FUNDAMENTAL GROUPS OF COMPLEX PROJECTIVE SPACES

A standard result in many graduate topology classes describes the fundamental groups of the real projective spaces $\mathbb{RP}^n$: If $n = 1$ the fundamental group is $\mathbb{Z}$, while if $n \geq 2$ the fundamental group is $\mathbb{Z}_2$. There is also a very simple description for the fundamental group of $\mathbb{CP}^n$:

Theorem 1. *For all $n \geq 1$ the fundamental group of $\mathbb{CP}^n$ is trivial.*

A similar result holds for quaternionic projective spaces. The proof for the latter is similar to the complex case, but we shall pass on giving details here. Our proof of Theorem 1 is essentially the one which appears in Section 3.4 of the following book:

> **E. L. Lima.** *Fundamental Groups and Covering Spaces* (Translated by J. Gomes). A K Peters, Natick, MA, 2003.

We shall use the notation and results from `projspaces1a.pdf` as needed. Note that the spaces $\mathbb{FP}^n$ are arcwise connected because they are quotient spaces of the arcwise connected spaces $\mathbb{F}^{n+1} - \{\mathbf{0}\}$.

Local triviality and path lifting

We shall begin by introducing an important generalization of a covering space projection.

Definition. A continuous map $q : E \to B$ of topological spaces is said to be *locally trivial* if for each point $x \in B$ there is an open neighborhood U of x and a homeomorphism $\varphi : U \times F \to q^{-1}[U]$ such that $q \circ \varphi$ is the projection mapping from $U \times F$ to U.

This is similar to one characterization of a smooth submersion of smooth manifolds, but one difference is the stipulation that the entire inverse image looks like a product; for a smooth submersion, one only knows that part of the inverse image looks like a product (however, if a smooth submersion is a proper mapping, for which inverse images of compact subsets are compact, a result of C. Ehresmann shows that the map is also locally trivial). A homeomorphism with the properties of φ is called a *local trivialization* of q.

A continuous mapping q satisfying the conditions in the definition is also said to be a *topological fiber bundle projection*. Two consequences of the definition are that q is a surjective open mapping (hence is a quotient map).

The key steps in the proof of Theorem 1 are given by the next two results:

PROPOSITION 2. (Weak Path Lifting Property) *Let $q : E \to B$ be a locally trivial continuous mapping. If $h : [0, 1] \to B$ is a continuous path and $y \in E$ satisfies $q(y) = h(0)$, then there is a continuous lifting of h to a continuous path $h' : [0, 1] \to E$ such that $h'(0) = y$ and $q \circ h' = h$.*

In general one cannot expect the lifted path h' to be unique. For example, if $q : B \times F \to B$ is coordinate projection and F is an arcwise connected space with more than one point, then h' can be any mapping $h'(x, t) = (\,h(x, t),\, g(t)\,)$, where $g : [0, 1] \to F$ is chosen so that $y = (x, a)$ and $g(0) = a$. Clearly there is more than one choice for g, and therefore there is more than one choice for h'.

Proof. (*Sketch*) The argument closely resembles the proof for the existence part of the Path Lifting Property for covering space projections. As in that proof, one can find a finite collection of open subsets $U_j \subset B$, where $1 \leq j \leq m$, such that there are local trivializations of the maps $U_j \times F \cong q^{-1}[U_j] \to U_j$ and h maps each subinterval $\left[\frac{j-1}{m}, \frac{j}{m}\right]$ into U_i. There is no problem constructing a lifting over the first interval starting at the point y, so by induction we can assume that we have a lifting over the interval $\left[\,0,\ \frac{j-1}{m}\,\right]$. To complete the inductive step, it is only necessary to construct a continuous lifting of the curve $\left[\,\frac{j-1}{m},\ \frac{j}{m}\,\right]$ starting at $h'\left(\frac{j-1}{m}\right)$. As in the case $j = 1$ we can find a lifting using local triviality. Ultimately we obtain the desired continuous lifting h'.∎

PROPOSITION 3. *For each $n \geq 1$, the quotient mapping $\pi : \mathbb{C}^{n+1} - \{\mathbf{0}\} \to \mathbb{CP}^n$ is locally trivial.*

Proof. We need to construct an open covering $\mathcal{V}$ of $\mathbb{CP}^n$ such that for each open subset $V_k \in \mathcal{V}$ the open subset $\pi^{-1}[V_k]$ is homeomorphic to $V_k \times (\mathbb{C} - \{0\})$ such that π corresponds to projection onto the first factor. For each k such that $1 \leq k \leq n+1$ take V_k be the set of all points in $\mathbb{CP}^n$ represented by homogeneous coordinates $(z_1, \cdots, z_{n+1})$ such that $z_k \neq 0$; these sets form an open covering because every element of $\mathbb{C}^{n+1} - \{\mathbf{0}\}$ has at least one nonzero coordinate. Note that if one set of homogeneous coordinates for a point has a nonzero k^{th} coordinate, then so does every other set.

If $k = n+1$ then V_k is the image of the mapping $j : \mathbb{C}^n \to \mathbb{CP}^n$ appearing in `projspaces1a.pdf`; by construction this map sends $(z_1, \cdots, z_n)$ to $(z_1, \cdots, z_n, 1)$. Furthermore, if $\Phi : \mathbb{C}^n \times (\mathbb{C} - \{0\}) \to \mathbb{C}^{n+1} - \{\mathbf{0}\}$ sends (z, t) to (tz, t), then Φ defines a homeomorphism onto the open subset $\pi^{-1}[V_{n+1}]$ such that $\pi \circ \Phi(z, t) = j(z)$. This means that the open set V_{n+1} satisfies the condition in the definition of local triviality.

To complete the proof, we need to show that each of the remaining open subsets V_k (where $k \leq n$) also satisfies the condition in the definition of local triviality. One quick way of doing this is to consider the invertible $(n+1) \times (n+1)$ matrices A_k obtained by interchanging columns number k and $n+1$ in the $(n+1) \times (n+1)$ identity matrix. Let $L(A_k)$ be the induced homeomorphism of $\mathbb{C}^{n+1} - \{\mathbf{0}\}$ as in Theorem 5 of `projspaces1a.pdf`, and let T_k be the associated projective collineation of $\mathbb{CP}^n$. Then as in the proof of the cited theorem we have $\pi \circ L(A_k) = T_k \circ \pi$. By construction T_k maps V_k to V_{n+1}, and $L(A_k)$ maps $\pi^{-1}[V_k]$ to $\pi^{-1}[V_{n+1}]$, and therefore there is a homeomorphism from $\mathbb{C}^n \times (\mathbb{C} - \{0\})$ to $\pi^{-1}[V_k]$ such that π restricted to the latter corresponds to projection onto the $\mathbb{C}^n$ coordinate. Since the sets V_k form an open covering of $\mathbb{CP}^n$, it follows that π is locally trivial.∎

Proof of Theorem 1

Let $u \in \pi_1(\mathbb{CP}^n, p_0)$, and let $h : [0,1] \to \mathbb{CP}^n$ be a closed curve representing u. By the preceding two results there is a continuous lifting $h' : [0,1] \to \mathbb{C}^{n+1} - \{\mathbf{0}\}$ such that $h'(0)$ is a predetermined point which projects to p_0. Since h' is a lifting of h and the latter is a closed curve, it follows that $h'(1)$ also projects to $p(0)$. Now the inverse image of $\{p_0\}$ in $\mathbb{C}^{n+1} - \{\mathbf{0}\}$ is homeomorphic to the arcwise connected space $\mathbb{C} - \{0\}$, so there is a continuous curve f joining $h'(1)$ to $h'(0)$. Consider the concatenation $h' + f$ of these two curves. Since $\mathbb{C}^{n+1} - \{\mathbf{0}\} \cong S^{2n+1} \times \mathbb{R}$ is simply connected, it follows that $h' + f$ represents the trivial element of $\pi_1(\mathbb{C}^{n+1} - \{\mathbf{0}\}, h'(0))$ and hence $(\pi \circ h') + (\pi \circ f) = h + (\text{constant})$ represents the trivial element of $\pi_1(\mathbb{CP}^n, p_0)$. Since h and $h + (\text{constant})$ represent the same element and h represents the original class u, it follows that u must be trivial. The class u was chosen arbitrarily, and therefore it follows that the fundamental group of $\mathbb{CP}^n$ must be trivial.∎