FIGURES FOR ALGEBRAIC TOPOLOGY LECTURE NOTES

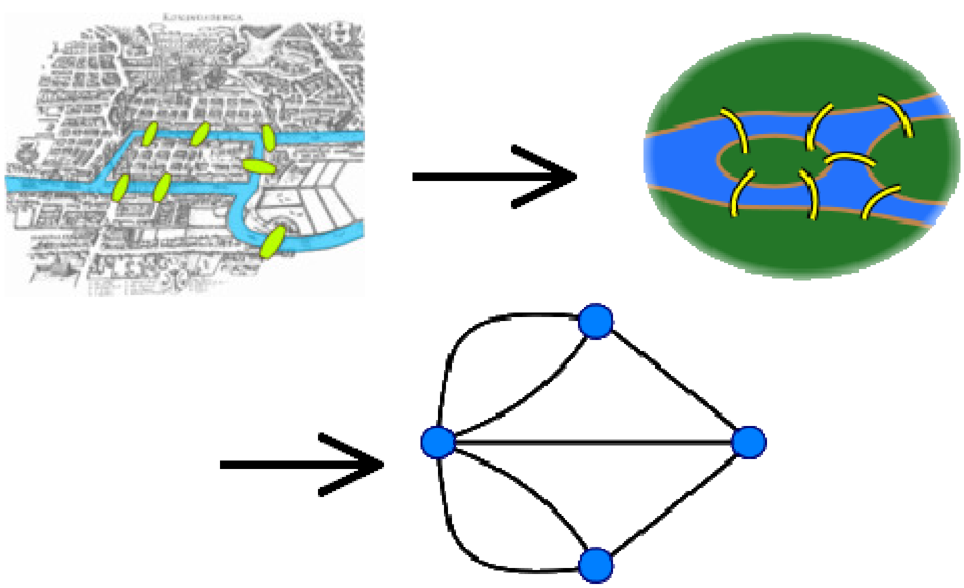
II : Homotopy and cell complexes

II.3 : Abstract cell complexes

The Königsberg Bridge Network. In the 18th century, the city of Königsberg (now known as Kaliningrad, on the Baltic Sea in a small sliver of Russian territory sandwiched between Poland and Lithuania) had seven bridges across the Pregel (or Pregolya) River, which runs through the city.

(***Source:*** http://news.bbc.co.uk/2/hi/europe/country_profiles/6177003.stm)

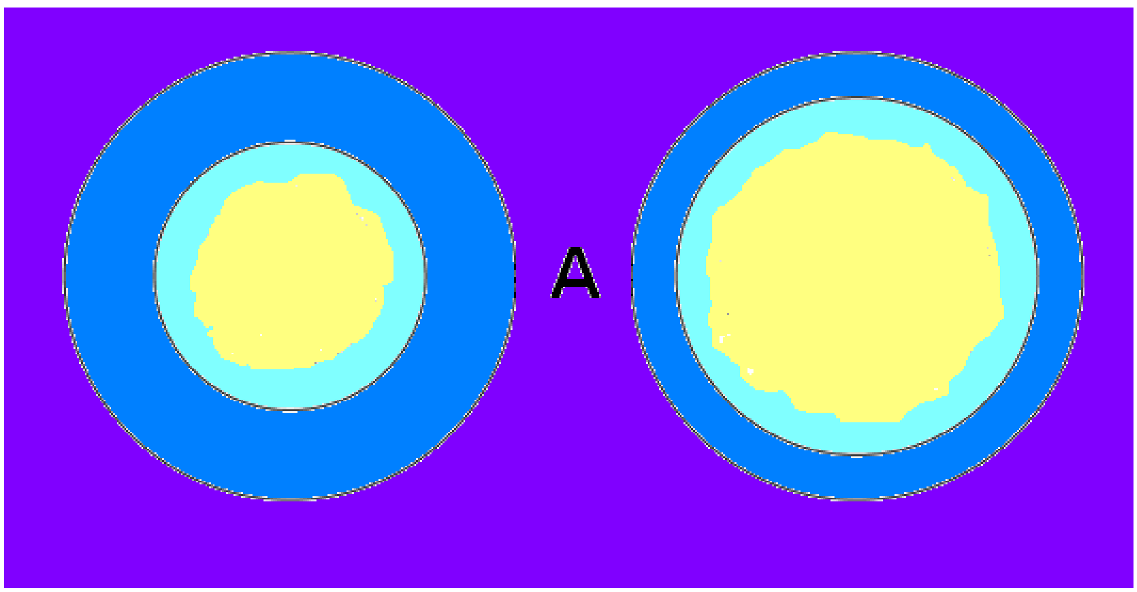
The original question was to find a path through the city which crosses over each of these seven bridges exactly once, and Euler reduced the problem to a question about an edge – vertex graph; specifically, one first eliminates all features but the landmasses and the bridges connecting them, and then one represents each landmass with a vertex and each bridge with an edge whose endpoints are the two landmasses it connects.



(***Source:*** http://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg)

In **1736** Leonhard Euler proved that it was impossible to find a path of the desired type; a description of his approach in modern terminology appears in Unit **III.**

***Cell attachments and NDR neighborhoods*.** The drawing below is supposed to represent a space **X** obtained by taking another space **A** and attaching two **2 –** cells. The subspace **A** is colored purple, and the blue and yellow regions correspond to the two cells. In this case **A** is a strong deformation retract of the open set **U** given by **A** together with the two ring – shaped regions colored in medium blue. Note that a more refined analysis of the situation shows that every open neighborhood **U** of **A** in **X** has a subneighborhood **V** such that the closure of **V** is contained in **U** and **A** is a strong deformation retract of both **V** and its closure (the complement of the yellow regions is supposed to represent a typical open neighborhood of **A** in **X**). Of course, the details are in the notes.

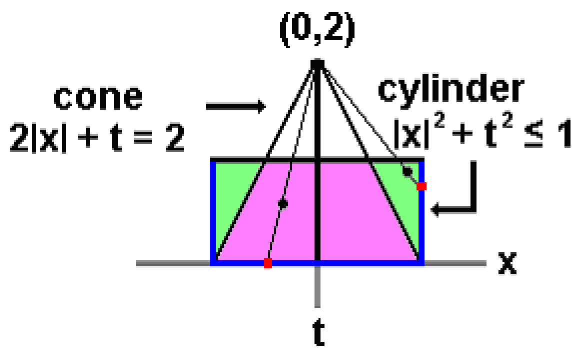


More generally, if A is a subset of **X** such that **A** is a strong deformation retract of some open neighborhood **U** of **A** in **X**, then one frequently says that **A** is a ***neighborhood deformation retract*** (***NDR***) in **X**. An extremely large (probably dominant) share of the subspaces studied in topology and geometry have this propery (in particular, this is true for subcomplexes of a simplicial complex, smooth submanifolds of a smooth manifold, and subsets of Euclidean space defined by finitely many polynomial equations and/or inequalities), and their properties are discussed further in the following reference:

> **N. E. Steenrod,** A convenient category of topological spaces. *Michigan Mathematical Journal* **14** (1967), pp. 133 – 152.

II.4 : The Homotopy Extension Property

Cell attachments and NDR neighborhoods. The crucial point is that the inclusion of $\mathbf{D}^n \times [0,1] \cup \mathbf{S}^{n-1} \times [0,1]$ in $\mathbf{D}^n \times [0,1]$ is a retraction; an illustration of this retraction when $n = 1$ is given below:



In this illustration, the retraction sends the points marked in black into the points marked in red on the corresponding lines. The explicit definition of the retraction has two cases, depending upon whether or not the original point lie in the pink colored region or the green colored region(s).

One can obtain the case $n = 2$ from the one – dimensional case by taking solids and surfaces of revolution about the t – axis, and likewise in higher dimensions one can view the drawing as a planar cross section of the general construction.