FIGURES FOR ALGEBRAIC TOPOLOGY LECTURE NOTES

I : Foundatonal and geometric background

I.2 : Barycentric coordinates and polyhedra

Barycentric coordinates. In the drawing below, each of the points **P, Q, R** lies in the plane determined by $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, and consequently each can be written as a linear combination $w_1\mathbf{P}_1 + w_2\mathbf{P}_2 + w_3\mathbf{P}_3$, where $w_1 + w_2 + w_3 = 1$. For the point **P**, the barycentric coordinates w_i are all positive, while for the point **R** the barycentric coordinates are such that $w_1 = 0$ but the other two are positive, and for the point **Q** the barycentric coordinates are such that w_1 is negative but the other two are positive.

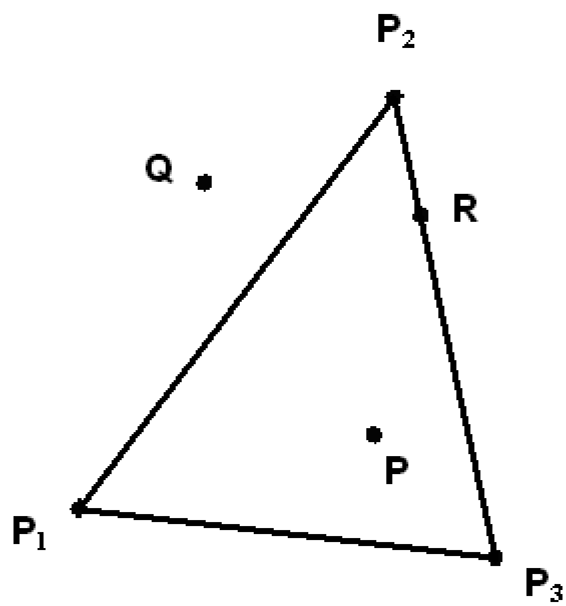


FIGURE 1

(***Source:*** http://graphics.idav.ucdavis.edu/education/GraphicsNotes/Barycentric-Coordinates/Barycentric-Coordinates.html)

Examples of points for which w_2 is positive but the remaining coordinates are negative can also be constructed using this picture; for example, if one takes the midpoint **M** of the segment $[\mathbf{P}_1\mathbf{P}_3]$, then the point $\mathbf{S} = 2\mathbf{P}_2 - \mathbf{M}$ will have this property (geometrically, $\mathbf{P}_2$ is the midpoint of the segment joining **M** and **S**).

Illustration of a $2-$ simplex. We shall use a modified version of Figure 1; the points of the $2-$ simplex with vertices $\mathbf{P_1}$, $\mathbf{P_2}$, and $\mathbf{P_3}$ consists of the triangle determined by these points and the points which lie inside this triangle (in the usual intuitive sense of the word).

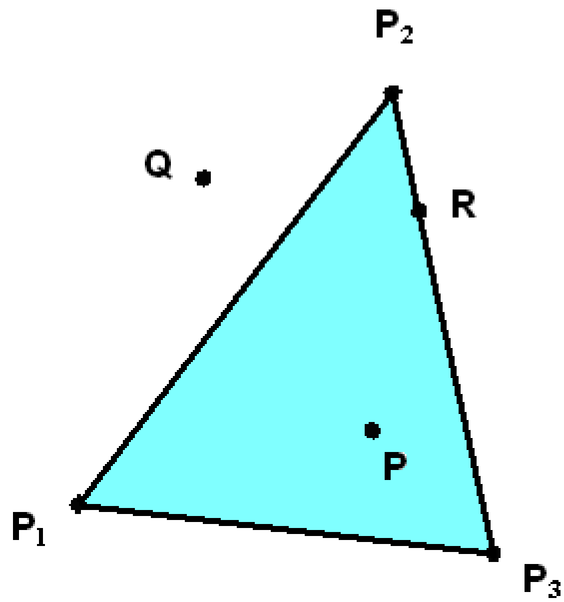
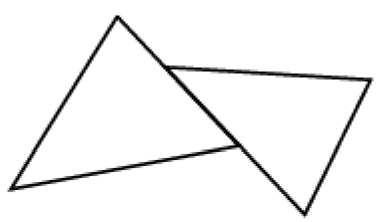


FIGURE 2

In this picture the points **P** and **R** lie on the simplex $\mathbf{P_1P_2P_3}$ because their barycentric coordinates are all nonnegative, but the point **Q** does not because one of its barycentric coordinates is negative.

Note that the **(*proper*) *faces*** of this simplex are the closed segments $\mathbf{P_1P_2}$, $\mathbf{P_2P_3}$, and $\mathbf{P_1P_3}$ joining pairs of vertices as well as the three vertices themselves (and possibly the empty set if we want to talk about an empty face with no vertices).

Simplicial decompositions. It is useful to look at a few spaces given as unions of $2-$ simplices, some of which determine simplicial complexes in the sense of the notes and others that do not.

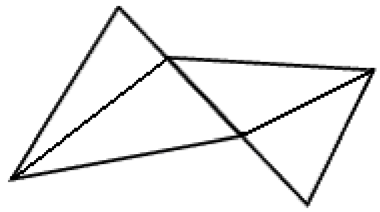


not a simplicial complex

FIGURE 3

(***Source:*** http://mathworld.wolfram.com/SimplicialComplex.html)

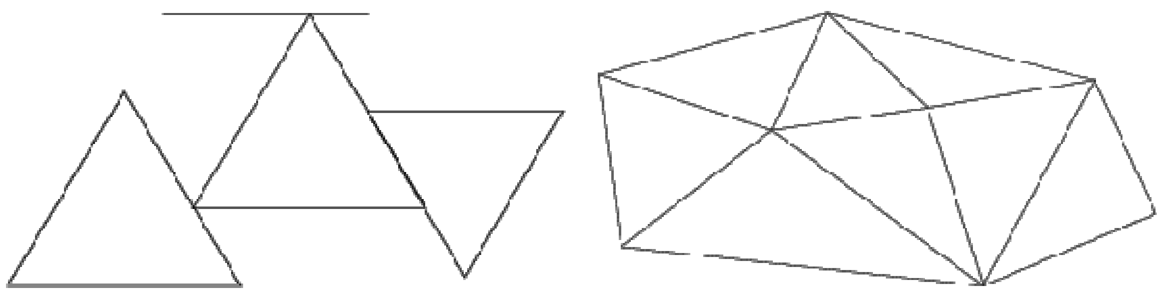
In the example above the intersection of the $\mathbf{2}$ – simplices is not a common face. On the other hand, we can split the two simplices into smaller pieces such that we do have a simplicial decomposition.



simplicial complex

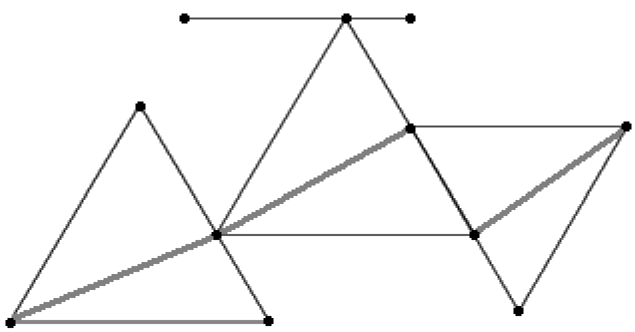
FIGURE 4

Here are two more examples; in the second case the simplices determine a simplicial complex and in the first they do not. As in the preceding example, one can subdivide the simplices in the first example to obtain a simplicial decomposition.



Not a simplicial complex   A simplicial complex

(***Source:*** http://planning.cs.uiuc.edu/node274.html)



A simplicial complex

FIGURE 5

Triangulations. In the example from page $\mathbf{523}$ of Marsden and Tromba, the annulus bounded by two circles is split into four isometric pieces as in the drawing on the next page.

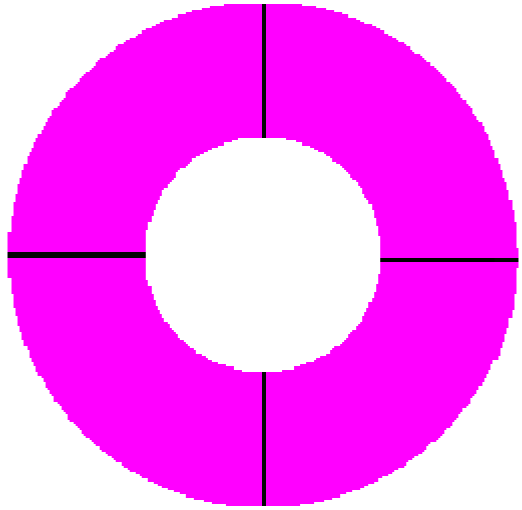


FIGURE 6

Each of the four pieces is homeomorphic to a solid rectangle. Since a solid rectangle has a simplicial decomposition into two $2-$ simplices, one can use such a decomposition to form a triangulation of the solid annulus.

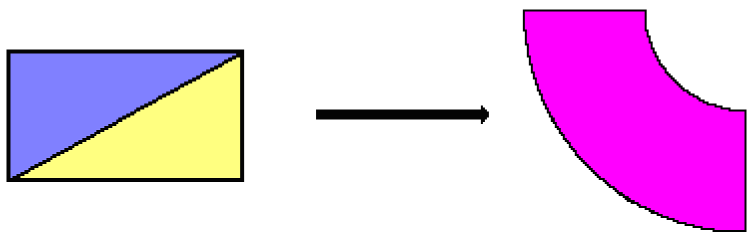


FIGURE 7

A closely related way of triangulating the annulus is suggested by the figure below:

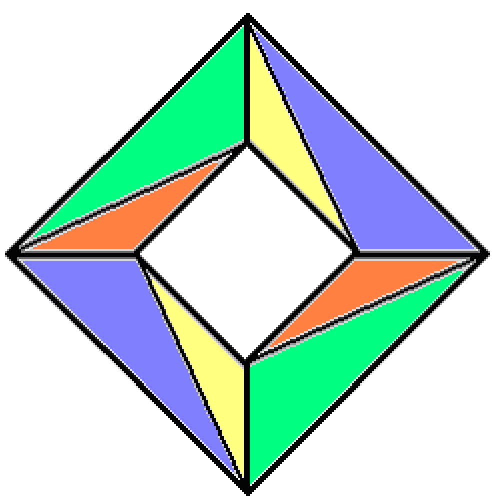


FIGURE 8

Similarly, many familiar closed polygonal regions can be triangulated fairly easily. Here is an example for a solid hexagon.

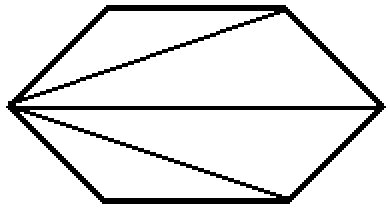


FIGURE 9

Triangulations of prisms. The drawings below illustrate the standard decomposition of a **3** – dimensional triangular prism.

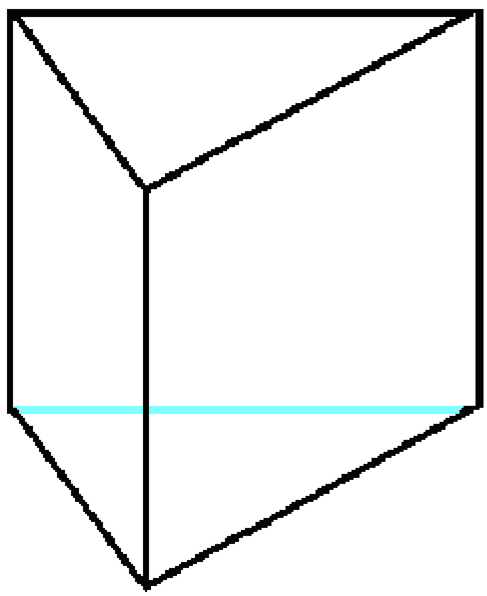


FIGURE 10

If we take $\mathbf{x_0}$, $\mathbf{x_1}$, and $\mathbf{x_2}$ to be the vertices of the bottom triangle and $\mathbf{y_0}$, $\mathbf{y_1}$, and $\mathbf{y_2}$ to be the vertices of the top triangle, then the decomposition is given as follows:

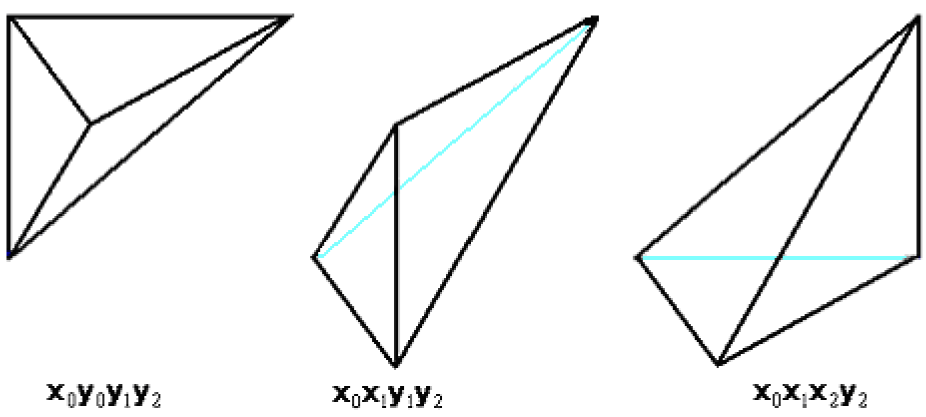


FIGURE 11

I.3 : Subdivisions

***Simple subdivisions* 1.** The drawing below depicts a subdivision of a $1-$ simplex given by a closed interval in the real line into three $1-$ simplices (which are just subintervals of the original interval).

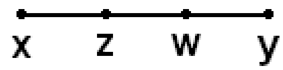


FIGURE 1

Similarly, every partition of an interval determines a subdivision.

***Simple subdivisions* 2.** The drawing below depicts a subdivision of a $2-$ simplex into two $2-$ simplices.

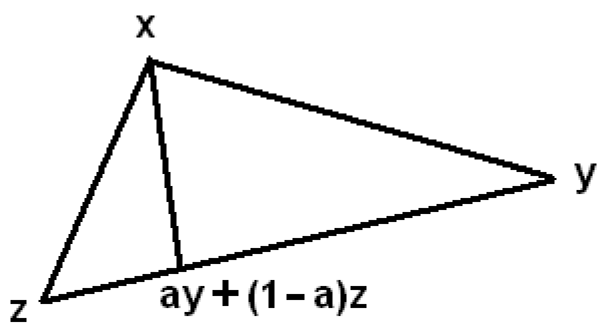


FIGURE 2

If $\mathbf{w} = a\mathbf{y} + (1-a)\mathbf{z}$ where $0 < a < 1$ and $p\mathbf{x} + q\mathbf{y} + r\mathbf{z}$ is a point on the simplex **xyz** (so that $p, q, r \geq 0$ and $p\mathbf{x} + q\mathbf{y} + r\mathbf{z} = 1$), then the point $p\mathbf{x} + q\mathbf{y} + r\mathbf{z}$ lies on the simplex **xwz** if and only if $p = 1$ or $p < 1$ and $q \geq a(1-p)$, and $p\mathbf{x} + q\mathbf{y} + r\mathbf{z}$ lies on the simplex **xwy** if and only if $p = 1$ or $p < 1$ and $q \leq a(1-p)$. The intersection of these simplices is the face with vertices **x** and **w.** Some other simple subdivisions of a $2-$ simplex are illustrated below.

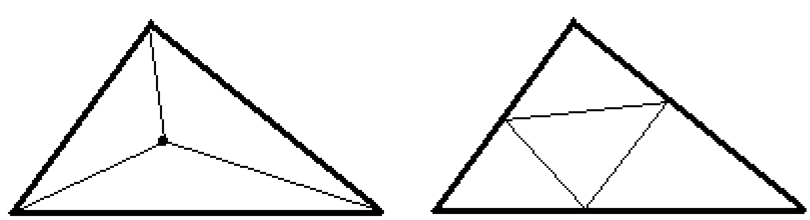


FIGURE 3

A nonexample. In general, if we given two simplicial decompositions, then neither is a subdivision of the other. For example, neither of the two simplicial decompositions of a rectangle described below is a subdivision of the other.

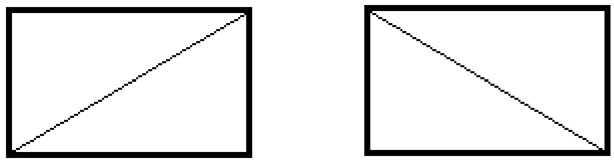


FIGURE 4

On the other hand, there is a decomposition which is a subdivision of ***both*** the decompositions shown above.

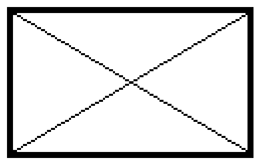
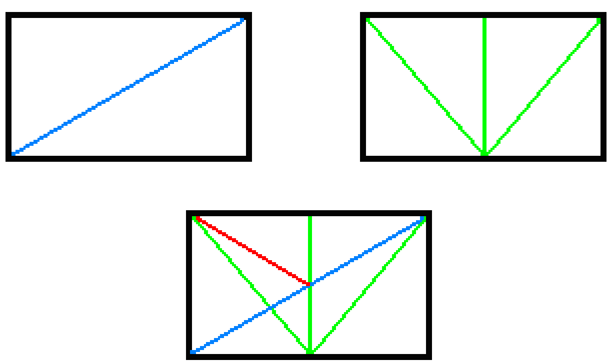


FIGURE 5

More generally, if we are given two simplicial decompositions **K** and **L** of a polyhedron **P** then one can ***always*** construct a third decomposition which is a subdivision of both **K** and **L**. This follows from results in the book, ***Elementary Differential Topology,*** by J. R. Munkres (see the notes for a more complete citation).

Here is a slightly more complicated pair of examples:



Examples from the previous section. Here are illustrations to indicate how one can subdivide the nonsimplicial decompositions from the figures in Section **I.2** to obtain simplicial decompositions.

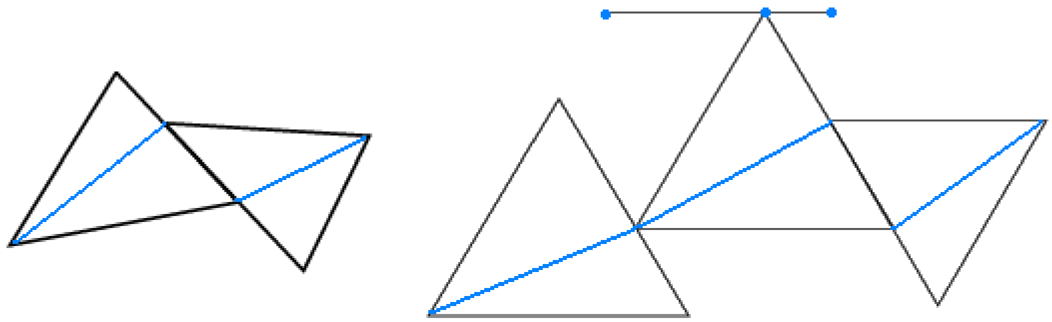


FIGURE 7

Barycentric subdivisions. Here is a drawing to illustrate the barycentric subdivision of a $2-$simplex.

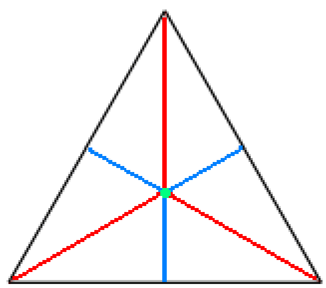


FIGURE 8

The vertices of a $2-$simplex in this subdivision are given by **a, b** and **c,** where **a** is a vertex of the original simplex, **b** is the midpoint of an edge which has **a** as a vertex, and **c** is the barycenter of the $2-$simplex itself. In this example, the diameters of the $2-$simplices in the barycentric subdivision are $2/3$ the diameter of the original simplex.

The drawing below illustrates the barycentric subdivision of a solid rectangular region with its basic decomposition into two $2-$simplices along a diagonal. Observe that the decompositions of the top and bottom $2-$simplices are just the barycentric subdivisions of the latter, and the decomposition of the edge where they intersect is just the barycentric subdivision of that edge.

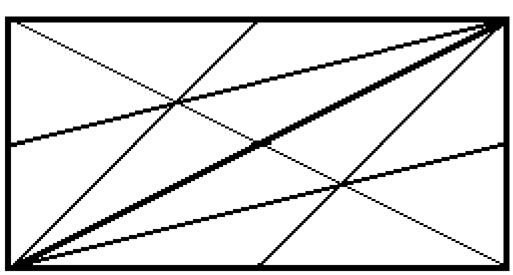


FIGURE 9

The next drawing illustrates the ***second barycentric subdivision*** of a $\mathbf{2}-$ simplex (however, the locations of several vertices are slightly inaccurate).

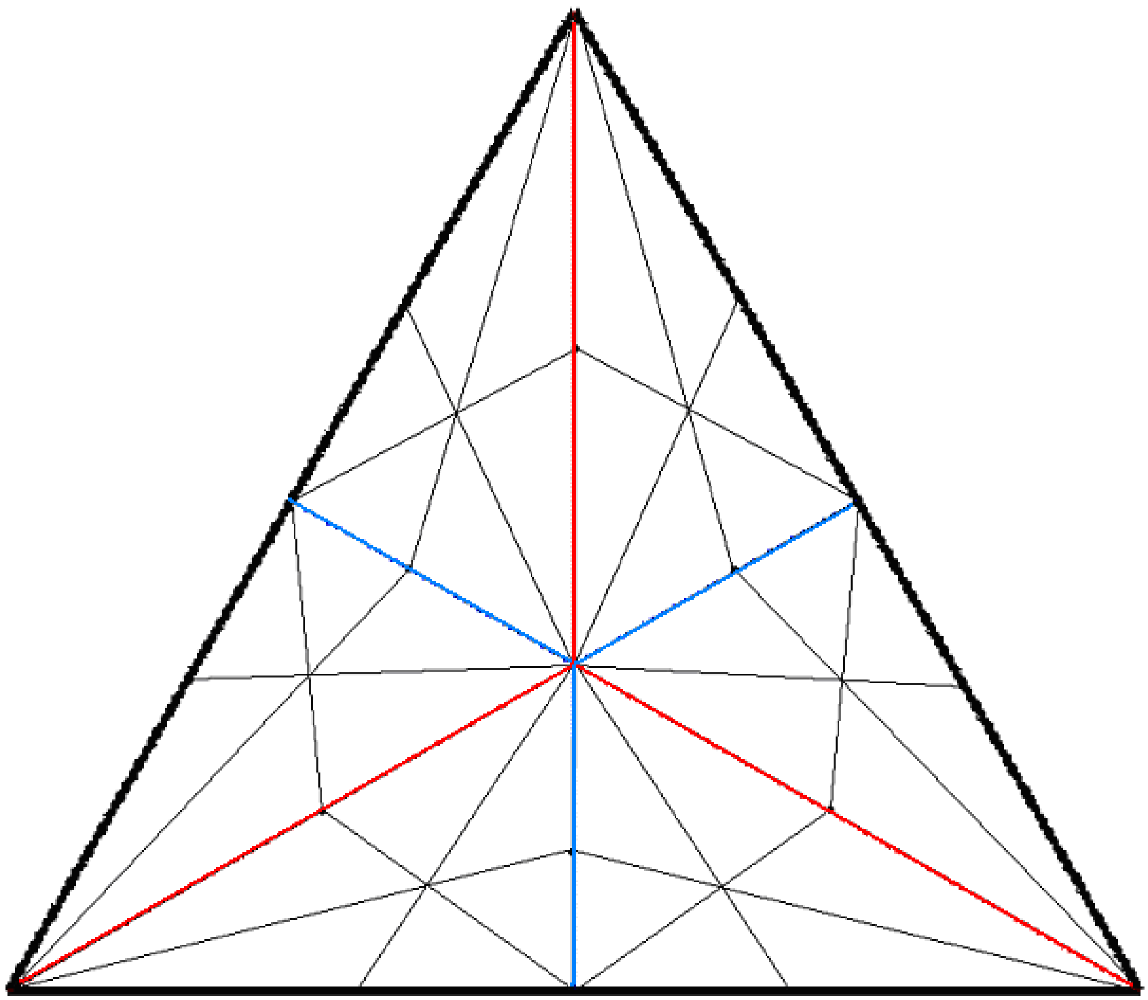


FIGURE 10

This decomposition of the $\mathbf{2}-$ simplex has **25** vertices, **60** edges and **36** simplices that are $\mathbf{2}-$ dimensional. Incidentally, in the third barycentric subdivision there are **121** vertices, **336** edges and **216** simplices that are $\mathbf{2}-$ dimensional.

Final comment on barycentric subdivisions. In the proof that the barycentric subdivision actually defines a simplicial decomposition of a simplex, the simplex containing a given point is determined by putting the barycentric coordinates in linear order. The drawing below indicates the correspondence between inequality chains and $\mathbf{2}$ – simplices in the barycentric subdivision of a $\mathbf{2}$ – simplex.

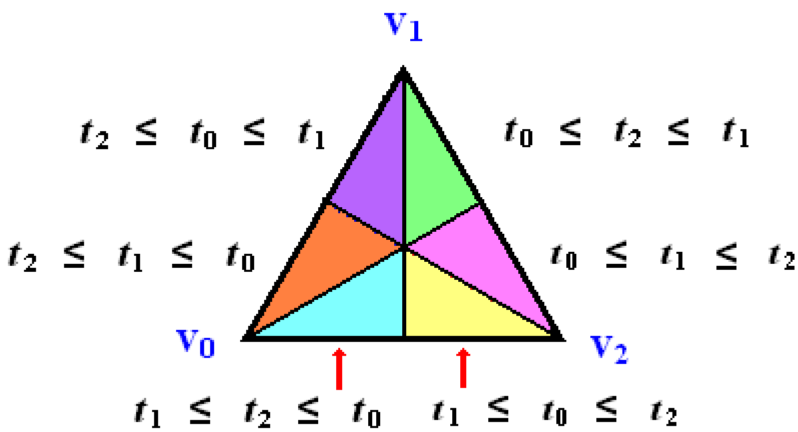


FIGURE 11

I.4 : Cones and suspensions

Cones. In the simplest cases, Proposition $\mathbf{1}$ implies that the topological cones on spaces are canonically homeomorphic to the standard cones of elementary geometry. For example, if $\mathbf{X}$ is the circle $\mathbf{S}^1$, then Proposition $\mathbf{1}$ shows that the cone on $\mathbf{X}$ is homeomorphic to the lateral (or top) surface of the cone illustrated below, and if $\mathbf{X}$ is the disk $\mathbf{D}^2$, then Proposition $\mathbf{1}$ shows that the cone on $\mathbf{X}$ is homeomorphic to the solid cone.

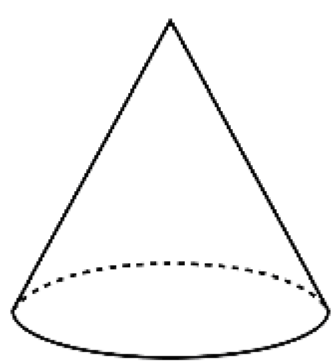
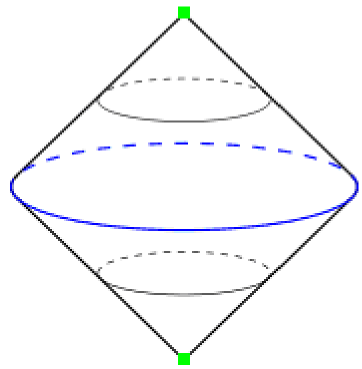


FIGURE 1

In this drawing, the original space is the base of the cone and the point at the top corresponds to the equivalence class of the subspace which is collapsed to a point.

Suspensions. The drawing below illustrates the suspension of the circle. Note that the closure of the piece above the xy – plane is merely the cone on the circle, while the closure of the piece below the xy – plane is the mirror image of that cone with respect to reflection about the xy – plane. Frequently these two subspaces are denoted by the symbols like $\mathbf{C}_+(\mathbf{X})$ and $\mathbf{C}_-(\mathbf{X})$.



(See the Wikipedia citation in http://www.answers.com/topic/suspension)

FIGURE 2

In this drawing, the original space is in blue, and the collapsed end points (the "poles") are both in green.