is the corresponding partition of $[0,1]$ given by shrinking Δ and Δ' to $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ respectively, and it follows immediately that the previously constructed mapping $\eta_{(U,u_0)}$ sends $(\gamma,\Delta)+(\gamma',\Delta')$ to $\eta_{(U,u_0)}(\gamma,\Delta)+\eta_{(U,u_0)}(\gamma',\Delta')$.

Consider now the following commutative diagram, in which the vertical map at the left sends (γ,Δ) to the homotopy class $[\gamma]$ and the map h is the Hurewicz homomorphism:

$$\begin{array}{ccc} \Theta(U,u_0) & \stackrel{\eta}{\longrightarrow} & H_1^{\rm smooth}(U)) \\ \downarrow & & \downarrow \\ \pi_1(U,u_0) & \stackrel{h}{\longrightarrow} & H_1(U)) \end{array}$$

We can now complete the proof as follows: The basic properties of line integrals imply that the line integral of ω along γ is equal to the integral of γ with respect to the chain $\mathrm{Chain}(\gamma,\Delta)=\sum_i\ T_i$. Since $d\omega=0$, the Generalized Stokes' Theorem implies that the integral $\int_{\mathrm{Chain}(\gamma,\Delta)}\ \omega$ only depends on the image of (γ,Δ) in $H_1^{\rm smooth}(U)$. If γ is basepoint preserving homotopic to the constant map whose value everywhere is u_0, then the class of γ in $\pi_1(U,u_0)$ is trivial and hence we can use the diagram to conclude that $h\circ[\gamma]=0$ and hence $\eta(\gamma,\Delta)=0$, and therefore by the preceding sentence we know that $\int_\gamma\ \omega=0$, which is what we wanted to prove.■

Proof that Theorem 1 implies Theorem 4. We do not know whether or not the freely homotopic closed curves γ_0 and γ_1 start and end at the same point, so assume that γ_i starts and ends at u_i for $i=0,1$. Choose appropriate partitions Δ_i such that γ_i is smooth on each subinterval determined by Δ_i for $i=0,1$. Since γ_0 and γ_1 are freely homotopic, the commutative diagram implies that

$$\eta_{(U,u_0)}(\gamma_0,\Delta_0) \quad = \quad \eta_{(U,u_1)}(\gamma_1,\Delta_1) \qquad \text{in} \qquad H_1^{\rm smooth}(U)\ .$$

As in the proof of Theorem 1, the integrand $\sum_i\ P_i\,dx_i$ corresponds to a closed 1-form ω, and therefore in this case the Generalized Stokes' Theorem implies that the integrals of ω over the chains $\int_{\mathrm{Chain}(\gamma_0,\Delta_0)}\ \omega$ and $\int_{\mathrm{Chain}(\gamma_1,\Delta_1)}\ \omega$ are equal As in the proof of Theorem 1, we know that these inegrals are respectively equal to the line integrals $\int_{\gamma_0}\ \omega$ and $\int_{\gamma_1}\ \omega$, and therefore these two line integrals must also be equal.■

Some classical implications

Frequently one sees the 3-dimensional case of the following result in multivariable calculus texts:

THEOREM 5. *Let $n\geq 3$, and suppose that U is obtained from $\mathbb{R}^n$ by removing finitely many points. If $\mathbf{F}=(P_1,\ \cdots\ ,P_n)$ is a smooth vector field on U such that*

$$\frac{\partial P_i}{\partial x_j} \quad = \quad \frac{\partial P_j}{\partial x_i}$$

for all $i \neq j$, then there is a smooth function g on U such that $\nabla g = \mathbf{F}$. In particular, if Γ is a regular piecewise smooth curve in U, then the value of the line integral

$$\int_\Gamma \sum_i P_i \, dx_i$$

depends only upon the endpoints of the curve Γ.

In particular, a result of this type is formulated as Theorem 7 on page 551 of Marsden and Tromba.

Theorem 5 contrasts sharply with the case $n = 2$, and the easiest way to explain the difference is to note that the complement of a finite subset in $\mathbb{R}^n$ is simply connected if $n = 3$ but is not simply connected if $n = 2$. We shall give a simpler (but less elementary) argument which only requires us to know that $H^1(U; \mathbb{R})$ is trivial if $n \geq 3$. By the Universal Coefficient Theorem relating integral homology to real cohomology, we only need to prove the following:

LEMMA 6. *Let $n \geq 3$, and suppose that U is obtained from $\mathbb{R}^n$ by removing a set X which contains exactly k points. Then the singular homology groups of $U = \mathbb{R}^n - X$ are given by $H_j(\mathbb{R}^n - X) \cong \mathbb{Z}$ if $k = 0$, $H_j(\mathbb{R}^n - X) \cong \mathbb{Z}^k$ if $k = n - 1$, and $H_j(\mathbb{R}^n - X) \cong 0$ otherwise.*

Proof that Lemma 6 implies Theorem 5. By Lemma 10 we know that $H_1(U = \mathbb{R}^n - X)$ is trivial because $n \geq 3$, and by the Universal Coefficient Theorem we know that $H^1(U; \mathbb{R}) \cong \mathrm{Hom}\,(H_1(U), \mathbb{R})$; therefore $H^1(U; \mathbb{R})$ is trivial. Since $H^*_{\mathrm{DR}}(U) \cong H^*(U; \mathbb{R})$ by de Rham's Theorem, it follows that $H^1_{\mathrm{DR}}(U)$ is trivial and therefore every closed 1-form over U is exact; *i.e.*, if $d\omega = 0$ then $\omega = dg$ for some g.

Let ω be the 1-form $\sum_i P_i \, dx_i$; the hypothesis on the functions P_i is equivalent to the identity $d\omega = 0$, and therefore if this identity holds we can apply the preceding paragraph to conclude that $\omega = dg$ for some smooth function g. If we translate this back into the language of vector fields, we see that the original vector field $\mathbf{F}$ is equal to ∇g, proving the first assertion in the conclusion of the theorem. The second assertion now follows because the line integral in question has the form $\int_\Gamma \nabla g \cdot d\mathbf{x}$ and we have already noted that the values of such line integrals only depend upon the endpoints of Γ.■

Proof of Lemma 6. For each $x \in X$ let V_x be the open neighborhood of radius r centered at x; choose r to be smaller than half the minimum distance between points of X (the minimum exists by the finiteness of X, and let $V = \cup_x U_x$, so that $\mathbb{R}^n = V \cup (\mathbb{R}^n - X))$ and $V \cap X = \cup_x V_x - \{x\}$. Then by excision, the splitting of the homology of X into the homology of its arc componenents, and Theorem VII.1.7 in `algtop-notes.pdf` we know that

$$H_j(\mathbb{R}^n, \mathbb{R}^n - X) \quad \cong \quad H_j(U, U - X) \quad \cong \quad H_j\left(\cup_x V_x, \cup_x V_x - \{x\}\right) \quad \cong$$

$$\bigoplus_x H_j\left(\, V_x, \, V_x - \{x\}\right) \quad \cong \quad \mathbb{Z}^k \quad \text{or} \quad 0$$

where the group is zero unless $j = n$, in which case it is isomorphic to $\mathbb{Z}^k$. We can now recover the homology groups $\mathbb{R}^n - X$ from the long exact homology sequence for $(\mathbb{R}^n, \mathbb{R}^n - X)$ and the fact that $H_j(\mathbb{R}^n)$ is $\mathbb{Z}$ if $j = 0$ and zero otherwise.■

Similar conclusions hold if U is obtained from $\mathbb{R}^n$ (where $n \geq 3$) by deleting an infinite sequence of isolated points $\{\mathbf{p}_1, \mathbf{p_2}, \cdots\}$. The main difference in the argument is that the open disk V_k centered at $\mathbf{p}_k$ must have a radius r_k such that for each $j \neq k$ we have $|\mathbf{p}_j - \mathbf{p}_k| > r_k$; we can always find such positive radii if we have a sequence of isolated points.■