and if we compose this with the isomorphism between smooth and ordinary singular cohomology we obtain an isomorphism from the de Rham cohomology of U to the ordinary singular cohomology of U with real coefficients. The aim of this section is to show that both $[J]$ and its composite with the inverse map from smooth to ordinary cohomology is an isomorphism for an arbitrary open subset of $\mathbb{R}^n$. As in Section II.2, an important step in this argument is to prove the result for open sets which are expressible as finite unions of convex open subsets of $\mathbb{R}^n$.

PROPOSITION 1. *If U is an open subset of $\mathbb{R}^n$ which is expressible as a finite union of convex open subsets, then the natural map from $H^*_{\mathrm{DR}}(U)$ to $H^*_{\mathrm{smooth}}(U;\mathbb{R})$ and the associated natural map to $H^*(U;\mathbb{R})$ are isomorphisms.*

Proof. If W is an open subset in $\mathbb{R}^n$ we shall let ψ^W denote the natural map from de Rham to singular cohomology. If we combine the Mayer-Vietoris sequence of the preceding section with the considerations of Section II.2, we obtain the following important principle:

> *If $W = U \cup V$ and the mappings ψ^U, ψ^V and $\psi(U \cap V)$ are isomorphisms, then $\psi^{U \cup V}$ is also an isomorphism.*

Since we know that ψ^V is an isomorphism if V is a convex open subset, we may prove the proposition by induction on the number of convex open subsets in the presentation $W = V_1 \cup \cdots \cup V_k$ using the same sorts of ideas employed in Section II.2 to prove a corresponding result for the map relating smooth and ordinary singular homology.■

Extension to arbitrary open sets

Most open subsets of $\mathbb{R}^n$ are not expressible as finite unions of convex open subsets, so we still need some method for extracting the general case. The starting point is the following observation, which implies that an open set is a *locally finite* union of convex open subsets.

THEOREM 2. *If U is an open subset of $\mathbb{R}^n$, then U is a union of open subsets W_n indexed by the positive integers such that the following hold:*

(1) *Each W_n is a union of finitely many convex open subsets.*

(2) *If $|m - n| \geq 3$, then $W_n \cap W_m$ is empty.*

Proof. Results from 205C imply that U can be expressed as an increasing union of compact subsets K_n such that K_n is contained in the interior of K_{n+1} and K_1 has a nonempty interior$^{(\star)}$. Define $A_n = K_n - \mathbf{Int}(K_{n-1})$, where K_{-1} is the empty set; it follows that A_n is compact. Let V_n be the open subset $\mathbf{Int}(K_{n+1}) - K_{n-1}$. By construction we know that V_n contains A_n and $V_n \cap V_m$ is empty if $|n-m| \geq 3$. Clearly there is an open covering of A_n by convex open subsets which are contained in V_n, and this open covering has a finite subcovering; the union of this finite family of convex open sets is the open set W_n that we want; by construction we have $A_n \subset W_n$, and since $U = \cup_n A_n$ we also have

$U = \cup_n W_n$. Furthermore, since $W_n \subset V_n$, and $V_n \cap V_m$ is empty if $|n-m| \geq 3$, it follows that $W_n \cap W_m$ is also empty if $|n-m| \geq 3$.∎

We shall also need the following result:

PROPOSITION 3. *Suppose that we are given an open subset U in $\mathbb{R}^n$ which is expressible as a countable union of pairwise disjoint subset U_k. If the map from de Rham cohomology to singular cohomology is an isomorphism for each U_k, then it is also an isomorphism for U.*

Proof. By construction the cochain and differential forms mappings determined by the inclusions $i_k : U_k \to U$ define morphisms from $\wedge^*(U)$ to the cartesian product $\Pi_k \wedge^* (U_k)$ and from $S^*_{\text{smooth}}(U)$ to $\Pi_k \; S^*_{\text{smooth}}(U_k)$. We claim that these maps are isomorphisms. In the case of differential forms, this follows because an indexed set of p-forms $\omega_k \in \wedge^p(U_k)$ determine a unique form on U (existence follows because the subsets are pairwise disjoint), and in the case of singular cochains it follows because every singular chain is uniquely expressible as a sum $\sum_k c_k$, where c_k is a singular chain on U_k and all but finitely many c_k's are zero (since the image of a singular simplex $T : \Delta_q \to U$ is pathwise connected and the open sets U_k are pairwise disjoint, there is a unique m such that the image of T is contained in U_m).

If we are given an abstract family of cochain complexes C_k then it is straightforward to verify that there is a canonical homomorphism

$$H^* \left(\prod_k C_k \right) \longrightarrow \prod_k H^*(C_k)$$

defined by the projection maps

$$\pi_j : \prod_k C_k \longrightarrow C_j$$

and that this mapping is an isomorphism. Furthermore, it is natural with respect to families of cochain complex mappings $f_k : C_k \to E_k$.

The proposition follows by combining the observations in the preceding two paragraphs$^{(\star)}$.∎

We are now ready to prove the main result, which G. de Rham (1903–1990) first proved in 1931:

THEOREM 4. (de Rham's Theorem.) *The natural maps from de Rham cohomology to smooth and ordinary singular cohomology are isomorphisms for every open subset U in an arbitrary $\mathbb{R}^n$.*

Proof. Express U as a countable union of open subset W_n as in the discussion above, and for $k = 0, 1, 2$ let $U_k = \cup_m \; W_{3m+k}$. As noted in the definition of the open sets W_j, the open sets W_{3m+k} are pairwise disjoint. Therefore by the preceding proposition and the first result of this section we know that the natural maps from de Rham cohomology to singular cohomology are isomorphisms for the open sets U_k.

We next show that the natural map(s) must define isomorphisms for $U_0 \cup U_1$. By the highlighted statement in the proof of the first proposition in this section, it will suffice to show that the same holds for $U_0 \cap U_1$. However, the latter is the union of the pairwise disjoint open sets $W_{3m} \cap W_{3m+1}$, and each of the latter is a union of finitely many convex open subsets. Therefore by the preceding proposition and the first result of this section we know that the natural maps from de Rham to singular cohomology are isomorphisms for $U_0 \cap U_1$ and hence also for $U^* = U_0 \cup U_1$.

Clearly we would like to proceed similarly to show that we have isomorphisms from de Rham to singular cohomology for $U = U_2 \cup U^*$, and as before it will suffice to show that we have isomorphisms for $U_2 \cap U^*$. But $U_2 \cap U^* = (U_2 \cap U_0) \cup (U_2 \cap U_1)$. By the preceding paragraph we know that the maps from de Rham to singular cohomology are isomorphisms for $U_0 \cap U_1$, and the same considerations show that the corresponding maps are isomorphisms for $U_0 \cap U_2$ and $U_1 \cap U_2$. Therefore we have reduced the proof of de Rham's Theorem to checking that there are isomorphisms from de Rham to singular cohomology for the open set $U_0 \cap U_1 \cap U_2$. The latter is a union of open sets expressible as $W_i \cap W_j \cap W_k$ for suitable positive integers i, j, k which are distinct. The only way such an intersection can be nonempty is if the three integers i, j, k are *consecutive* (otherwise the distance between two of them is at least 3). Therefore, if we let

$$S_m \quad = \quad \bigcup_{0 \leq k \leq 2} W_{3m-k} \cap W_{3m+1-k} \cap W_{3m+2-k}$$

it follows that S_m is a finite union of convex open sets, the union of the open sets S_m is equal to $U_0 \cap U_1 \cap U_2$, and if $m \neq p$ then $S_m \cap S_p$ is empty (since the first is contained in W_{3m} and the second is contained in the disjoint subset W_{3p}). By the first result of this section we know that the maps from de Rham to singular cohomology define isomorphisms for each of the open sets S_m, and it follows from the immediately preceding proposition that we have isomorphisms from de Rham to singular cohomology for $\cup_m S_m = U_o \cap U_1 \cap U_2$. As noted before, this implies that the corresponding maps also define isomorphisms for U.■

Some applications

In Section 6 we shall use de Rham's Theorem to generalize results multivariable calculus on path independence for line integrals in open subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$. For the time being we shall limit ourselves to verifying another result which sometimes appears in multivariable calculus texts.

PROPOSITION 5. *Suppose that $U \subset \mathbb{R}^3$ is a contractible open set and $\mathbf{F}$ is a smooth vector field on U whose divergence $\nabla \cdot \mathbf{F}$ is zero. Then $\mathbf{F} = \nabla \times \mathbf{P}$ for some vector field $\mathbf{P}$ on U.*

Proof. Given $\mathbf{F} = (F_1, F_2, F_3)$ as in the statement of the proposition, let $\theta_{\mathbf{F}}$ be the 2-form

$$F_1\,dx_2 \wedge dx_3 \;+\; F_2\,dx_3 \wedge dx_1 \;+\; F_3\,dx_1 \wedge dx_2$$

and note that $d\theta_{\mathbf{F}} = (\nabla \cdot \mathbf{F})\,dx_1 \wedge dx_2 \wedge dx_3$. Therefore the divergence condition translates into $d\theta_{\mathbf{F}} = 0$. Since $H^1_{\mathrm{DR}}(U) \cong H^2(U;\mathbb{R})$ by de Rham's Theorem and the latter is trivial by contractibility, it follows that $\theta_{\mathbf{F}} = d\omega$ for some 1-form ω. Expand ω as $\sum_i P_i\,dx_i$ and write $\omega = \omega_{\mathbf{P}}$ to reflect this expansion. Then direct calculation shows that $d\omega_{\mathbf{P}}$ is the 2-form $\theta_{\nabla\times\mathbf{P}}$ in the notation at the beginning of the proof. Therefore $\theta_{\mathbf{F}} = \theta_{\nabla\times\mathbf{P}}$, and by construction this means that $\mathbf{F} = \nabla \times \mathbf{P}$.■

Generalization to arbitrary smooth manifolds

In fact, one can state and prove de Rham's Theorem for every (second countable) smooth manifold if we use Conlon's approach to define differential forms (and related constructions) more generally; details are given in Chapters 6–8 of Conlon. The details of this generalization are beyond the scope of this course, so we shall only give a purely formal method for deriving the general case of de Rham's Theorem from the special case of open sets in $\mathbb{R}^n$ and a generalization of differential forms satisfying a few simple properties.

FACT 6. *The category of (second countable) smooth manifolds and smooth mappings has the following properties:*

(i) *It contains the category of open sets in $\mathbb{R}^n$ as a full subcategory.*

(ii) *The cochain complex functors $\wedge^*$ and S^*_{smooth} extend to this category, and likewise for the natural transformation $\theta^* : \wedge^* \to S^*_{\mathrm{smooth}}$.*

(iii) *Every smooth manifold M^m is a smooth retract of some open set $U \subset \mathbb{R}^N$ for sufficiently large values of N.*

Property (i) follows directly from the construction of the category of smooth manifolds and smooth mappings, while (ii) clearly must hold in any reasonable extension of differential forms to smooth manifolds. Finally, (iii) is an immediate consequence of the Tubular Neighborhood Theorem for a smooth embedding of M in some $\mathbb{R}^N$; one reference is Lee, Proposition 10.20, page 256.

In view of the preceding discussion, the general case of de Rham's Theorem will be a consequence of the following very general result:

THEOREM 7. *Let $\mathcal{A}$ be a category, let $\mathcal{W} \subset \mathcal{A}$ be a full subcategory, and assume that every object in $\mathcal{A}$ is an $\mathcal{A}$-retract of an object in $\mathcal{W}$. Assume further that E and F are contravariant functors from $\mathcal{A}$ to the category of abelian groups and that $\theta : E \to F$ is a natural transformation. Then $\theta(X)$ is an isomorphism for all objects X in $\mathcal{A}$ if and only if it is an isomorphism for all objects X in $\mathcal{W}$.*

Proof. One implication is trivial, so we shall only look at the other case in which $\theta(X)$ is an isomorphism for all objects X in $\mathcal{W}$.

Suppose that X is an object of $\mathcal{A}$, choose a retract $i : X \to Y$, where Y is an object of $\mathcal{W}$, and let $r : Y \to x$ be such that $r \circ i = \mathrm{id}(X)$. Consider the following commutative diagram:

$$\begin{array}{ccccc} E(Y) & \xrightarrow{i^*} & E(X) & \xrightarrow{r^*} & E(Y) \\ \downarrow \theta_Y & & \downarrow \theta_X & & \downarrow \theta_X \\ F(Y) & \xrightarrow{i^*} & F(X) & \xrightarrow{r^*} & F(Y) \end{array}$$

Since $i^* \circ r^*$ is the identity on $E(X)$ and $F(X)$, it follows that i^* is onto and r^* is 1–1. To see that θ_X is 1–1, notice that $\theta_X(u) = \theta_X(v)$ implies $\theta_Y r^*(u) = r^*\theta_X(u) = r^*\theta_X(v) = \theta_Y r^*(v)$. Since θ_Y is an isomorphism it follows that $r^*(u) = r^*(v)$, which in turn implies $u = v$ because r^* is 1–1. To see that θ_X is onto, given $u \in F(X)$ use the surjectivity of i^* to write $u = i^*(v)$. Since θ_Y is an isomorphism it follows that $v = \theta_Y(w)$ for some w, and thus we have $u = i^*\theta_Y(v) = \theta_X i^*(w)$.■

V.5 : Multiplicative properties of de Rham cohomology

(Hatcher, §§ 3.1–3.2; Conlon, § D.3; Lee, Ch. 15)

DEFAULT HYPOTHESIS. Unless specifically stated otherwise, all cochain complexes, modules in this section are vector spaces over the real numbers, all algebraic morphisms are linear transformations, and all tensor products are taken over the real numbers.

As in the case of simplicial cup products, the Leibniz rule for for differential forms

$$d(\omega_1 \wedge \omega_2) \quad = \quad (d\omega_1) \wedge \omega_2 \ \pm \ \omega_1 \wedge d(\omega_2)$$

implies that the wedge of two closed forms is closed and the wedge of a closed form with an exact form is exact. Consequently there is a well defined (bilinear) cohomology wedge product

$$H^p_{\mathrm{DR}}(M) \otimes_{\mathbb{R}} H^q_{\mathrm{DR}}(M) \ \longrightarrow \ H^{p+q}_{\mathrm{DR}}(M)$$

sending $[\omega] \otimes [\theta]$ to $[\omega \wedge \theta]$ (where ω and θ are closed forms). It follows imeediately that this product makes the de Rham cohomology of a smooth manifold into a graded algebra and this structure is functorial. Since the de Rham and singular cohomology groups of a smooth manifold are isomorphic, it is natural to ask if the wedge product and cup product correspond under the isomorphism in de Rham's theorem, and it turns out that this is the case. We shall not give all the details of the argument; the references mentioned at appropriate points contain the omitted steps. Our approach will involve some explicit constructions involving simplicial chains and cochains which are taken from Eilenberg and Steenrod and also from the following classic text (which we shall simply call *Homology*):

S. MacLane. *Homology* (Reprint of the first edition). Grundlehren der mathematischen Wissenschaften Bd. 114. Springer-Verlag, Berlin-New York, 1967.

LEMMA 1. *Let A be a p-simplex in $\mathbb{R}^n$ with vertices $a_0, \cdots, a_p$, and let B be a p-simplex in $\mathbb{R}^m$ with vertices $b_0, \cdots, b_q$. Then there is a simplicial decomposition of $A \times B \subset \mathbb{R}^n \times \mathbb{R}^m$ such that every point likes on at one $(p+q)$-simplex and an arbitrary $(p+q)$-simplex of the decomposition has vertices*

$$(a_{i_0}, b_{j_0}), \cdots, (a_{i_{p+q}}, b_{j_{p+q}})$$

where $i_t \geq i_{t+1}$, $j_t \geq j_{t+1}$ for all t and exactly one *of these two inequalities is* strict *for each t.*

For future reference, we note that the vertices of this decomposition for $A \times B$ have a standard lexicographic ordering obtained from the given orderings for the vertices of A and B.

Lemma 1 is a special case of the construction appearing in Section II.8 of Eilenberg and Steenrod.■

We shall also need an explicit singular (in fact, simplicial) chain

$$X(p,q) \quad \in \quad C_{p+q}(\Lambda_p \times \Lambda_q) \quad \subset \quad S_{p+q}(\Lambda_p \times \Lambda_q)$$

defined on page 243 of *Homology*. This chain contains plus or minus each of the affine ordered simplices mentioned in Lemma 1, and the sign is that of $\det T_\alpha$, where T_α is the unique affine map sending the vertices of Δ_{p+q} monotonically to those of the $(p+q)$-simplex $\alpha \subset \Lambda_p \times \Lambda_q$ (the assertion about signs requires a little work). The choice of signs leads to the following result:

LEMMA 2. *Let f be a smooth real valued function defined on an open neighborhood of $\Lambda_p \times \Lambda_q \subset \mathbb{R}^p \times \mathbb{R}^q$. Then*

$$\int_{\Lambda_p \times \Lambda_q} f(t)\,dt \quad = \quad \int_{X(p,q)} f(t)\,dt^1 \wedge \cdots \wedge dt^{p+q}$$

where the left hand side is the usual Riemann or Lebesgue integral and the right hand side is the differential forms integral.■

Again turning to page 743 of *Homology*, we see that there is a natural chain transformation

$$\gamma : S_*^{\text{smooth}}(M) \otimes S_*^{\text{smooth}}(N) \longrightarrow S_*^{\text{smooth}}(M \times N)$$

such that if M and N are open neighborhoods of Λ_p and Λ_q in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively and $\sigma_k : \Lambda_k \to \mathbb{R}^k$ is the standard inclusion, then $\gamma(\sigma_p \otimes \sigma_q) = X(p,q)$; in fact, the map γ is an explicit chain inverse to the Alexander-Whitney map (see pages 743–744 of *Homology*).

NOTATIONAL CONVENTIONS. Given forms $\omega \in \wedge^p(M)$ and $\eta \in \wedge^q(N)$, the *external wedge* $\omega \times \eta \in \wedge^{p+q}(M \times n)$ is equal to

$$\left(p_M^{\#}\omega\right) \wedge \left(p_M^{\#}\omega\right) \quad \in \quad \wedge^{p+q}(M \times N)\ .$$

In coordinates, if $\omega = f(x)\,dx^1 \wedge \cdots \wedge dx^p$ and $\eta = g(y)\,dy^1 \wedge \cdots \wedge dy^q$, then

$$\omega \times \eta \quad = \quad f(x)\,g(y)\,dx^1 \wedge \cdots \wedge dx^p \wedge dy^1 \wedge \cdots \wedge dy^q \ .$$

Given a chain complex S_*, a commutative ring with unit $\mathbb{R}$, and cochains $f : S_p \to \mathbb{R}$, $g : S_q \to \mathbb{R}$, define the map $f \bowtie g : S_p \otimes S_q \to \mathbb{R}$ by the formula

$$f \bowtie g(u \otimes v) \quad = \quad f(u) \cdot g(v) \ .$$

We can now state and prove a key fact relating the cross product in singular cohomology and the external wedge product in de Rham cohomology.

PROPOSITION 3. *The following diagram is commutative:*

$$\begin{array}{ccc}
\wedge^p(M) \otimes \wedge^q(N) & \xrightarrow{\times} & \wedge^{p+q}(M \times N) \\
\Big\downarrow \theta_M \otimes \theta_N & & \Big\downarrow \theta_{M \times N} \\
S^p_{\mathrm{smooth}}(M) \otimes S^q_{\mathrm{smooth}}(N) & \longrightarrow & S^{p+q}_{\mathrm{smooth}}(M \times N) \\
\Big\downarrow \bowtie & & \Big\downarrow (\gamma | S_p \otimes S_q)^* \\
[S_p \otimes S_q]^* & \xrightarrow{=} & [S_p \otimes S_q]^*
\end{array}$$

In this diagram W^* denotes the dual space to the vector space W.

Proof. By the naturality properties of the constructions in the diagram, it suffices to consider the case in which M and N are open neighborhoods of the simplices Λ_p, $\Lambda_q \in \mathbb{R}^p, \mathbb{R}^q$ and to evaluate both composites applied to a tensor product of forms $\omega \otimes \eta$ on the universal example $\sigma_p \otimes \sigma_q$. Assume ω and η are given as in the notational conventions. Then the value of the composite $\bowtie \circ (\theta_M \otimes \theta_N) \circ (\omega \otimes \eta)$ at $\sigma_p \otimes \sigma_q$ is equal to

$$\int_{\sigma_p} \omega \ \cdot \int_{\sigma_q} \eta \quad = \quad \int_{\Lambda_p} f(x)\,dx \ \cdot \int_{\Lambda_q} g(y)\,dy \quad = \quad \int_{\Lambda_p \times \Lambda_q} f(x) \cdot g(y)\ \,dx\,dy$$

(the last equation follows from Fubini's Theorem). By Lemma 2, the last integral in the display is equal to

$$\int_{X(p,q)} f(x)\,g(y)\,dx^1 \wedge \cdots \wedge dy^q \quad = \quad \int_{X(p,q)} \omega \times \eta$$

and by definition the latter is equal to

$$(\gamma | S_p \otimes S_q)^* \circ \theta_{M \times N} \circ (\omega \otimes \eta) \quad \text{evaluated at} \quad \sigma_p \otimes \sigma_q$$

which is what we wanted to prove.∎

The next result is nearly as important as the previous one for relating the cup and wedge products.

PROPOSITION 4. *Suppose that $r + t = p + q$ but $(s, t) \neq (p, q)$. Then $(\gamma | S_r \otimes S_t)^* \theta(\omega \otimes \eta) = 0$.*

Proof. Again by naturality it suffices to consider the case where M and N are open neighborhoods of the simplices Λ_r, $\Lambda_t \in \mathbb{R}^r, \mathbb{R}^t$ and to evaluate at $\sigma_r \otimes \sigma_t$. The hypothesis implies that either $r < p$ or $t < q$. Since $\dim M = r$ and $\dim N = t$, it follows that either $\wedge^p(M)$ or $\wedge^q(N)$ is trivial.∎

The preceding results give us a cochain level formula relating the cross and external wedge products (and thus also for the cup and ordinary wedge products).

PROPOSITION 5. *In the setting above, let $\psi : S_*(M \times N) \to S_*(M) \otimes S_*(N)$ be the Alexander-Whitney map. Then $\theta_M(\omega) \times \theta_N(\eta) = \psi^* \circ \gamma^* \circ \theta_{M \times N}(\omega \times \eta)$.*

Proof. The left hand side is equal to $\psi^* \circ \rho(p, q) \circ \theta_M \otimes \theta_N(\omega \otimes \eta)$, where $\rho(p, q)$ projects $[S_*^{\mathrm{smooth}}(M) \otimes S_*^{\mathrm{smooth}}(N)]_{p+q}$ onto the direct summand $S_p^{\mathrm{smooth}}(M) \otimes S_q^{\mathrm{smooth}}(N)$. By Proposition 4 the composite $\psi^* \circ \gamma^* \circ \theta_{M \times N}(\omega \times \eta)$ is equal to $\rho(p, q)^* \circ (\gamma | S_p \otimes S_q)^* \circ \theta_{M \times N}(\omega \times \eta)$, and therefore $\theta_M(\omega) \times \theta_N(\eta) = \psi^* \circ \gamma^* \circ \theta_{M \times N}(\omega \times \eta)$ by Proposition 3.∎

We can now state and prove the main result of this section:

THEOREM 6. *Let $\theta : H^*_{\mathrm{DR}}(M) \to H^*(M)$ be the isomorphism in de Rham's Theorem, and let ω and η be closed forms on M. Then $\theta_M([\omega] \wedge [\eta]) = \theta_M([\omega]) \cup \theta_M([\eta])$.*

Proof. Let $M = N$ in the preceding discussion, and let $\Delta_M : M \to M \times M$ be the diagonal. Applying the cochain mapping $\Delta_M^{\#}$ to the right hand side of the equation in Proposition 5, we get $\theta_M(\omega) \cup \theta_M(\eta)$ on the cochain level. Applying $\Delta_M^{\#}$ to the left hand side, we get $\Delta_M^{\#} \circ \psi^* \circ \gamma^* \circ \theta_{M \times M}(\omega \times \eta)$. Since $\gamma \circ \psi$ is chain homotopic to the identity (γ is a a chain homotopy inverse to ψ), the conditions $d\omega = d\eta = 0$ imply that

$$\psi^* \circ \gamma^* \circ \theta_{M \times M}(\omega \times \eta) \quad = \quad \theta_{M \times M}(\omega \times \eta) \; + \; \delta z \qquad \text{for some} \quad z \; .$$

Therefore we have

$$\theta_M(\omega) \cup \theta_M(\eta) \quad = \quad \Delta_M^{\#}\left(\theta_{M \times M}(\omega \times \eta) \; + \; \delta z\right) \quad =$$

$$\theta_M \circ \Delta_M^{\#}(\omega \times \eta) \; + \; \delta \Delta_M^{\#} z \quad = \quad \theta_M(\omega \wedge \eta) \; + \; \delta \Delta_M^{\#} z$$

where the last equation follows because $\omega \wedge \eta = \Delta_M^{\#}(\omega \times \eta)$. This means that if ω and η are closed forms, then the closed forms $\theta_M(\omega \wedge \eta)$ and $\theta_M(\omega) \cup \theta_M(\eta)$ determine the same singular cohomology class.∎

V.6 : Path independence of line integrals

(Conlon, § 8.2; Lee, Chs. 11, 16)