V.2 : Generalized Stokes' Formula

(Conlon, §§ 2.6, 8.1–8.2; Lee, Ch. 14)

At the end of Section 0 we discussed a far-reaching extension of the classical theorems of vector analysis (including the Fundamental Theorem of Calculus) to higher dimensions. In this section we shall formulate a version of this generaliztion which plays the key role in relating smooth singular cochains to differential forms.

Integration over smooth singular chains

If U is an open subset of $\mathbb{R}^n$ and $T\Lambda_q \to U$ is a smooth singular q-simplex, then the basic integration formula in Section V.0 provides a way of defining an integral $\int_T \omega$ if $\omega \in \wedge^q(U)$. There is a natural extension of this to singular chains; if $\mathbf{c}$ is the smooth singular chain $\sum_i n_i T_i$ where the n_i are integers, then since the group of smooth singular q-chains is free abelian on the smooth singular q-simplices the following is well defined:

$$\int_{\mathbf{c}} \omega \quad = \quad \sum_i n_i \int_{T_i} \omega$$

This definition has the following invariance property with respect to smooth mappings $f : U \to V$.

PROPOSITION 1. *Let $\mathbf{c} \in S_q^{\mathrm{smooth}}(U)$, where U is above, let $f : U \to V$ be smooth and let $\omega \in \wedge^q(V)$. Then we have*

$$\int_{f_{\#}^{\mathrm{smooth}}(\mathbf{c})} \omega \quad = \quad \int_{\mathbf{c}} f^{\#}\omega \ .$$

This follows immediately from the definition of integrals and the Chain Rule.■

The combinatorial form of the **Generalized Stokes' Formula** is a statement about integration of forms over smooth singular chains.

THEOREM 2. (Generalized Stokes' Formula, combinatorial version) *Let* $\mathbf{c}$, U, ω...(*etc.*) *be as above. Then we have*

$$\int_{d\,\mathbf{c}} \omega \quad = \quad \int_{\mathbf{c}} d\,\omega \ .$$

Full proofs of this result appear on pages 251–253 of Conlon and also on pages 272–275 of Rudin, *Principles of Mathematical Analysis* (3^{rd} Ed.)$^{(\star)}$. Here is an outline of the basic steps: First of all, by additivity it is enough to prove the result when $\mathbf{c}$ is given by a smooth singular simplex T. Next, by Proposition 1 and the identity $f^{\#} \circ d \ = \ d \circ f^{\#}$, we know that it suffices to prove the result when T is the universal singular simplex $\mathbf{1}_q$ defined by the inclusion of Λ_q — the simplex in $\mathbb{R}^q$ whose vertices are $\mathbf{0}$ and the unit vectors — into some small open neighborhood W_0 of Λ_q. In this case the integrals reduce to ordinary integrals in $\mathbb{R}^q$. We can reduce the proof even further as follows: Let $\theta_i \in \wedge^{q-1}(W_0)$ be the basic $(q-1)$-form $dx^{i_1} \wedge \cdots dx^{i_{q-1}}$, where $i_1 < \cdots < i_{q-1}$ runs over all elements of $\{1, \cdots, q\}$ **except** i. By additivity it will suffice to prove the theorem for $(q-1)$-forms expressible as $g\,\theta_i$, where g is a smooth function on W_0. Yet another change of variables argument shows that it suffices to prove the result for $(q-1)$-forms expressible as $g\,dx^2 \wedge \cdots \wedge dx^q$. Now the exterior derivative of the latter form is equal to

$$\frac{\partial g}{\partial x^1} \cdot dx^1 \wedge \cdots dx^q$$

so the proof reduces to evaluating the integral of the left hand factor in this expression over Λ_q, and this is done by viewing this multiple integral as an interated integral via Fubini's Theorem (see Rudin, *Real and complex analysis* or almost any text discussing Lebesgue integration) and applying the Fundamental Theorem of Calculus.■

RELATION TO CLASSICAL VECTOR ANALYSIS. The identifications in (i) – (iii) lead to a general statement that includes the following three basic results:

(1) The standard path independence result stating that the line integral $\int \nabla f \cdot d\mathbf{x}$ is equal to f(final point on curve) − f(initial point on curve).

(2) Stokes' Theorem (*note the spelling!!*) relating line and surface integrals.

(3) The so-called Gauss or Divergence Theorem relating surface and volume integrals.

In each case the result can be stated in terms of differential forms and p-surfaces (where $p = 1, 2, 3$) as follows: *If we are given a p-surface σ that has a reasonable notion of boundary $\partial\sigma$ such that $\partial\sigma$ is somehow a sum of $(p-1)$-surfaces with coefficients of ± 1, then*

$$\int_{\partial\sigma} \omega \quad = \quad \int_{\sigma} d\omega$$

for all $(p-1)$-forms ω.

In all cases the relationship to the Generalized Stokes' Formula depends upon the existence of *piecewise smooth triangulations* for the domains in which the various integrals are defined. More precisely, these are families of mappings T_α from the standard simplices Λ_q satisfying the following conditions:

(a) The union of the images of the simplices is the entire domain of integration, and the intersection of two images is a common face.

(b) Each map T_α is smooth and 1–1, and the derivative matrix at each point (*i.e.*, the matrix of partial derivatives of the coordinate functions of T_α) normally has rank $q(\alpha)$; in some cases these statements can be weakened slightly to allow some irregular behavior on the boundaries.

(c) The structure described above induces a similar structure on the boundary of the domain of integration.

(d) The sum of the integrals with respect to the mappings T_α are the standard notion of integral for the domain under consideration, and likewise for the boundary.

One way of restating the final condition is to say that if one forms a triangulating chain for the domain of integration by adding the symbols $\pm T_\alpha$ formally , then the algebraic boundary of this chain (in the sense of singular homology) will be a triangulating chain for the boundary of the domain. Standard examples in multivariable calculus amount to saying that such a condition does not hold for a smooth bounded surface in $\mathbb{R}^3$ corresponding to a Möbius strip.

Results (1)–(3) are special cases of more general result which hold in all finite dimensions. Unfortunately, precise formulations of such generalizations require more background than we have developed (mainly from [MunkresEDT]), so we shall not try to state such results explicitly here.

V.3 : Definition and properties of de Rham cohomology

(Hatcher, §§ 2.1, 2.3, 3.1; Conlon, §§ 2.6, 8.1, 8.3–8.5; Lee, Ch. 15)

Let U be an open subset of $\mathbb{R}^n$ for some n. Since the exterior derivative on $\wedge^p(U)$ satisfies $d \circ d = 0$, it follows that $(\wedge^*(U), d^*)$ is a cochain complex, which we shall call the **de Rham (cochain) complex**.

Definition. The **de Rham cohomology groups** $H^q_{\rm DR}(U)$ are the cohomology groups of the de Rham complex of differential forms.

The Generalized Stokes' Formula in Theorem 2.2 implies that integration of differential forms defines a morphism J of chain complexes from $\wedge^*(U)$ to $S^*(U;\mathbb{R})$, where U is an arbitrary open subset of some Euclidean space. The aim of this section and the next is to show that the associated cohomology map $[J]$ defines an isomorphism from $H^*_{\rm DR}(U)$ to $H^*_{\rm smooth}(U;\mathbb{R})$; by the results of the preceding section, it will also follow that the de Rham cohomology groups are isomorphic to the ordinary singular cohomology groups $H^*(U;\mathbb{R})$. In order to prove that $[J]$ is an isomorphism, we need to show that the de Rham cohomology groups $H^*_{\rm DR}(U)$ satisfy analogs of certain formal properties that hold for (smooth) singular cohomology.

One of these properties is a homotopy invariance principle, and the other is a Mayer-Vietoris sequence. Extremely detailed treatments of these results are given in Conlon, so at several points we shall be rather sketchy.

The following abstract result will be helpful in proving homotopy invariance. There are obvious analogs for other subcategories of topological spaces and continuous mappings, and also for covariant functors.

LEMMA 1. *Let T be a contravariant functor defined on the category of open subsets of $\mathbb{R}^n$ and smooth mappings. Then the following are equivalent:*

(1) *If f and g are smoothly homotopic mappings from U to V, then $T(f) = T(g)$.*

(2) *If U is an arbitrary open subset of $\mathbb{R}^n$ and $i_t : U \to U \times \mathbb{R}$ is the map sending u to (u, t), then $T(i_0) = T(i_1)$.*

Proof. (1) $\Longrightarrow$ (2). The mappings i_0 and i_1 are smoothly homotopic, and the inclusion map defines a homotopy from $U \times (-\varepsilon, 1 + \varepsilon)$ to $U \times \mathbb{R}$.

(2) $\Longrightarrow$ (1). Suppose that we are given a smooth homotopy $H : U \times (-\varepsilon, 1+\varepsilon) \to V$. Standard results from 205C imply that we can assume the homotopy is "constant" on some sets of the form $(-\varepsilon, \eta) \times U$ and $(1 - \eta, 1 + \varepsilon) \times U$ for a suitably small positive number η. One can then use this property to extend H to a smooth map on $U \times \mathbf{R}$ that is "constant" on $(-\infty, \eta) \times U$ and $(1 - \eta, \infty) \times U$. By the definition of a homotopy we have $H \circ i_1 = g$ and $H \circ i_0 = f$. If we apply the assumption in (1) we then obtain

$$T(g) \quad = \quad T(i_1) \circ T(H) \quad = \quad T(i_0) \circ T(H) \quad = \quad T(f)$$

which is what we wanted.■

A simple decomposition principle for differential forms on a cylindrical open set of the form $U \times \mathbb{R}$ will be useful. If U is open in $\mathbb{R}^n$ and I denotes the k-element sequence $i_1 < \cdots < i_k$, we shall write

$$\xi_I \quad = \quad dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and say that such a form is a *standard basic monomial k-forms* on U. Note that the wedge of two standard basic monomials $\xi_J \wedge \xi_I$ is either zero or ± 1 times a standard basic monomial, depending upon whether or not the sequences J and I have any common wedge factors.

PROPOSITION 2. *Every k-form on U is uniquely expressible as a sum*

$$\sum_I f_I(x,t)\, dt \wedge \xi_I \quad + \quad \sum_J g_J(x,t)\, \xi_J$$

where the index I runs over all sequences $0 < i_1 < \cdots < i_{k-1} \leq n$, the index J runs over all sequences $0 < j_1 < \cdots < j_k \leq n$, and f_I, g_J are smooth functions on $U \times \mathbb{R}$.■

We then have the following basic result.

THEOREM 3. *If U is an open subset of some $\mathbb{R}^n$ and $i_t : U \to U \times \mathbb{R}$ is the map $i_t(x) = (x, t)$, then the associated maps of differential forms $i_0^{\#}, i_1^{\#} : \wedge^*(U \times \mathbb{R}) \to \wedge^*(U)$ are chain homotopic.*

In this example the chain homotopy is frequently called a *parametrix*.

COROLLARY 4. *In the setting above the maps i_0^* and i_1^* from $H^*_{\mathrm{DR}}(U \times \mathbb{R})$ to $H^*_{\mathrm{DR}}(U)$ are equal.*■

Proof of Theorem 3. The mappings $P^q : \wedge^q(U \times \mathbb{R}) \to \wedge^{q-1}(U)$ are defined as follows. If we write a q-form over $U \times \mathbb{R}$ as a sum of terms $\alpha_I = f_I(x,t)\, dt \wedge \xi_I$ and $\beta_J = g_J(x,t)\, \xi_J$ using the lemma above, then we set $P^q(\beta_J) = 0$ and

$$P^q\big(\,\alpha_I\,\big) \quad = \quad \left(\int_0^1 f_I(x,u)\, du\right) \cdot \xi_I \ ;$$

we can then extend the definition to an arbitrary form, which is expressible as a sum of such terms, by additivity.

We must now compare the values of $dP + Pd$ and $i_1^{\#} - i_0^{\#}$ on the generating forms α_I and β_J described above. It follows immediately that $i_1^{\#}(\alpha_I) - i_0^{\#}(\alpha_I) = 0$ and

$$i_1^{\#}(\beta_J) \ - \ i_0^{\#}(\beta_J) \quad = \quad [g(x,1) \ - \ g(x,0)]\, \beta_J \ .$$

Next, we have $d \circ P(\beta_J) = d(0) = 0$ and

$$d \circ P(\alpha_I) \quad = \quad d\left(\int_0^1 f_I(x,u) du)\right) \cdot \xi_I \quad =$$

$$\sum_j \left(\int_0^1 \frac{\partial f_I}{\partial x^j}(x,u)\, du\right) \wedge dx^j \wedge \omega_I \ .$$

Similarly, we have

$$P \circ d(\alpha_I) \quad = \quad P\left(\sum_j \frac{\partial f_I}{\partial x^j}\, dx^j \wedge dt \wedge \xi_I \ + \ \frac{\partial f_I}{\partial t}\, dt \wedge dt \wedge \xi_I\right)$$

in which the final summand vanishes because $dt \wedge dt = 0$. If we apply the definition of P to the nontrivial summation on the right hand side of the displayed equation and use the identity $dx^j \wedge dt = -dt \wedge dx^j$, we see that the given expression is equal to $-d \circ P(\alpha_I)$; this shows that the values of both $dP + Pd$ and $i_1^{\#} - i_0^{\#}$ on α_I are zero. It remains to compute $P \circ d(\beta_J)$ and verify that it is equal to $i_1^{\#}(\beta_J) - i_0^{\#}(\beta_J)$. However, by definition we have

$$P \circ d(g_J\, \xi_J) \quad = \quad P\left(\sum_i \frac{\partial g_J}{\partial x^i}\, dx^i \wedge \xi_J \ + \ \frac{\partial g_J}{\partial t}\, dt \wedge \xi_J\right)$$

and in this case P maps the summation over i into zero because each form $dx^i \wedge \xi_J$ is either zero or ± 1 times a standard basic monomial, depending on whether or not dx^i appears as a factor of ξ_J. Thus the right hand side collapses to the final term and is given by

$$P\left(\frac{\partial g_J}{\partial t}\,dt \wedge \xi_J\right) \quad = \quad \left(\int_0^1 \frac{\partial g_J}{\partial u}(x,u)\,du\right)\xi_J \quad =$$

$$[\,g(x,1) \;-\; g(x,0)\,]\;\xi_J$$

which is equal to the formula for $i_1^{\#}(\beta_J) - i_0^{\#}(\beta_J)$ which we described at the beginning of the argument.■

COROLLARY 5. *If U is a convex open subset of some $\mathbb{R}^n$, then $H^q_{\mathrm{DR}}(U)$ is isomorphic to $\mathbb{R}$ if $q = 0$ and is trivial otherwise.*

This follows because the constant map from U to $\mathbb{R}^0$ is a smooth homotopy equivalence if U is convex, so that the de Rham cohomology groups of U are isomorphic to the de Rham cohomology groups of $\mathbb{R}^0$, and by construction the latter are isomorphic to the groups described in the statement of the Corollary.■

COROLLARY 6. (Poincaré Lemma) *Let U be a convex open subset of some $\mathbb{R}^n$ and let $q > 0$. The a differential q-form ω on U is closed ($d\omega = 0$) if and only if it is exact ($\omega = d\theta$ for some θ).*■

Both of the preceding also hold if we merely assume that U is star-shaped with respect to some point $\mathbf{v}$ (*i.e.*, if $\mathbf{x} \in U$, then the closed line segment joining $\mathbf{x}$ and $\mathbf{v}$ is contained in U), for in this case the constant map is again a smooth homotopy equivalence.■

The Mayer-Vietoris sequence

Here is the main result:

THEOREM 7. *Let U and V be open subsets of $\mathbb{R}^n$. Then there is a long exact Mayer-Vietoris sequence in de Rham cohomology*

$$\cdots \to H^{q-1}_{\mathrm{DR}}(U\cap V) \to H^q_{\mathrm{DR}}(U\cup V) \to H^q_{\mathrm{DR}}(U)\oplus H^q_{\mathrm{DR}}(V) \to H^q_{\mathrm{DR}}(U\cap V) \to H^{q+1}_{\mathrm{DR}}(U\cup V) \to \cdots$$

and a commutative ladder diagram relating the long exact Mayer-Vietoris sequences for $\{U, V\}$ in de Rham cohomology and smooth singular cohomology with real coefficients.

Proof. The existence of the Mayer-Vietoris sequence will follow if we can show that there is a short exact sequence of chain complexes

$$0 \to \wedge^*(U\cup V) \longrightarrow \wedge^*(U) \oplus \wedge^*(V) \longrightarrow \wedge^*(U\cap V) \to 0$$

where the map from $\wedge^*(U \cup V)$ is given on the first factor by the $i_U^\#$ (where i_U denotes inclusion) and on the second factor by $-i_V^\#$, and the map into $\wedge^*(U \cap V)$ is given by the maps $j_U^\#$ and $j_V^\#$ defined by inclusion of $U \cap V$ into U and V.

The exactness of this sequence at all points except $\wedge^*(U \cap V)$ follows immediately. Therefore the only thing to prove is that the map to $\wedge^*(U \cap V)$ is surjective. This turns out to be less trivial than one might first expect (in contrast to singular cochains, a differential form on $U \cap V$ need not extend to either U or V), but it can be done using smooth partitions of unity. Specifically, let $\{\varphi_U, \varphi_V\}$ be a smooth partition of unity subordinate to the open covering $\{U, V\}$ of $U \cup V$, and let $\omega \in \wedge^p(U \cap V)$. Consider the forms $\varphi_U \cdot \omega$ and $\varphi_V \cdot \omega$ on $U \cap V$. By definition of a partition of unity there are open subsets $U_0 \subset U$ and $V_0 \subset V$ whose closures in $U \cup V$ are contained in U and V respectively, and such that φ_U and φ_V are zero off the closures of U_0 and V_0. This means that we can define a smooth form θ_U on U such that

$$\theta_U | U \cap V \quad = \quad \varphi_U \cdot \omega \ , \qquad \theta_U | U - \overline{U_0}$$

because both restrict to zero on $U \cap V - \overline{U_0}$. The same reasoning also yields a similar form θ_V on V, and it follows that

$$(\theta_U, \theta_V) \quad \in \quad \wedge^p(U) \ \oplus \ \wedge^p(V)$$

maps to $\omega \in \wedge^p(U \cap V)$. Additional details are given in Conlon (specifically, the last four lines of the proof for Lemma 8.5.1 on page 267).

The existence of the commutative ladder follows because the Generalized Stokes' Formula defines morphisms from the objects in the de Rham short exact sequence into the following analog for smooth singular cochains:

$$0 \to S^*_{\mathrm{smooth},\mathcal{U}}(U \cup V) \longrightarrow S^*_{\mathrm{smooth}}(U) \oplus S^*_{\mathrm{smooth}}(V) \longrightarrow S^*_{\mathrm{smooth}}(U \cap V) \to 0$$

The first term in this sequence denotes the cochains for the complex of $\mathcal{U}$-small chains on $U \cup V$, where $\mathcal{U}$ denotes the open covering $\{U, V\}$.

Since the displayed short exact sequence yields the long exact Mayer-Vietoris sequence for (smooth) singular cohomology, the statement about a commutative ladder in the theorem follows.■

V.4 : De Rham's Theorem

(Conlon, § 8.9; Lee, Chs. 15–16)

The results of the preceding section show that the natural map $[J] : H^*_{\mathrm{DR}}(U) \to H^*_{\mathrm{smooth}}(U; \mathbb{R})$ is an isomorphism if U is a convex open subset of some Euclidean space, and if we compose this with the isomorphism between smooth and ordinary singular cohomology we obtain an isomorphism from the de Rham cohomology of U to the ordinary singular cohomology of U with real coefficients. The aim of this section is to show that both $[J]$ and its composite with the inverse map from smooth to ordinary cohomology is an isomorphism for an arbitrary open subset of $\mathbb{R}^n$. As in Section II.2, an important step in this argument is to prove the result for open sets which are expressible as finite unions of convex open subsets of $\mathbb{R}^n$.

PROPOSITION 1. *If U is an open subset of $\mathbb{R}^n$ which is expressible as a finite union of convex open subsets, then the natural map from $H^*_{\mathrm{DR}}(U)$ to $H^*_{\mathrm{smooth}}(U;\mathbb{R})$ and the associated natural map to $H^*(U;\mathbb{R})$ are isomorphisms.*

Proof. If W is an open subset in $\mathbb{R}^n$ we shall let ψ^W denote the natural map from de Rham to singular cohomology. If we combine the Mayer-Vietoris sequence of the preceding section with the considerations of Section II.2, we obtain the following important principle:

> *If $W = U \cup V$ and the mappings ψ^U, ψ^V and $\psi(U \cap V)$ are isomorphisms, then $\psi^{U \cup V}$ is also an isomorphism.*

Since we know that ψ^V is an isomorphism if V is a convex open subset, we may prove the proposition by induction on the number of convex open subsets in the presentation $W = V_1 \cup \cdots \cup V_k$ using the same sorts of ideas employed in Section II.2 to prove a corresponding result for the map relating smooth and ordinary singular homology.∎

Extension to arbitrary open sets

Most open subsets of $\mathbb{R}^n$ are not expressible as finite unions of convex open subsets, so we still need some method for extracting the general case. The starting point is the following observation, which implies that an open set is a *locally finite* union of convex open subsets.

THEOREM 2. *If U is an open subset of $\mathbb{R}^n$, then U is a union of open subsets W_n indexed by the positive integers such that the following hold:*

(1) *Each W_n is a union of finitely many convex open subsets.*

(2) *If $|m - n| \geq 3$, then $W_n \cap W_m$ is empty.*

Proof. Results from 205C imply that U can be expressed as an increasing union of compact subsets K_n such that K_n is contained in the interior of K_{n+1} and K_1 has a nonempty interior$^{(\star)}$. Define $A_n = K_n - \mathbf{Int}(K_{n-1})$, where K_{-1} is the empty set; it follows that A_n is compact. Let V_n be the open subset $\mathbf{Int}(K_{n+1}) - K_{n-1}$. By construction we know that V_n contains A_n and $V_n \cap V_m$ is empty if $|n - m| \geq 3$. Clearly there is an open covering of A_n by convex open subsets which are contained in V_n, and this open covering has a finite subcovering; the union of this finite family of convex open sets is the open set W_n that we want; by construction we have $A_n \subset W_n$, and since $U = \cup_n A_n$ we also have

$U = \cup_n W_n$. Furthermore, since $W_n \subset V_n$, and $V_n \cap V_m$ is empty if $|n - m| \geq 3$, it follows that $W_n \cap W_m$ is also empty if $|n - m| \geq 3$.■

We shall also need the following result:

PROPOSITION 3. *Suppose that we are given an open subset U in $\mathbb{R}^n$ which is expressible as a countable union of pairwise disjoint subset U_k. If the map from de Rham cohomology to singular cohomology is an isomorphism for each U_k, then it is also an isomorphism for U.*

Proof. By construction the cochain and differential forms mappings determined by the inclusions $i_k : U_k \to U$ define morphisms from $\wedge^*(U)$ to the cartesian product $\Pi_k \wedge^* (U_k)$ and from $S^*_{\text{smooth}}(U)$ to $\Pi_k\ S^*_{\text{smooth}}(U_k)$. We claim that these maps are isomorphisms. In the case of differential forms, this follows because an indexed set of p-forms $\omega_k \in \wedge^p(U_k)$ determine a unique form on U (existence follows because the subsets are pairwise disjoint), and in the case of singular cochains it follows because every singular chain is uniquely expressible as a sum $\sum_k c_k$, where c_k is a singular chain on U_k and all but finitely many c_k's are zero (since the image of a singular simplex $T : \Delta_q \to U$ is pathwise connected and the open sets U_k are pairwise disjoint, there is a unique m such that the image of T is contained in U_m).

If we are given an abstract family of cochain complexes C_k then it is straightforward to verify that there is a canonical homomorphism

$$H^* \left(\prod_k C_k \right) \longrightarrow \prod_k H^*(C_k)$$

defined by the projection maps

$$\pi_j : \prod_k C_k \longrightarrow C_j$$

and that this mapping is an isomorphism. Furthermore, it is natural with respect to families of cochain complex mappings $f_k : C_k \to E_k$.

The proposition follows by combining the observations in the preceding two paragraphs$^{(\star)}$.■

We are now ready to prove the main result, which G. de Rham (1903–1990) first proved in 1931:

THEOREM 4. (de Rham's Theorem.) *The natural maps from de Rham cohomology to smooth and ordinary singular cohomology are isomorphisms for every open subset U in an arbitrary $\mathbb{R}^n$.*

Proof. Express U as a countable union of open subset W_n as in the discussion above, and for $k = 0, 1, 2$ let $U_k = \cup_m W_{3m+k}$. As noted in the definition of the open sets W_j, the open sets W_{3m+k} are pairwise disjoint. Therefore by the preceding proposition and