T of Λ_q in $\mathbb{R}^q$ such that T extends to a map $W_T \to U$ which is smooth in the usual sense (the coordinate functions have continuous partial derivatives of all orders). The group of *smooth singular q-chains* $\mathbf{S}_q^{\text{smooth}}(U)$ is the free abelian group on all smooth singular q-simplices in U.

There is an obvious natural relationship between the smooth and ordinary singular chain groups which is given by the standard affine isomorphism φ from Δ_q to Λ_q defined on vertices by $\varphi(\mathbf{e}_1) \ = \ \mathbf{0}$ and $\varphi(\mathbf{e}_i) \ = \ \mathbf{e}_{i-1}$ for all $i > 1$. Specifically, each smooth singular q-simplex $T : \Lambda_q \to U$ determines the continuous singular q-simplex $T \circ \varphi : \Delta_q \to U$. The resulting map of singular chain groups will be denoted by

$$\varphi^{\#} : \mathbf{S}_q^{\text{smooth}}(U) \ \longrightarrow \ \mathbf{S}_q(U)$$

with subscripts or superscripts added if it is necessary to keep track of q or U.

One important feature of the ordinary singular chain groups is that they can be made into a chain complex, and it should not be surprising to learn that there is a compatible chain complex structure on the groups of smooth singular chains. We recall the definition of the chain complex structure on $\mathbf{S}_*(X)$ for a topological space X, starting with the preliminary constructions. If Δ_q is the standard q-simplex, then for each i such that $0 \leq i \leq q$ there is an i^{th} face map $\partial_i : \Delta_{q-1} \to \Delta_q$ sending the domain to the face of Δ_q opposite the vertex $\mathbf{e}_{i+1}$ with $\partial_i(\mathbf{e}_j) = \mathbf{e}_j$ if $j \leq i$ and $\partial_i(\mathbf{e}_j) = \mathbf{e}_{j+1}$ if $j \geq i+1$. Then each face map ∂_i defines function from singular q-simplices to singular $(q-1)$-simplices by the formula $\partial_i(T) = T \circ \partial_i$, and the formula

$$d_q \quad = \quad \sum_{i=0}^{q} (-1)^i \, \partial_i$$

defines a homomorphism from $\mathbf{S}_q(X)$ to $\mathbf{S}_{q-1}(X)$ with some important formal properties like $d_{q-1} \circ d_q = 0$.

For the analogous constructions on smooth singular chain groups, we first need compatible face maps on Λ_q. The simplest way to do this is to relabel the vertices of the latter as $\mathbf{0} = \mathbf{v}_0$ and $\mathbf{e}_i = \mathbf{v}_{i+1}$ for all i; then we may define ∂_i^Λ in the same way as ∂_i, the only difference being that we replace the vertices $\mathbf{e}_j$ for Δ_q by the vertices $\mathbf{v}_j$ for Λ_q.

We claim that if $T : \Lambda_q \to U$ is a smooth singular simplex then are all of the faces given by the composites $T \circ \partial_i^\Lambda$; this follows because each of maps ∂_i^Λ is an affine mapping and hence is smooth.

It follows immediately that the preceding constructions are compatible with the simplex isomorphisms φ constructed above, so that $\varphi^\# \circ \partial_i = \partial_i^\Lambda \circ \varphi^\#$, and if we define

$$d_q^{\text{smooth}} : \mathbf{S}_q^{\text{smooth}}(U) \longrightarrow \mathbf{S}_{q-1}^{\text{smooth}}(U)$$

to be the sum of the terms $(-1)^i \partial_i^\Lambda$, then one has the following compatibility between smooth and singular chains.

PROPOSITION 1. *Let U be an open subset of $\mathbb{R}^n$ for some n, and let $\varphi^\# : \mathbf{S}_q^{\text{smooth}}(U) \to \mathbf{S}_q(U)$ and d_*^{smooth} be the map given by the preceding constructions. Then the latter map makes $\mathbf{S}_*^{\text{smooth}}(U)$ into a chain complex such that $\varphi^\#$ is a morphism of chain complexes.*

The assertion in the first sentence can be verified directly from the definitions, and the first assertion in the second sentence follows from the same sort of argument employed earlier in these notes. Finally, the fact that $\varphi^\#$ is a chain complex morphism is an immediate consequence of the assertion in the first sentence and the definitions of the differentials in the two chain complexes in terms of the maps ∂_i and ∂_i^Λ.■

We shall denote the homology of the complex of smooth singular chains by $H_*^{\mathrm{smooth}}(U)$ and call the associated groups the *smooth singular homology groups* of the open set $U \subset \mathbb{R}^n$. Later in this section we shall prove the following fundamentally important result.

ISOMORPHISM THEOREM. *For all open subsets $U \subset \mathbb{R}^n$, the associated homology morphism $\varphi_*^{\#}$ from the smooth singular homology groups $H_*^{\mathrm{smooth}}(U)$ to the ordinary singular homology groups $H_*(U)$.*

Functoriality properties

In order to prove the Isomorphism Theorem, we need to establish additional properties of smooth singular chain and homology groups that are similar to basic properties of ordinary singular chain and homology groups. The first of these is a basic naturality property:

PROPOSITION 2. *Let $U \subset \mathbb{R}^n$, (etc.) be as above, let $V \subset \mathbb{R}^m$ be open, and let $f : U \to V$ be a smooth mapping from U to V (the coordinates have continuous partial derivatives of all orders). Then there is a functorial chain map $f_{\#}^{\mathrm{smooth}} : \mathbf{S}_*^{\mathrm{smooth}}(U) \to \mathbf{S}_*^{\mathrm{smooth}}(V)$ such that $f_{\#}^{\mathrm{smooth}}$ maps a smooth singular q-simplex T to $f \circ T$ and we have the naturality property*

$$f_{\#} \circ \varphi^{\#} \quad = \quad \varphi^{\#} \circ f_{\#}^{\mathrm{smooth}}$$

where $f_{\#}$ is the corresponding map of smooth singular chains from $\mathbf{S}_*(U)$ to $\mathbf{S}_*(V)$.■

COROLLARY 3. *In the setting of the preceding result, one has functorial homology homomorphisms on smooth singular homology, and the maps $\varphi_*^{\#}$ define natural transformations from smooth singular homology to ordinary singular homology.*■

Combining this with the Isomorphism Theorem mentioned earlier, we see that the construction $\varphi_*^{\#}$ determines a natural isomorphism from smooth singular homology to ordinary singular homology for open subsets of Euclidean spaces.

Since we are already discussing functoriality, this is a good point to mention some properties of this sort which hold for differential forms but were not formulated in Section 0:

THEOREM 4. *Let $f : U \to V$ and $g : V \to W$ be smooth mappings of open subsets in Cartesian (Euclidean) spaces $\mathbb{R}^n$ where n need not be the same for any of U, V, W. Then the pullback construction on differential forms satisfies the identity $(g \circ f)^{\#} \;=\; f^{\#} \circ g^{\#}$. Furthermore, if f is the identity on U then $f^{\#}$ is the identity on $\wedge^*(U)$.*

The second of these is trivial, and the first is a direct consequence of the definitions and the Chain Rule for derivatives of composite maps.■

Our objective is to show that the natural map from smooth singular chains to ordinary chains

$$S_*^{\text{smooth}}(U) \longrightarrow S_*(U)$$

defines isomorphisms in homology and in cohomology with real coefficients if U is an arbitrary open subset of some $\mathbb{R}^n$.

It will be convenient to extend the definition of smooth singular chain complexes to arbitrary subsets of $\mathbb{R}^n$ for some n. Specifically, if $A \subset \mathbb{R}^n$ then the smooth singular chain complex $S_*^{\text{smooth}}(A)$ is defined so that each group $S_q(A)$ is free abelian on the set of continuous mappings $T : \Lambda_q \to A$ which extend to smooth mappings T' from some open neighborhood $W(T')$ of Λ_q to $\mathbb{R}^n$. If A is an open subset of $\mathbb{R}^n$, then this is equivalent to the original definition, for if we are given T' as above we can always find an open neighborhood V of Λ_q such that T' maps V into A.

Clearly the definitions of smooth and ordinary singular chains are similar, and in fact many properties of ordinary singular chain complexes extend directly to smooth singular chain complexes. The following two are particularly important:

(0) If A is a convex subset of $\mathbb{R}^n$ (which is not necessarily open), then the constant map defines an isomorphism from $H_q^{\text{smooth}}(A)$ to $H_q^{\text{smooth}}(\mathbb{R}^0)$ for all q; in particular, these groups vanish unless $q = 0$.

(1) If we are given two smooth maps $f, g : U \to V$ such that f and g are smoothly homotopic, then the chain maps from $S_*^{\text{smooth}}(U)$ to $S_*^{\text{smooth}}(V)$ determined by f and g are chain homotopic.

(2) The construction of barycentric subdivision chain maps $\beta : S_*(U) \to S_*(U)$ in Section I.2 of these notes, and the related chain homotopy from β to the identity, determine compatible mappings of the same type on smooth singular chain complexes.

The first two of these follow because the chain homotopy constructions from Section I.5 clearly send smooth chains to smooth chains. The proof of the final assertion has two parts. First, the barycentric subdivision chain map in Section I.2 takes singular chains in the images of the canonical mappings

$$S_*^{\text{smooth}}(W) \longrightarrow S_*(W)$$

into chains which also lie in the images of such mappings. However, the construction of the chain homotopy must be refined somewhat in order to ensure that it sends smooth chains to smooth chains. In order to construct such a refinement, one needs to know that the homology of $S_*^{\text{smooth}}(\Lambda_q)$ is isomorphic to the homology of a point (hence is zero in positive dimensions). The latter is true by Property (0).■

As in the ordinary case, if $\mathcal{W}$ is an open covering of an open set $U \subset \mathbb{R}^n$, then one can define the complex $\mathcal{W}$-small singular chains

$$S_*^{\mathrm{smooth},\mathcal{W}}(U)$$

generated by all smooth singular simplices whose images lie inside a single element of $\mathcal{W}$, and the argument for ordinary singular chains implies that the inclusion map

$$S_*^{\mathrm{smooth},\mathcal{W}}(U) \longrightarrow S_*^{\mathcal{W}}(U)$$

defines isomorphisms in homology. The latter in turn implies that one has long exact Mayer-Vietoris sequences relating the smooth singular homology groups of U, V, $U \cap V$ and $U \cup V$, where U and V are open subsets of (the same) $\mathbb{R}^n$, and in fact one has a long commutative ladder diagram relating the Mayer-Vietoris sequences for (U, V) with smooth singular chains and ordinary singular chains.

The smooth and ordinary singular chain groups for $\mathbb{R}^0$ are identical, and therefore their smooth and ordinary singular homology groups are isomorphic under the canonical map from smooth to ordinary singular homology. By the discussion above, it follows that the canonical map

$$\varphi_*^U : S_*^{\mathrm{smooth}}(U) \longrightarrow S_*(U)$$

is an isomorphism if U is a convex open subset of some $\mathbb{R}^n$. The next step is to extend the class of open sets for which φ_*^U is an isomorphism.

THEOREM 5. *The map φ_*^U is an isomorphism if U is a finite union of convex open subsets in $\mathbb{R}^n$.*

Proof. Let (C_k) be the the statement that φ_*^U is an isomorphism if U is a union of at most k convex open subsets. Then we know that (C_1) is true. Assume that (C_k) is true; we need to show that the latter implies (C_{k+1}).

The preceding statements about ladder diagrams and the Five Lemma imply the following useful principle: *If we know that φ_*^U, φ_*^V, and $\varphi_*^{U \cap V}$ are isomorphisms in all dimensions, then the same is true for $\varphi_*^{U \cup V}$.* — Suppose now that we have a finite sequence of convex open subsets $W_1, \cdots, W_{k+1}$, and take U and V to be $W_1 \cup \cdots \cup W_k$ and W_{k+1} respectively. Then we know that φ_*^U and φ_*^V are isomorphisms by the inductive hypotheses. Also, since

$$U \cap V \quad = \quad (W_1 \cap W_{k+1}) \cup \cdots \cup (W_k \cap W_{k+1})$$

and all intersections $W_i \cap W_j$ are convex, it follows from the induction hypothesis that $\varphi_*^{U \cap V}$ is an isomorphism in all dimensions. Therefore by the observation at the beginning of this paragraph we know that $\varphi_*^{U \cup V}$ is an isomorphism, which is what we needed in order to complete the inductive step.■

To complete the proof that φ_*^U is an isomorphism for all U, we need the so-called *compact carrier properties* of singular homology. There are two versions of this result.

THEOREM 6. *Let X be a topological space, and let $u \in H_q(X)$. Then there is a compact subset $K \subset X$ such that u lies in the image of the canonical map from $H_q(K)$ to $H_q(X)$. Furthermore, if K is a compact subset of X, and v and w are classes in $H_q(K)$ whose images in $H_q(X)$ are equal, then there is a compact subset L such that $K \subset L \subset X$ such that the images of v and w are equal in $H_q(L)$.*

Proof. Choose a singular chain $\sum_i n_i T_i$ representing u, where each T_i is a continuous mapping $\Delta_q \to X$. If K is the union of the images $T_i[\Delta_q]$, then K is compact, and it follows that u lies in the image of $H_q(K)$ (because the chain lies in the subcomplex $S_*(K) \subset S_*(X)$.

To prove the second assertion in the proposition, note that by additivity it suffices to prove this when $w = 0$. Once again choose a representative singular chain $\sum_i n_i T_i$ for v; since the image of v in $H_q(X)$ is a boundary, there is a $(q+1)$-chain $\sum_j m_j U_j$ on X whose boundary is $\sum_i n_i T_i$. Let L be the union of K and the compact sets $U_j[\Delta_{q+1}]$; then L is compact and it follows immediately that v maps to zero in $H_q(L)$.■

We shall need a variant of the preceding result.

THEOREM 6. *Let U be an open subset of some $\mathbb{R}^n$, and let $u \in H_q^{\mathrm{CAT}}(U)$, where CAT denotes either ordinary singular homology or smooth singular homology. Then there is a finite union of convex open subsets $V \subset U$ such that u lies in the image of the canonical map from $H_q^{\mathrm{CAT}}(V)$ to $H_q^{\mathrm{CAT}}(U)$. Furthermore, if V is a finite union of convex open subsets of U, and v and w are classes in $H_q^{\mathrm{CAT}}(V)$ whose images in $H_q^{\mathrm{CAT}}(U)$ are equal, then there is a finite union of convex open subsets W such that $V \subset W \subset U$ such that the images of v and w are equal in $H_q^{\mathrm{CAT}}(W)$.*

Proof. The argument is similar, so we shall merely indicate the necessary changes. We adopt all the notation from the preceding discussion.

For the first assertion, by compactness we know that there is a finite union of convex open subsets V such that $K \subset V \subset U$, and it follows that u lies in the image of the homology of V. For the second assertion, take W to be the union of V and finitely many convex open subsets whose union contains L. It then follows that v maps to zero in the homology of W.■

We can now prove the following general result.

THEOREM 7. *The map φ_*^U is an isomorphism for arbitrary open subsets of some $\mathbb{R}^n$.*

Proof. If $u \in H_q(U)$, then we know there is some finite union of convex open subsets V such that $u = i_*(u_1)$, where $i : V \subset U$ is inclusion. By our previous results we know that $u_1 = \varphi_*^V(u_2)$ for some $u_2 \in H_q^{\rm smooth}(V)$, and since $i_* \circ \varphi_*^V = \varphi_*^U \circ i_*$, it follows that $u = \varphi_*^U i_*(u_2)$, so that φ_*^U is onto.

To show that φ_*^U is 1–1, suppose that v lies in its kernel. By the previous results we know that v lies in the image of $H_q^{\rm smooth}(V)$; suppose that v_1 maps to v. Then it follows that $v_2 = \varphi_*^V(v_1) \in H_q(V)$ maps to zero in $H_q(U)$, so that there is a finite union of convex open subsets W such that $V \subset W$ and v_2 maps to zero in $H_q(W)$. If $j : V \to W$ is inclusion, then it follows that $j_*(v_1)$ lies in the kernel of φ_*^W; however, we know that the latter map is 1–1 and therefore it follows that $j_*(v_1) = 0$. Since the image of the latter element in $H_*^{\rm smooth}(U)$ is equal to v, it follows that $v = 0$ and hence φ_*^U is 1–1, which is what we wanted to prove.∎

Smooth singular cochains

As in Unit IV, we can dualize the construction of smooth singular chains to obtain smooth singular cochain groups for an open subset $U \subset \mathbb{R}^n$. Specifically, if M is an abelian group then the smooth singular cochain complex is defined by

$$S^*_{\rm smooth}(U;M) \quad = \quad {\rm Hom}\left(S_*^{\rm smooth}(U), M\right)$$

with the coboundary δ^* given by ${\rm Hom}(d_*, M)$.

If we are given a smooth map of open subsets in Euclidean spaces $f : U \to V$ and its associated map of smooth singular chain complexes $f_\#$, then we have maps of singular cochain complexes

$$f^\# \;=\; {\rm Hom}(f^\#, M) : S^*_{\rm smooth}(V;M) \to S^*_{\rm smooth}(U;M)$$

and morphisms of cohomology groups $f^* : H^*_{\rm smooth}(V;M) \to H^*_{\rm smooth}(U;M)$ which are contravariantly functorial with respect to smooth mappings. Furthermore, for open subsets in Euclidean spaces the canonical natural transformation from $S_*^{\rm smooth}(U)$ to $S_*(U)$ defines natural transformations of cochain complexes

$$\varphi^{\#\#} : S^*(U;M) \longrightarrow S^*_{\rm smooth}(U;M)$$

and cohomology groups $H^*(U;M) \to H^*_{\rm smooth}(U;M)$ which are natural with respect to smooth maps.

Theorem 7 and the weak Universal Coefficient Theorem of Unit IV immediately yields the following result for cohomology with field coefficients:

THEOREM 8. *If $\mathbb{F}$ is a field and U is an arbitrary open subset of $\mathbb{R}^n$, then the map $\varphi_U^* : H^*(U;\mathbb{F}) \longrightarrow H^*_{\rm smooth}(U;\mathbb{F})$ is an isomorphism of real vector spaces.*∎