Real and complex projective spaces — 2

This is a sequel to the file `projspaces1.pdf`, and the purpose here is to compute the homology and cohomology of real and complex projective spaces. We shall frequently refer to Hatcher for some of the details, but there are also places in which we give alternative approaches.

Throughout this document $\mathbb{F}$ will denote the real or complex numbers, and d will denote the real dimension $\dim_{\mathbb{R}} \mathbb{F}$, which of course is 1 if $\mathbb{F} = \mathbb{R}$ and 2 if $\mathbb{F} = \mathbb{C}$. Similar considerations yield analogous results for projective spaces over the quaternions (this is noted at various places in Hatcher), but these are omitted in order to simplify the discussion at a few places.

DEFAULT CONVENTION. If no coefficient groups are specified for homology or cohomology groups, the coefficients are assumed to be some commutative ring with unit.

Cell decompositions of projective spaces

These are described explicitly in Examples 0.4 and 0.6 on pages 6–7 of Hatcher. For each k such that $0 \leq k \leq n$, this cell decomposition has exactly one k-cell e^{dl} in dimension $d \cdot k$, and the union of all cells through dimension $d \cdot k$ is just the standardly embedded $\mathbb{FP}^k$ corresponding to the image of the composite $S^{dk+d-1} \subset S^{dn+d-1} \to \mathbb{FP}^k$. The explicit attaching maps $S^{dk} \to \mathbb{FP}^k$ are described on the cited pages of Hatcher.

Suppose now that $n \geq 2$ and $\mathbb{F} = \mathbb{R}$, so that $\pi_1(\mathbb{FP}^n) \cong \mathbb{Z}_2$. Then the give cell decomposition $\mathcal{E}$ lifts to a cell decomposition $\mathcal{E}'$ of the double covering S^n with the following properties:

(*i*) For each k such that $0 \leq k \leq n$ there are two k-cells $e^k_\pm \subset S^k$ in $\mathcal{E}'$, and they correspond to the upper and lower hemispheres in S^k consisting of all points whose $(k+1)^{\text{st}}$ coordinates are nonnegative for e^k_+ and nonpositive for e^k_-.

(*ii*) If $Tx = -x$ is the antipodal involution on S^n and $\varphi : S^n \to \mathbb{RP}^n$, then T interchanges $e^k_\pm$ and φ maps $e^k_\pm - \partial e^k_\pm$ homeomorphically onto $e^k - \partial e^k$ (by construction $\varphi \circ T = \varphi$).

The map φ is a *cellular map* with respect to the cell decompositions on S^n and $\mathbb{RP}^n$ in the sense that it sends the k-skeleton of $\mathcal{E}'$ to the k-skeleton of $\mathcal{E}$ for all k, and similarly the antipodal involution T is cellular with respect to the cell decomposition on S^n. One reason for interest in cellular maps is the following result about cellular chains and homology.

THEOREM. *Let $(X_1, \mathcal{E}_1)$ and $(X_2, \mathcal{E}_2)$ be cell complexes, and let $\theta : X_1 \to X_2$ be a cellular map. Then φ induces a chain map of cellular chain groups, and its associated homology map is the homology map of θ.*

In this document we shall not need the final assertion in the theorem, so the proof of this statement will be omitted.

The existence of the chain map follows from the fact that φ sends k-skeleta to k-skeleta and the definitions of the cellular chain complexes via the relative homology groups $C_k(Y, \mathcal{F}) = H_k(Y_k, Y_{k-1})$.■

This result implies the following additional consequence in the main example of interest to us:

(*iii*) The map φ induces a cellular complex chain map $\varphi_\# : C_*(S^n) \to C_*(\mathbb{RP}^n)$, and this map takes the algebraic generators of $C_k(S^n)$ corresponding to $e^k_\pm$ to $\pm$ the algebraic generator of $C_k(\mathbb{RP}^n)$ corresponding to e^k.

This follows from the final assertion in *(ii)* and the description of cellular homology groups as free abelian groups on the cells in each dimension.

The homology of $\mathbb{FP}^n$

The computation for complex projective spaces is very easy to state and prove.

PROPOSITION. *If* $\mathbb{D}$ *is a commutative ring with unit, then* $H^q(\mathbb{CP}^n;\mathbb{D})$ *is isomorphic to* $\mathbb{D}$ *if* q *is even and* $0 \leq q \leq 2n$*, and it is zero otherwise.*

Proof. The homology groups are isomorphic to the homology groups of the cellular chain complex. Since the nonzero groups are concentrated in even dimensions and boundary maps involve adjacent dimensions, it follows that either the domain or the codomain of every boundary map is trivial, and hence every boundary map is trivial, so that the homology groups are isomorphic to the cellular chain groups. By construction the cellular chain groups are given as in the statement of the proposition.■

The real case is considerably less trivial to analyze. Our approach will be more algebraic than Hatcher's.

We begin by summarizing some algebraic consequences of the preceding discussion. In each nontrivial dimension we have the cellular chain map

$$\varphi_{\#} : \mathbb{D} \oplus \mathbb{D} \cong C_k(S^n) \quad\longrightarrow\quad C_k(\mathbb{RP}^n) \cong \mathbb{D}$$

and by choosing suitable units in $\mathbb{D}$ we can choose the standard free generators so that the chain map $T_{\#}$ interchanges the free generators $\alpha^k_{\pm}$ of $C_k(S^n)$ and maps each of these to the free generator $\alpha^k \in C_k(\mathbb{RP}^n)$. This information provides us with enough clues to reconstruct the differentials in both cellular chain groups.

THEOREM A. *The boundary classes* $d_k(\alpha^k_+)$ *and* $d_k(\alpha_k)$ *in cellular homology are given as follows:*

(1) *We have* $d_k(\alpha^k_+) = u(\alpha^{k-1}_+ \ + \ (-1)^k\alpha^{k-1}_-$ *for some unit* $u \in \mathbb{D}$*.*

(2) *We have* $d_k(\alpha^k) = u(1 \ + \ (-1)^k)\alpha^{k-1}$ *for some unit* $u \in \mathbb{D}$*.*

Before proving these results, we shall use the second conclusion to compute the homology of real projective space with various coefficients.

THEOREM B. *The homology groups* $H_k(\mathbb{RP}^n;\mathbb{D})$ *are given as follows:*

(1) $H_k(\mathbb{RP}^n;\mathbb{D})$ *is isomorphic to* $\mathbb{D}$ *if* $k = 0$ *or if* n *is odd and* $k = n$*.*

(2) $H_k(\mathbb{RP}^n;\mathbb{D})$ *is isomorphic to* $\mathbb{D}/2\cdot\mathbb{D}$ *if* n *is odd and* $1 \leq k < n$*.*

(3) $H_k(\mathbb{RP}^n;\mathbb{D})$ *is zero otherwise.*

This follows immediately from the formula $d_k(\alpha^k) = u(1 \ + \ (-1)^k)\alpha^{k-1})$; the details of checking this are left to the reader.■

It is instructive to look at the meaning of Theorem A when $\mathbb{D}$ is one of the standard examples.

CASE A1: The case $\mathbb{D} = \mathbb{Z}$. In this case (1) $H_k(\mathbb{RP}^n;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ if $k = 0$ or if n is odd and $k = n$, (2) $H_k(\mathbb{RP}^n;\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$ if n is odd and $1 \leq k < n$, (3) $H_k(\mathbb{RP}^n;\mathbb{Z})$ is zero otherwise.■

CASE A2: The case $\mathbb{D} = \mathbb{Q}$ or $\mathbb{D} = \mathbb{Z}_p$ where p is an odd prime. For these examples $2 \cdot \mathbb{D} = \mathbb{D}$, and hence the computation reduces to $H_k(\mathbb{RP}^n; \mathbb{D})$ is isomorphic to $\mathbb{D}$ if $k = 0$ or if n is odd and $k = n$, and $H_k(\mathbb{RP}^n; \mathbb{D})$ is zero otherwise.■

CASE A3: The case $\mathbb{D} = \mathbb{Z}_2$. For this example $2 \cdot \mathbb{D} = 0$, and therefore $H_k(\mathbb{RP}^n; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}/2$ if $0 \leq k \leq n$, and $H_k(\mathbb{RP}^n; \mathbb{Z}_2)$ is zero otherwise.■

Proof of Theorem A. Note that the cell decomposition of S^n contains the hemisphere D^n_+ as a subcomplex, and since cellular chains yield the homology of a space it follows that the chain subcomplex $C_*(D^n_+) \subset C_*(S * n)$ is acyclic. We shall see that this provides enough information to recover the differentials on $C_*(S^n)$.

We can directly verify the formula in Theorem A when $k = 1$, for in this case the cell $D^1_+ \subset D^n_+$ defines a cell subcomplex which is in fact a simplicial complex. Assume now that we have verified the formula in all positive dimensions $\leq k$, and consider the class $d_{k+1}(e^{k+1}_+)$. We know this class is a cycle. Furthermore, by the induction hypothesis we can compute directly that the cycle group $\mathfrak{z}_k(D^n_+)$ is generated by $\alpha^k_+ + (-1)^k \alpha^k_-$ (details are left to the reader; recall that the chain map $T_\#$ switches generators). The only choices for $d_{k+1}(e^{k+1}_+)$ consistent with $H_k(D^n_+) = 0$ are given by units in $\mathbb{D}$ times the generating cycle. We can now map everything into the chain complex $C_*(S^n)$, where we use the chain map $T_\#$ to derive the desired formula for $d_{k+1}(e^{k+1}_-)$. This completes the proof of the inductive step, yielding (1).

Statement (2) follows from (1) and the fact that the chain map $\varphi_\#$ sends $\alpha^k_\pm$ to α_k.■

The complement of $\mathbb{FP}^m$ in $\mathbb{FP}^{m+k}$

If we take the standard "linear" embedding of $\mathbb{FP}^m$ in $\mathbb{FP}^{m+k}$ as the set of all points whose last k homogeneous coordinates vanish, then the set of all points C_m whose first $(m+1)$ homogeneous coordinates vanish is a projective $(k-1)$-plane which is disjoint from the standard copy of $\mathbb{FP}^m$, and standard results in projective geometry imply that there are no projective k-planes in $\mathbb{FP}^{m+k}$ which are disjoint from $\mathbb{FP}^m$. We are going to need the following result about the inclusion of C_m in $\mathbb{FP}^{m+k} - \mathbb{FP}^m$.

THEOREM C. *The inclusion of C_m in $\mathbb{FP}^{m+k} - \mathbb{FP}^m$ is a deformation retract.*

Proof. Throughout this proof we shall view $\mathbb{F}^{m+k+1}$ as the product $\mathbb{F}^{m+1} \times \mathbb{F}^k$, so that $\mathbb{FP}^m$ is the set of all points with homogeneous coordinates of the form $(x, 0)$ and C_m is the set of all points with homogeneous coordinates of the form $(0, y)$. The disjointness of $\mathbb{FP}^m$ and C_m reflects the fact that a nonzero vector (x, y) in $\mathbb{F}^{m+k+1}$ either has $x \neq 0$ or $y \neq 0$. By construction, the complement $\mathbb{FP}^{m+k} - \mathbb{FP}^m$ consists of all points whose homogeneous coordinates lie in $\mathbb{F}^{m+1} \times (\mathbb{F}^k - \{\mathbf{0}\})$.

Define a straight line homotopy

$$H' : \Big(\mathbb{F}^{m+1} \times (\mathbb{F}^k - \{\mathbf{0}\})\Big) \times [0,1] \;\longrightarrow\; \mathbb{F}^{m+1} \times (\mathbb{F}^k - \{\mathbf{0}\})$$

by $H'(x, y; t) = \big((1-t)x, y\big)$. By construction H'_0 is the identity, the map H'_1 maps into $\mathbf{0}\} \times (\mathbb{F}^k - \{\mathbf{0}\})$, and the restriction of H'_t to the latter subspace is the identity for all f (hence H' shows that the subspace is a strong deformation retract of the entire domain of H'_t). We claim that this passes to a homotopy of quotient spaces

$$H : \Big(\mathbb{FP}^{m+k} - \mathbb{FP}^m\Big) \times [0,1] \;\longrightarrow\; \mathbb{FP}^{m+k} - \mathbb{FP}^m$$

with similar properties. One step in proving this claim is to check that H' passes to a map of sets; this is routine and amounts to verifying that equivalent points in the domain go to equivalent points in the codomain under H'. The second and final step is to prove that H is continuous, and this is essentially a special case of the following result, which appears in Munkres, *Topology*, ??????.

> *If X is a locally compact Hausdorff space and $\mathcal{R}$ is an equivalence relation on X, then the natural continuous and $1-1$ map from the quotient space $(X \times [0,1])/(\mathcal{R} \times \mathrm{Diag}_{[0,1]})$ to $(X/\mathcal{R}) \times [0,1]$ is a homeomorphism.*

The existence and continuity of H implies that C_m is a strong deformation retract of $\mathbb{FP}^{m+k} - \mathbb{FP}^m$.∎

Cup product computations

On page 213 of Hatcher there is an assertion implying that the relative cup product map

$$H^s\left(\mathbb{R}^{s+t}, \mathbb{R}^{s+t} - \{\mathbf{0}\} \times \mathbb{R}^t, \mathbb{F}\right) \otimes_{\mathbb{F}} H^t\left(\mathbb{R}^{s+t}, \mathbb{R}^{s+t} - \mathbb{R}^s \times \{\mathbf{0}\}; \mathbb{F}\right) \longrightarrow H^{s+t}\left(\mathbb{R}^{s+t}, \mathbb{R}^{s+t} - \{\mathbf{0}\}; \mathbb{F}\right)$$

is algebraically equivalent to the standard isomorphism $\mathbb{F} \otimes_{\mathbb{F}} \mathbb{F} \cong \mathbb{F}$, but unfortunately the reference for a proof (to Example 3.11 on pages 210–211) is at best very cryptic and at worst unjustified. Therefore we shall give a proof here.

The assertion in Hatcher is clearly related to the fact that the relative cross product

$$H^s(\mathbb{R}^s, \mathbb{R}^s - \{\mathbf{0}\}; \mathbb{F}) \ \otimes_{\mathbb{F}} \ H^t(\mathbb{R}^s, \mathbb{R}^t - \{\mathbf{0}\}; \mathbb{F}) \ \longrightarrow \ H^{s+t}\left(\mathbb{R}^{s+t}, \mathbb{R}^{s+t} - \{\mathbf{0}\}; \mathbb{F}\right)$$

is algebraically equivalent to the standard isomorphism $\mathbb{F} \otimes_{\mathbb{F}} \mathbb{F} \cong \mathbb{F}$ by an appropriate version of the Künneth Theorem (note that

$$\mathbb{R}^{s+t} - \{\mathbf{0}\} \ = \ \mathbb{R}^s \times \{\mathbf{0}\} \ \cup \ \{\mathbf{0}\} \times \mathbb{R}^t$$

so that the relative cross product has the form described in Unit IV of the course notes `advanced-notes2012.pdf`), and the derivation of Hatcher's assertion from this isomorphism is given by the following result, in which all cohomology groups and tensor products are assumed to be over some field $\mathbb{F}$:

COMPATIBILITY LEMMA. *There is a commutative diagram of the form*

$$\begin{array}{ccc} H^s(\mathbb{R}^s, \mathbb{R}^s - \{\mathbf{0}\}) \otimes H^t(\mathbb{R}^t, \mathbb{R}^t - \{\mathbf{0}\}) & \rightarrow & H^{s+t}\left(\mathbb{R}^{s+t}, \mathbb{R}^{s+t} - \{\mathbf{0}\}\right) \\ \downarrow \pi_1^* \otimes \pi_2^* & & \downarrow = \\ H^s\left(\mathbb{R}^{s+t}, \mathbb{R}^{s+t} - \{\mathbf{0}\} \times \mathbb{R}^t\right) \otimes H^t\left(\mathbb{R}^{s+t}, \mathbb{R}^{s+t} - \mathbb{R}^s \times \{\mathbf{0}\}\right) & \rightarrow & H^{s+t}\left(\mathbb{R}^{s+t}, \mathbb{R}^{s+t} - \{\mathbf{0}\}\right) \end{array}$$

in which the upper and lower horizontal arrows are given by the relative cross and cup products respectively and the maps $\pi_1, \pi_2 : \mathbb{R}^{s+t} \cong \mathbb{R}^s \times \mathbb{R}^t \to \mathbb{R}^s, \mathbb{R}^t$ are projections onto the first and second factors.

In algebraic terms, if we let $\omega_k \in H^k(\mathbb{R}^k, \mathbb{R}^k - \{\mathbf{0}\}; \mathbb{F})$ be the image of a standard generator in the integral group $H^k(\mathbb{R}^k, \mathbb{R}^k - \{\mathbf{0}\}; \mathbb{Z})$ under the unique unit preserving coefficient homomorphism $\mathbb{Z} \to \mathbb{F}$, then

$$\pi_1^*\omega_s \ \cup \ \pi_2^*\omega_t \ = \ \omega_s \times \omega_t$$

which corresponds to the assertion in Hatcher.

Proof. As usual, the cup product is the composite of the cross product and a suitable diagonal mapping:

$$H^s\left(\mathbb{R}^{s+t}, \mathbb{R}^{s+t} - \{\mathbf{0}\} \times \mathbb{R}^t\right) \otimes H^t\left(\mathbb{R}^{s+t}, \mathbb{R}^{s+t} - \mathbb{R}^s \times \{\mathbf{0}\}\right)$$

$$\Big\downarrow \times$$

$$H^{s+t}\left(\mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^s \times \mathbb{R}^t, (\mathbb{R}^s - \{\mathbf{0}\} \times \mathbb{R}^t \times \mathbb{R}^s \times \mathbb{R}^t) \cup (\mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^s \times \mathbb{R}^t - \{\mathbf{0}\})\right)$$

$$\Big\downarrow \Delta^*$$

$$H^{s+t}\left(\mathbb{R}^{s+t}, \mathbb{R}^{s+t} - \{\mathbf{0}\}\right)$$

In this example the diagonal map Δ sends the pair $(\mathbb{R}^{s+t}, \mathbb{R}^{s+t} - \{\mathbf{0}\})$ to

$$H^{s+t}\left(\mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^s \times \mathbb{R}^t, (\mathbb{R}^s - \{\mathbf{0}\} \times \mathbb{R}^t \times \mathbb{R}^s \times \mathbb{R}^t) \cup (\mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^s \times \mathbb{R}^t - \{\mathbf{0}\})\right) .$$

It is straightforward to verify that

$$\pi_1^*\omega_s \ \times \ \pi_2^*\omega_t \quad = \quad (\pi_1 \times \pi_2)^*(\omega_s \times \omega_t)$$

and if we apply Δ^* to both sides we obtain the identity

$$\pi_1^*\omega_s \ \cup \ \pi_2^*\omega_t \quad = \quad \Delta^*(\pi_1 \times \pi_2)^*(\omega_s \times \omega_t)$$

so the proof of the lemma reduces to showing that the composite of Δ and $\pi_1 \times \pi_2$ is the identity; but this follows because

$$(\pi_1 \times \pi_2)^\circ \Delta(v, w) \quad = \quad \pi_1 \times \pi_2(v, w, v, w) = (v, w) \ .■$$

The main results

NOTATIONAL CONVENTIONS. As before, $\mathbb{F}$ will denote the real or complex numbers, while d denotes $\dim_{\mathbb{R}} \mathbb{F}$ and the coefficients $\mathbb{D}$ will be an arbitrary field if $\mathbb{F} = \mathbb{C}$ and $\mathbb{D} = \mathbb{Z}_2$ if $\mathbb{F} = \mathbb{R}$. With respect to these conventions our previous computations can be restated in the form

$$H^k(\mathbb{FP}^n; \mathbb{D}) \quad \cong \quad \mathbb{D} \qquad \text{if} \quad \mathrm{k} \quad = \quad 0, \mathrm{d}, 2\mathrm{d}, \ \cdots \ \mathrm{nd}$$

and $H^k(\mathbb{FP}^n; \mathbb{D})$ is trivial otherwise. Let ξ_j denote a generator of $H^{dj}(\mathbb{FP}^n; \mathbb{D})$ for $0 \leq j \leq n$. The key step in Hatcher's computation of cup products is the following result:

NONDEGENERACY LEMMA. *In the preceding notation, for all j we have $\xi_j \cup \xi_{n-j} \neq 0$.*

Proof. We shall follow Hatcher fairly closely and use it as a reference for some details.

We shall first summarize some previous observations and their consequences. The results on cell decomposition imply that the standardly embedded $\mathbb{FP}^m$ in $\mathbb{FP}^n$ is the dm-skeleton of the latter. Since the differentials in $C_*(\mathbb{FP}^n; \mathbb{D})$ are all trivial by our computations, it follows that the maps $H^k(\mathbb{FP}^n; \mathbb{D}) \to H^k(\mathbb{FP}^m; \mathbb{D})$ are isomorphisms if $k \leq dm$ (and trivial otherwise for dimensional reasons). Combining this with our results on the complements $\mathbb{FP}^n - \mathbb{FP}^m$, we also have the following:

(*a*) The maps $H^k(\mathbb{FP}^n;\mathbb{D}) \to H^k(\mathbb{FP}^n - \mathbb{FP}^m;\mathbb{D})$ are isomorphisms if $k \leq d(n-m-1)$ and trivial otherwise.

(*b*) The maps $H^k(\mathbb{FP}^n, \mathbb{FP}^n - \mathbb{FP}^m;\mathbb{D}) \to H^k(\mathbb{FP}^n;\mathbb{D})$ are isomorphisms if $dn \geq k \geq d(n-m)$ and trivial otherwise.

Now let $Q \subset \mathbb{FP}^n$ be the $(n-m)$-plane consisting of all points whose first $m-1$ homogeneous coordinates are zero; it follows immediately that $Q \cap \mathbb{FP}^m$ consists of the point p with homogeneous coordinates given by the unit vector $\mathbf{e}_m \in \mathbb{F}^{n+1}$. If $W \subset \mathbb{FP}^n$ denotes the complement of the hyperplane defined by the linear homogeneous equation $x_m = 0$, then every point in W has a unique set of homogeneous coordinates $(t_1, \cdot, t_{n+1})$ such that $t_m = 1$, and if we delete this homogeneous coordinate we obtain a homeomorphism from W to $\mathbb{F}^n$ (verify these assertions; the inverse map sends $(x_1, \cdots, x_n)$ to the class $[t_1, \cdot, t_{n+1}]$ where $t_i = x_i$ if $i < m$, $t_m = 1$ and $t_i = x_{i-1}$ if $i > m$). Under this homeomorphism $W \leftrightarrow \mathbb{F}^n$ the intersections $\mathbb{FP}^m \cap W$ and $Q \cap W$ correspond to $\{\mathbf{0}\} \times \mathbb{F}^{n-m}$ and $\mathbb{F}^m \times \{\mathbf{0}\}$ respectively, and p correspond to $\mathbf{0} \in \mathbb{F}^n$.

One key assertion in Hatcher is that the maps

$$H^{dm}(\mathbb{FP}^n, \mathbb{FP}^n - Q;\mathbb{D}) \;\to\; H^k(\mathbb{FP}^n;\mathbb{D})\;, \qquad H^{d(n-m)}(\mathbb{FP}^n, \mathbb{FP}^n - \mathbb{FP}^m;\mathbb{D}) \;\to\; H^{d(n-m)}(\mathbb{FP}^n;\mathbb{D})$$

are isomorphisms, and therefore we may choose cohomology classes ζ_m and ζ_{n-m} in the respective domains so that ζ_j maps to ξ_j (in fact, there are unique choices).

The second key assertion in Hatcher is that the relative cup product $\zeta_m \cup \zeta_{n-m}$ is nontrivial, and this is done using the following commutative diagram on page 213 of that reference:

$$\begin{array}{ccc}
H^{dm}(\mathbb{FP}^n) \otimes H^{d(n-m)}(\mathbb{FP}^n) & \stackrel{\cup}{\longrightarrow} & H^{dn}(\mathbb{FP}^n) \\
\uparrow & & \uparrow \\
H^{dm}\,(\mathbb{FP}^n, \mathbb{FP}^n - Q) \otimes H^{d(n-m)}\,(\mathbb{FP}^n, \mathbb{FP}^n - \mathbb{FP}^m) & \stackrel{\cup}{\longrightarrow} & H^{dn}(\mathbb{FP}^n, \mathbb{FP}^n - \{p\}) \\
\downarrow & & \downarrow \\
H^{dm}(\mathbb{F}^n, \mathbb{F}^n - \mathbf{0} \times \mathbb{F}^{n-m}) \otimes H^{d(n-m)}(\mathbb{F}^n, \mathbb{F}^n - \mathbb{F}^m \times \mathbf{0}) & \stackrel{\cup}{\longrightarrow} & H^{dn}(\mathbb{F}^n, \mathbb{F}^n - \mathbf{0})
\end{array}$$

We have verified that the cup product map in the bottom row defines an isomorphism, and as noted above we also know that the upward pointing vertical arrows define isomorphisms. Also, by excision we know that the downward pointing vertical arrows define isomorphisms (here we are identifying $\mathbb{F}^n$ with the open set W described above). If we combine these observations, we see that the top and middle horizontal arrows must also define isomorphisms. Since the conclusion of the result is equivalent to saying that the top horizontal arrow defines an isomorphism, this completes the proof.■

MAIN THEOREM. *In the setting above the (possibly iterated) cup product* ξ_1^j *generates* $H^{dj}(\mathbb{FP}^n;\mathbb{D})$.

We can reformulate this result to state that the cohomology ring is a truncated polynomial ring on one variable ξ such that $\xi^{n+1} = 0$.

Proof. If $n = 1$ this follows because ξ_1 is a generator. Assume by induction that the result is true for $\mathbb{FP}^{n-1}$ where $n \geq 2$. By statement (*a*) in the proof of the preceding lemma and the fact that maps in cohomology preserve cup products, the conclusion of the theorem follows provided $j \leq n-1$, so it is only necessary to check the case $j = n$. The preceding lemma showed that

$\xi_{n-1} \cup \xi_1 = \xi_n$, and the induction hypothesis implis that $\xi_1^{n-1} = u \cdot \xi_{n-1}$ for some (nonzero) unit $u \in \mathbb{D}$. Therefore it follows that $\xi_1^n = u \cdot \xi_{n-1} \cup \xi_1 = u \cdot \xi_n$, completing the inductive step.∎

Topological applications

We can combine the preceding computations with the results of Section IV.6 in `advanced-notes2012.pdf` to yield the following results on covering $\mathbb{FP}^n$ — which is a topological dn-manifold — by contractible open sets.

COROLLARY. *If $\mathcal{U}$ is an open covering of $\mathbb{FP}^n$ by contractible open subsets, then $\mathcal{U}$ contains at least $n+1$ distinct open subsets. Conversely, $\mathbb{FP}^n$ has an open covering consisting of $n+1$ (distinct) open subsets.*

Proof. The first statement follows because $\xi_1^n \in H^{dn}(\mathbb{FP}^n;\mathbb{D})$ is a nonzero cup product of n positive dimensional cohomology classes, so the results of Section IV.6 imply that an open covering of $\mathbb{FP}^n$ by contractible open sets must contain at least $n+1$ open subsets. To see the converse, let $U_i \subset \mathbb{FP}^n$ be the open set of all points whose i^{th} homogeneous coordinate in $\mathbb{F}^{n+1}$ is nonzero. There are $n+1$ such open subsets, and U_{n+1} is homeomorphic to $\mathbb{F}^n$ by the homeomorphism sending $(x_1, \cdots, x_n) \in \mathbb{F}^n$ to the point with homogeneous coordinates $(x_1, \cdots, x_n, 1) \in \mathbb{F}^{n+1}$; an explicit inverse is the map sending $[t_1, \cdots, t_n, t_{n+1}]$ to

$$(t_1/t_{n+1}, \cdots, t_n/t_{n+1}) \quad \in \quad \mathbb{F}^n$$

(verify that this point does not depend upon the choice of homogeneous coordinates). If we permute the roles of the coordinates in $\mathbb{F}^{n+1}$ so that v_i corresponds to t_{n+1}, then the same method shows that each U_i is homeomorphic to $\mathbb{F}^n$ and hence all the sets U_i are contractible, yielding the type of open covering described in the second sentence of the proposition.∎

We also have the following application, which is a topological version of the basic principle that a p-plane and an $(n-p)$-plane in $\mathbb{FP}^n$ always have a common point.

NONDISJUNCTION PROPERTY. *Assume the conditions on $\mathbb{F}$ and $\mathbb{D}$ in the Main Theorem, and let $1 \leq m \leq n-1$. Then there is no topological embedding of $\mathbb{FP}^{n-m}$ in $\mathbb{FP}^n - \mathbb{FP}^m$ which induces an isomorphism in d-dimensional cohomology.*

Proof. Assume the contrary, and let $Y \subset \mathbb{FP}^n - \mathbb{FP}^m$ be homeomorphic to $\mathbb{FP}^{n-m}$ such that the inclusion induces an isomorphism in d-dimensional cohomology. Since induced maps in cohomology preserve cup products and $\xi_1^k \neq 0$ if $H^{dk}(\mathbb{FP}^r;\mathbb{D}) \neq 0$, it follows that the composite of inclusion induced mappings

$$H^{d(n-m)}(\mathbb{FP}^n;\mathbb{D}) \longrightarrow H^{d(n-m)}(\mathbb{FP}^n - \mathbb{FP}^m;\mathbb{D}) \longrightarrow H^{d(n-m)}(\mathbb{FP}^{n-m};\mathbb{D})$$

must be nontrivial. On the other hand, we know that $H^{d(n-m)}(\mathbb{FP}^n - \mathbb{FP}^m;\mathbb{D}) = 0$ so this is impossible. Therefore our assumption about the existence of a topological embedding must be false, and this proves the result.∎

The preceding result reflects a fundamentally important phenomenon in algebraic topology: *Frequently cup products can be used to obtain information on whether (and, if so, how) two submanifolds $M, N \subset P$ intersect*. The following books provide additional information on this topic. Chapters 2 and 3 of the book by Guillemin and Pollack describe a more geometrical approach to such questions, and Section VIII.13 of Dold, *Lectures on Algebraic Topology*, describes a more algebraic approach.

V. Guillemin and A. Pollack. *Differential topology* (Reprint of the 1974 original). AMS Chelsea Publishing, Providence, RI, 2010.