EXERCISES FOR MATHEMATICS 246A

FALL 2010

Hatcher's book is the default source for references.

I. Foundational material

I.1 : Categories and functors

1. Definition. A morphism $f : A \to B$ in a category is a *monomorphism* if for all $g, h : C \to A$ we have that $f \circ h = f \circ g$ only if $h = g$. Dually, a morphism $f : A \to B$ in a category is an *epimorphism* if for all $u, v : B \to D$ we have that $u \circ f = v \circ f$ only if $u = v$.

(a) Prove that a monomorphism in the category **Set** is $1 - 1$ and an epimorphism in **Set** is onto. [*Hint:* Prove the contrapositives.]

(b) Prove that in the category of Hausdorff topological spaces (and continuous maps) a morphism $f : A \to B$ is an epimorphism if $f(A)$ is dense in B.

(c) Prove that the composite of two monomorphisms is a monomorphism and the composite of two epimorphisms is an epimorphism.

(d) A morphism $r : X \to Y$ in a category is called a *retract* if there is a morphism $q : Y \to X$ such that $qr = \mathrm{id}_X$. For example, in the category of sets or topological spaces the diagonal map $d_X : X \to X \times X$ is a retract with $q =$ projection onto either factor. Prove that every retract is a monomorphism.

(e) A morphism $p : A \to B$ in a category is called a *retraction* if there is a morphism $s : B \to A$ such that $q \circ r = \mathrm{id}_B$. For example, if r and q are as in (d) then q is a retraction. Prove that every retract is a monomorphism and every retraction is an epimorphism.

2. Let $\mathbf{A}$ be a category, and let $f : A \to B$ be a morphism in $\mathbf{A}$ such that

$$\mathsf{Morph}\,(f, C) : \mathsf{Morph}\,(B, C) \to \mathsf{Morph}\,(A, C)$$

is an isomorphism for all objects C in $\mathbf{A}$. Prove that f is an isomorphism. [*Hint:* Choose $C = B$ or A and consider the preimages of the identity elements.] Also prove the (relatively straightforward) converse.

3. An object $\mathbf{0}$ is called an *initial object* in the category $\mathbf{A}$ if for each object A in $\mathbf{A}$ there is a unique morphism $\mathbf{0} \to A$. An object $\mathbf{1}$ is a *terminal object* in $\mathbf{A}$ if for each object A there is a unique morphism $A \to \mathbf{1}$.

(a) Prove that the empty set is initial and every one point set is terminal in **Set**.

(b) Prove that a zero-dimensional vector space is both initial and terminal in the category **Vec**–F of vector spaces over a field F.

(c) Prove that every two initial objects in a category $\mathbf{A}$ are uniquely isomorphic (there is a unique isomorphism from one to the other), and similarly for terminal objects.

(d) If $\mathbf{A}$ contains an object Z that is both initial and terminal (a *null object*), prove that for each pair of objects A, B in $\mathbf{A}$ there is a unique morphism $A \to B$ that factors as $A \to Z \to B$. Also, if W is any other such object, prove that this composite equals the composite $A \to W \to B$. [*Hint:* Consider the unique maps from W to Z and vice versa.]

4. Prove that a covariant functor takes retracts to retracts and retractions to retractions. State the corresponding result for contravariant functors.

5. If E is a terminal object in the category $\mathbf{A}$ and $f : E \to X$ is a morphism in $\mathbf{A}$, prove that f is a monomorphism (in fact, something stronger is true—what is it?).

6. Let $\mathbf{A} = (\mathbb{N}^+, \mathsf{Morph}\,, \varphi)$, where $\mathbb{N}^+$ denotes the positive integers, $\mathsf{Morph}\,(p, q)$ denotes all $p \times q$ matrices with integer coefficients, and

$$\varphi : \mathsf{Morph}\,(p, q) \times \mathsf{Morph}\,(q, r) \to m(p, r)$$

is matrix multiplication. Verify that $\mathbf{A}$ is a category.

7. If f is a morphism in a category $\mathbf{A}$, a morphism g (in the same category) is called a *quasi-inverse* for f if and only if $f \circ g \circ f = f$. Prove that every morphism that has a quasi-inverse is itself the quasi-inverse of some morphism in the category.

8. In the category of sets, show that the Axioms of Choice implies that every mapping has a quasi-inverse. Also, in the matrix category of Exercise 6, show that every matrix has a quasi-inverse. [*Hint:* Look at the associated linear transformations, and choose bases in a suitable manner.]

NOTE. In fact, there are canonical choices of quasi-inverses. See the following Wikipedia articles for further information on generalizations of matrix inverses:

`http://en.wikipedia.org/wiki/Moore-Penrose_inverse`

`http://en.wikipedia.org/wiki/Group_inverse`

`http://planetmath.org/encyclopedia/DrazinInverse.html`

9. Suppose that $\mathbf{C}$ is a category in which every map has a quasi-inverse. Prove that every monomorphism in $\mathbf{C}$ is a retract. Using this, give examples of mappings in the category of topological spaces (and continuous mappings) which do not have quasi-inverses.

10. Let $\mathbf{A}$ and $\mathbf{B}$ be small categories. Prove that one can define a *product category* $\mathbf{A} \times \mathbf{B}$ whose objects are given by ordered pairs (X, Y), where X and Y are objects of $\mathbf{A}$ and $\mathbf{B}$ respectively, whose morphisms are given by ordered pairs (f, g) of morphisms f in $\mathbf{A}$ and g in $\mathbf{B}$, and whose domain, codomain and composition operations are given as follows:

$$\begin{aligned} \mathrm{Domain}(f, g) &= \big(\mathrm{Domain}(f), \mathrm{Domain}(g)\big) \\ \mathrm{Codomain}(f, g) &= \big(\mathrm{Codomain}(f), \mathrm{Codomain}(g)\big) \\ (f_1, g_1) \circ (f_0, g_0) &= (f_1 \circ f_0, g_1 \circ g_0) \end{aligned}$$

Prove that $\mathbf{A}\times\mathbf{B}$ with these definitions of objects, morphisms, domains, codomains and composition forms a category, and show that "projections onto the first and second coordinates" define covariant functors from this category into $\mathbf{A}$ and $\mathbf{B}$ respectively.

11. Suppose that we are in a category $\mathbf{C}$ with morphisms $f : X \to Y$ and $g : Y \to Z$. Prove that if any two of f, g and $g \circ f$ are isomorphisms, then so is the third.

12. Let $\mathbf{IC_0}$ be the category whose objects are open intervals in the real line and whose morphisms are continuous mappings, and let $\mathbf{IC_1}$ be the subcategory with the same objects, but whose morphisms are maps with continuous first derivatives. Give an example of a morphism in $\mathbf{IC_1}$ which is an isomorphism in $\mathbf{IC_0}$ but not in $\mathbf{IC_1}$ (hence subcategories are not necessarily closed under taking inverses).

13. Let $\{X_\alpha\}$ be an indexed family of objects in a category $\mathbf{C}$. Then a *categorical product* of the X_α is given by an object P and morphisms $p_\alpha : P \to X_\alpha$ such that for each indexed family of maps f_α from a fixed object Y into the objects X_α, there is a unique $f : Y \to P$ such that $p_\alpha \circ f = f_\alpha$ for all α. — All the standard examples of product constructions turn out to have this property.

(a) Prove that if (P, p_α) and (Q, q_α) are categorical products, then there is a unique isomorphism $h : Q \to P$ such that $q_\alpha = p_\alpha \circ h$ for all α. [*Hint:* The only morphism φ from P to itself satisfying $p_\alpha = p_\alpha \circ \varphi$ for all α is the identity.]

(b) Formulate the dual notion of coproduct in a category (a product in the opposite category), and state the dual of the conclusion in (a).

(c) Show that the (external) direct sum is both a product and coproduct in $\mathbf{VEC}_{\mathbb{F}}$ for finite families of vector spaces, and show that the coproduct can be viewed as a proper subspace of the product for infinite families.

14. Let **FLD** be the category of (commutative) fields with morphisms given by field homomorphisms. Show that the category **FLD** does not have products. [*Hints:* Suppose we could construct a product $\mathbb{A}$ of the complex numbers with itself in this category, and consider the morphisms from $\mathbb{C}$ to itself given by the identity and complex conjugation. Recall that every homomorphism of fields is injective.]

15. Let **TOP** be the category of topological spaces and continuous mappings. Show that there is a *homotopy category* **HTP** whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps from one space to another. [*Hint:* The key thing to note is that one has identities and a decent well-defined notion of composition in **HTP**.]

16. We have mentioned that the reason for specifying codomains as part of the structure for morphisms is that functors to not necessarily preserve the injectivity of mappings. Illustrate this for the fundamental group functor $\pi_1(X, x)$ on pointed topological spaces by giving an example of a continuous map of pointed spaces $f : (X, x) \to (Y, y)$ such that f is injective but f_* is surjective and not injective, and also give an example of a continuous map of pointed spaces $f : (X, x) \to (Y, y)$ such that f is surjective but f_* is injective and not surjective.

I.2 : Barycentric coordinates and polyhedra

1. Suppose that P is a polyhedron which has a simplicial decomposition $\mathbf{K}$ with N vertices. Prove that P is homeomorphic to a subset of the simplex Δ_N such that the simplices in $\mathbf{K}$ correspond to sub-simplices of Δ_n.

2. Suppose that $(P, \mathbf{K})$ is a simplicial complex, and let $\mathbf{L}$ be a subcollection of $\mathbf{K}$ which is closed under taking faces. If Q is the union of all the simplices in $\mathbf{L}$, prove that $(Q, \mathbf{L})$ is a polyhderon.

> **Definition.** Let X be a metrizable topological space, let n be a nonnegative integer, and let $x \in X$. Then x is said to be an n-fold branch point of X if there is an open neighborhood base $U_1 \supset U_2 \cdots$ of x in X such that each U_k is connected, each deleted neighborhood $U_k - \{x\}$ has exactly n components, and if $m < k$ then the inclusion mappings $U_k - \{x\} \subset U_m - \{x\}$ induce 1–1 correspondences between the connected components of these spaces (hence different components of $U_k - \{x\}$ map to different components of $U_m - \{x\}$); see Example 5 on page 11 of the notes for the precise definition of the map of connected components associated to a continuous function $f : X \to Y$.

3. (*a*) Let X and x be as above. Explain why x is a 0-fold branch point of X if and only if x is isolated in X (in other words $\{x\}$ is open).

(*b*) Suppose that x is an n-fold branch point of X. Prove that for every sufficiently small open neighborhood V of x, the deleted neighborhood $V - \{x\}$ contains at least n connected components.

(*c*) Suppose that $(P, \mathbf{K})$ is a connected 1-dimensional polyhedron in some $\mathbb{R}^n$ such that every vertex of $\mathbf{K}$ is contained in a 1-simplex. Prove that for each $x \in P$ there is some positive integer n such that x is an n-fold branch point of P. [*Hint:* Why can we take $n = 2$ if x is not a vertex? If x is a vertex, then x lies on some finite number of 1-simplices.]

(*d*) Suppose that x is an n-fold branch point of X and $m \neq n$ is another nonnegative integer. Prove that x cannot be an m-fold branch point of X. [*Hint:* Use (*b*).]

(*e*) Use the preceding two parts of the exercise to show that if $(P, \mathbf{K})$ satisfies the conditions in (*c*) then for each $x \in X$ there is a unique positive integer n_x such that x is an n_x-fold branch point of x. Also, explain why the set $V_n(P)$ of n-fold branch points is finite if and only if $n \neq 2$.

> *Notation.* If x is an n-fold branch point in X for some nonnegative integer n, we shall set $n_x = n_{(x;P)}$ denote the unique integer for which this is true.

(*f*) Let $X \subset \mathbb{R}^2$ be the union of the circles of radius $1/n$ centered at the points $(0, 1/n)$, where n is a positive integer. Show that there is no $n \geq 0$ such that n is an n-fold branch point of the origin. [*Hint:* For each $M > 0$ show that there is some open neighborhood U_M of $(0,0)$ such that if $V \subset U_M$ then $V - \{x\}$ contains at least M components.]

4. (*a*) Suppose that $(P, \mathbf{K})$ and $(Q, \mathbf{L})$ are connected 1-dimensional polyhedra in some $\mathbb{R}^n$ such that every vertex in eiher polyhedron is contained in a 1-simplex, and let $f : P \to Q$ be a homeomorphism. Prove that for all positive integers n the map h sends $V_n(P)$ to $V_n(Q)$. In particular, show that if $n \neq 2$ then $V_n(P)$ and $V_n(Q)$ have the same numbers of elements and that $V_2(P)$ and $V_2(Q)$ have the same (finite) numbers of components.

(*b*) Using the notion of n-fold branch points, show that there are at least **7** homeomorphism types represented by the standard hexadecimal digits as written below (in sans-serif type):

0 1 2 3 4 5 6 7 8 9 A B C D E F

Are new homeomorphism types added if we consider the remaining letters of the alphabet? Explain. — Obviously, one can formulate similar questions for a more or less arbitrary set of printed characters.

(c) As noted in the next to last paragraph on page 358 of Munkres, the Figure 8 and Figure Theta spaces, corresponding to 8 and θ respectively, have the same homotopy type, but neither is a deformation retract of the other, and in fact neither is homeomorphic to a subspace of the other. Prove the last assertion in the preceding sentence. [*Hint:* Suppose more generally that we have 1-dimensional polyhedra P and Q such that P is homeomorphic to a subset of Q, and let $x \in P$. Modify earlier arguments to show that $n_{(x;P)} \leq n_{(x;Q)}$, and explain why this shows that the Figure Eight cannot be a subset of the Figure Theta and vice versa by describing the sets V_n(Figure Eight) and V_n(Figure Theta) for $n > 2$.]

5. Let $(P, \mathbf{K})$ be a 1-dimensional complex satisfying the conditions in previous exercises. Prove that $V_2(P)$ is an open subset with finitely many connected (equivalently, arc/path) components, prove that each of these components is homeomorphic to an open interval, and prove that the closure of each component is homeomorphic to a closed interval.

> **Note.** Using this result it is not difficult to prove the following statement, which is often called the **Hauptvermutung for 1-complexes**: *If $(P, \mathbf{K})$ and $(Q, \mathbf{L})$ are 1-dimensional simplicial complexes such that P and Q are homeomorphic, then there are* linear subdivisions *(as defined in the next section) $\mathbf{K}_1$ of $\mathbf{K}$ and $\mathbf{L}_1$ of $\mathbf{L}$ such that $(P, \mathbf{K}_1)$ and $(Q, \mathbf{L}_1)$ are isomorphic simplicial complexes.* — Although the proof is somewhat lengthy and inelegant, it can be done only using the methods and results described above. — The history of such statements dates back to at least 1908, when E. Steinitz and H. Tietze raised the question of whether this holds for polyhedra of arbitrary dimensions in connection with the constructions for simplicial homology groups in Unit III of these notes. Studies of the Hauptvermutung and related issues have had an enormous impact on geometric topology, and a fairly comprehensive bibliography is given on the Hauptvermutung website `http://www.maths.ed.ac.uk/~aar/haupt`; one other important reference is the following paper of E. M. Brown: *The Hauptvermutung for 3-complexes*, Transactions of the American Mathematical Society Vol. **144** (1969), 173–196. — To summarize known results, the Hauptvermutung is true for complexes of dimension ≤ 3 and false in all higher dimensions. In fact, for **every** simplicial complex $(P, \mathbf{K})$ of dimension ≥ 5, there is another complex $(Q, \mathbf{L})$ of the same dimension such that P and Q are homeomorphic but $\mathbf{K}$ and $\mathbf{L}$ do not have isomorphic subdivisions.

6. A simplicial complex $(P, \mathbf{K})$ is said to be a *star complex* if there is some vertex v of $\mathbf{K}$ such that every maximal simplex σ of $\mathbf{K}$ has v as one of its vertices. Prove that if $(P, \mathbf{K})$ is a star complex, then P is contractible (and in fact $\{v\}$ is a deformation retract of P).

I.3 : Subdivisions

1. Suppose that $(P, \mathbf{K})$ is a simplicial complex of dimension ≥ 1. Prove that P has infinitely many different simplicial decompositions, and in fact, if M is an arbitrary positive number then there is a simplicial decomposition of P with more than M vertices.

2. (a) Suppose that $(P, \mathbf{K})$ is a polyhedron and $(Q, \mathbf{L})$ is a subpolyhedron. If U is an open neighborhood of Q in P, prove at there is some $r > 0$ such that in the r^{th} barycentric subdivision, every simplex of $B^r(\mathbf{K})$ which contains points of Q is a subset of U.

(b) Using the preceding and the methods and results from Section II.9 of Eilenberg and Steenrod, prove that there is an open set V such that $Q \subset V \subset \overline{V}$ and Q is a strong deformation retract of both V and $\overline{V}$.

3. (a) Prove that $\mathbb{R}^n$ contains an infinite sequence of points such that any $n+1$ points in the set are affinely independent.

(b) Let A be a simplex with vertices v_i, and let $f : A \to \mathbb{R}^n$ be the affine-linear map

$$f\left(\sum_i t_i\, v_i\right) \quad = \quad \sum_i t_i\, w_i$$

for $w_i \in \mathbb{R}^n$. Prove that f is an isomorphism of simplices preserving barycentric coordinates if the vectors w_i are affinely independent.

(c) Using the preceding observations, prove that if $(P, \mathbf{K})$ is a simplicial complex of dimension n, then it is isomorphic to a polyhedron in $\mathbb{R}^{2n+1}$. — Later in this course we shall give examples of 1-dimensional complexes which cannot be even topologically embedded in $\mathbb{R}^2$.

I.4 : Cones and suspensions

1. In the category of spaces with basepoints one generally wants a slightly different version of cones and suspensions. In particular, if (X, x) is a Hausdorff topological space with basepoint, then the *reduced suspension* $\mathbf{S}(X, x)$ is defined to be the quotient of $S^1 \times X$ obtained by collapsing the subspace $\{1\} \times X \cup S^1 \times \{x\}$ to a point.

(a) Prove that there is a functorial quotient map from the unreduced suspension to the reduced suspension, and it corresponds to collapsing the "meridian" $[-1, 1] \times \{x\}$ in the unreduced suspension to a point.

(b) If X is a compact subset of some $\mathbb{R}^n$, explain why its unreduced suspension is metrizable (this requires input from 205A).

(c) If $X = S^n$ prove that the reduced suspension is homeomorphic to S^{n+1}. [*Hint:* Results from 205A imply that the unreduced suspension will be the one point compactification of the complement of $\{1\} \times X \cup S^1 \times \{x\}$ in $S^1 \times X$. Show that this complement is homeomorphic to $\mathbb{R}^{n+1}$.]

2. Similarly, if (X, x) is a Hausdorff topological space with basepoint, then the *reduced cone* $\mathbf{C}(X, x)$ is defined to be the quotient of $[0, 1] \times X$ obtained by collapsing the subspace $\{1\} \times X \cup [0, 1] \times \{x\}$ to a point.

(a) Formulate and prove analogs of the first two parts of the preceding exercise, and explain why the reduced suspension is homeomorphic to a union of two copis of the reduced cones, identified along their bases in the obvious fashion.

(b) If $X = S^n$ prove that the reduced suspension is homeomorphic to D^{n+1}. [*Hint:* Let $A \subset D^{n+1}$ denote the set of points for which the coordinate x_n is nonnegative, and use stereographic projection to show that $A - \{\mathbf{e}_{n+1}\}$ is homeomorphic to the set $\mathbb{R}^n_+ \subset \mathbb{R}^n$ of points such that $x_n \geq 0$, and hence A is homeomorphic to the one point compactification of the latter. Next, show that the reduced cone is homeomorphic to the one point compactification of $(S^n - \{\mathbf{e}_{n+1}\}) \times [0, 1)$; the latter can be identified with the set of all points in $D^n + 1$ that are not on the closed line segment joining $\mathbf{0}$ to $\mathbf{e}_{n+1}$. Compare these observations to obtain the assertion in this part of the problem.]

3. Let A be an $n \times n$ orthogonal matrix, and let B be the $(n+1) \times (n+1)$ orthogonal matrix in the block form

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} .$$

Then A and B determine homeomorphisms of S^{n-1} and S^n to themselves. Prove that the homeomorphism determined by B corresponds to the suspension of the homeomorphism determined by A if we identify S^n with $\Sigma(S^{n-1})$ as in the notes. Similarly, show that the homeomorphism determined by A on D^n corresponds to the cone of the homeomorphism determined by A on S^{n-1}.

NOTE. Of course, there are also many good exercises in Hatcher. For each section of the notes, it is worthwhile to look at the exercises in the corresponding sections of Hatcher and see which ones are related to the material covered in the notes. Suggestions for working exercises in various sections of Hatcher will be given in the course directory file `hwsuggestions.txt`.

II. Homotopy and cell complexes

II.1 : Homotopic mappings

1. Suppose that we are given continuous mappings $f : X \to Y$ and $g : Y \to Z$. Prove that if any two of the maps f, g and $g \circ f$ are homotopy equivalences, then so is the third.

2. Prove the following transitivity property (sometimes jokingly called the "Gertrude Stein" property) of deformation retracts: If B is a deformation retract of A and A is a deformation retract of X, then B is a deformation retract of X.

II.2 : The fundamental group

Since this material is covered in earlier courses, no problems are listed. However, some of the problems in the file

`http://math.ucr.edu/~res/math205B/math205Bexercises.pdf`

may be worth checking for review purposes. Also, the first exercise in the next section is relevant to this topic.

II.3 : Abstract cell complexes

0. Prove that a connected finite cell complex is arcwise connected. [*Hint:* Every closed cell lies in a single arc component.]

1. A group G is said to be *finitely presented* if it is isomorphic to a quotient group of the form F/R, where F is a free group on finitely many generators and R is the normal subgroup generated by a finite number of elements $r_1, \cdots, r_m$ in F.

(a) Prove that every finitely presented group is isomorphic to the fundamental group of some connected finite 2-dimensional cell complex.

(b) Suppose that X is an arcwise connected finite cell complex, and let X_2 be its 2-skeleton; choose a base point x of X to be one of the vertices. Prove that the inclusion of X_2 in X induces an isomorphism of fundamental groups. [*Hint:* Use the Seifert-van Kampen Theorem, the simple connectivity of S^{k-1} for $k \geq 3$, and Proposition II.3.4 in the notes.]

(c) Suppose that X is an arcwise connected 2-dimensional finite cell complex with a single 0-cell. Prove that the fundamental group of X is finitely presented. [*Hint:* Explain why the 1-skeleton X_1 is a one-point union of finitely many circles. The complex X is obtained by repeated adjunctions of 2-cells, and the restrictions of their attaching maps to the boundaries correspond to classes in the fundamental group of X_1, Show that if Y_j is obtained by adjoining the first j 2-cells, then the fundamental group of Y_j is isomorphic to the quotient of the fundamental group of X_1 by the normal subgroup generated by the first j "relations" $r_1, \cdots, r_j$.]

NOTE. One can use results from the Mathematics 205B directory to show that the statement in the third part holds for connected complexes with more than a single 0-cell, so we in fact have the following result: *A group G is isomorphic to the fundamental group of a connected finite complex if and only if G is finitely presented.*

2. Let $(X, \mathcal{E})$ be a finite cell complex. Show that $X \times [0,1]$ has a finite cell complex structure whose cells have the form $A \times \{0\}$, $B \times \{1\}$ and $C \times [0,1]$, where A, B and C are cells in $\mathcal{E}$.

3. Show that the cone and suspension of a finite cell complex have finite cell complex structures such that the "base" of the cone and the "equator" of the suspension are subcomplexes.

II.4 : The Homotopy Extension Property

1. A closed subspace $A \subset X$ is said to be *collared* in X if it has an open neighborhood homeomorphic to $A \times [0,1)$ such that A corresponds to $A \times \{0\}$ in the obvious fashion. Prove that a pair of spaces $A \subset X$ satisfying this condition also has the Homotopy Extension Property. Similarly, one says that A is *bicollared* in X if it has a neighborhood of the form $A \times (-1, 1)$ such that A corresponds to $A \times \{0\}$ in the obvious fashion. Prove that a pair of spaces $A \subset X$ satisfying this bicollaring condition also has the Homotopy Extension Property.

2. Let P be a polyhedron, suppose that $x \in P$ is a vertex in P with respect to some simplicial decomposition, and let $\gamma : [0,1] \to P$ be a continuous curve in P such that $\gamma(0) = x$. Prove that there is a continuous mapping $f : P \to P$ such that f is homotopic to the identity and $f(x) = \gamma(1)$.

3. A topological space S is said to be *solid* if the following holds: *If X is a compact metric space and $A \subset X$ is a closed subspace, then every continuous mapping $f : A \to S$ extends to X.* The Tietze Extension Theorem implies that every interval in the real line is solid.

(a) Prove that if S has the indiscrete topology, then S is solid.

(b) Prove that a product of solid spaces is solid.

(c) Prove that if a compact metric space is solid, then it is contractible.

4. Suppose that X is a compact metric space and $A \subset X$ is homeomorphic to D^n for some n. Prove that A is a retract of X.

NOTE. Books by K. Borsuk andn S.-T. Hu, both with the title *Theory of Retracts*, discuss some important further topics in the direction of the preceding two exercises.

III. Simplicial homology

III.1 : Exact sequences and chain complexes

1. Let (A_*, d^A_*) and (B_*, d^B_*) be chain complexes, let $f : A_* \to B_*$ be a chain complex map, and for each n let K_n denote the kernel of $f_n : A_n \to B_n$. Prove that $(K_*, 0)$ is a chain subcomplex of (A_*, d^A_*).

2. Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let $\wedge^k(V)$ be the exterior k^{th} power of V (the *Wikipedia* article on exterior algebra summarizes the main points and is mathematically reliable). Set $n = \dim V$, and let $S_k = \wedge^{n-k}(V)$. Given a nonzero vector $\mathbf{u} \in V$, define $\Delta_k : S_k \to S_{k-1}$ — equivalently, from $\wedge^{n-k}(V)$ to $\wedge^{(n-k)+1}(V)$ — by setting $\Delta_k(\mathbf{x}) = \mathbf{u} \wedge \mathbf{x}$. Prove that the maps w_{k+1} and w_k define an exact sequence $S_{k+1} \to S_k \to S_{k-1}$. [*Hint:* What do we know about $\mathbf{u} \wedge \mathbf{u} \wedge \mathbf{x}$? To prove that the kernel of w_k is contained in the image of w_{k+1}, pick a basis for V whose first element is $\mathbf{u}$.]

3. Let (C_*, d_*) be a chain complex of A-modules, where A is a commutative ring with unit, and let B be a fixed A-module. Show that the construction

$$\Big(\mathrm{Hom}_A(B, C_*),\ \mathrm{Hom}_A(B, d_*) \Big)$$

also defines a chain complex of A-modules.

4. (*a*) Let (C_*, d_*) be a chain complex, and define the *cylinder* (M_*, δ_*) by letting $M_k = C_k \oplus C_k \oplus C_{k-1}$ with

$$\delta_k(x, y, z) \quad = \quad \big(dx - (-1)^k z, dy + (-1)^k z, dz \big) \ .$$

Prove that (M_*, δ_*) is a chain complex with subcomplexes given by $0 \oplus C_* \oplus 0$ and $C_* \oplus 0 \oplus 0$.

(*b*) Let $t = 0, 1$, and let $i_0, i_1 : C_* \to M_*$ be the obvious maps to $0 \oplus C_* \oplus 0$ and $C_* \oplus 0 \oplus 0$. Explain why these mappings identify C_* with two distinct chain subcomplexes of M_*.

III.2 : Homology groups

1. Prove that if the maps d_k in a chain complex C_* are all zero then $H_q(C) \cong C_q$ for all integers q.

2. If C_* is a chain complex, explain why $H_q(C) = 0$ if ane only if $C_{q+1} \to C_q \to C_{q-1}$ is exact.

3. (a) In Exercise 4 from the preceding section, show that the chain maps i_0 and i_1 define monomorphisms in homology.

(b) In the same exercise, show that the chain maps i_0 and i_1 define the same map in homology. [*Hint:* Let U denote their difference, and show that U maps every cycle in C_* to a boundary in M_*.]

4. Give an example to show that if $B_* \subset A_*$ is a chain subcomplex, then the map $H_*(B) \to H_*(A)$ induced by inclusion is not necessarily injective. [*Hint:* Take B to be $\mathbb{Z}$ in dimension zero and zero elsewhere, and take A to be $\mathbb{Z}$ in dimensions 0 and 1 and zero elsewhere. Find a map from A_1 to A_0 for which the conditions in the exercise are met. In fact, there are many possibilities.]

III.3 : Homology and simplicial complexes

1. Let $(P, \mathbf{K})$ be a simplicial complex of dimension n. Prove that $H_n(P, \mathbf{K})$ is a finitely generated free abelian group. [*Hint:* Using ordered or oriented simplicial chains, explain why $H_n(P, \mathbf{K})$ is just the kernel of the boundary mapping d_n and hence is isomorphic to a subgroup of the finitely generated free abelian group $C_n(P, \mathbf{K})$.]

2. If $(P_i, \mathbf{K}_i)$ are disjoint simplicial complexes for $i = 1, 2$, show that the homology groups of their union is isomorphic to the direct sum of the homology groups of $(P_1, \mathbf{K}_1)$ and $(P_2, \mathbf{K}_2)$.

3. Let $(P, \mathbf{K})$ be a connected 1-dimensional simplicial complex with n_i simplices of dimension i. Prove that $H_1(P, \mathbf{K})$ is a free abelian group on $1 + n_1 - n_0$ generators.

III.4 : Comparison principles

1. Prove that the following potential generalizations of the Five Lemma (III.4.5) are false:

(a) A statement in which "isomorphism(s)" is replaced by "epimorphisms."

(b) A statement in which "isomorphism(s)" is replaced by "monomorphisms."

[*Hints:* Consider examples where the second or first row is all zeros but the other row is not.]

2. Suppose that we are given chain complexes B_* and B'_* with subcomplexes A_* and A'_*, and suppose that $f : B_* \to B'_*$ is a chain complex map which sends A_* to A'_*. Denote the associated map of subcomplexes by g, and let $h : A_*/B_* \to A'_*/B'_*$ denote the chain complex map obtained from f and g by passage to quotients. Prove that if two of the mapping sequences f_*, g_*, h_* are isomorphisms in homology (for all dimensions), then so is the third.

III.5 : Chain homotoopies

1. Suppose that C_* is a chain complex and M_* is the cylinder complex constructed in Exercise III.1.4. Prove that the maps i_0 and i_1 are chain homotopic.

2. A *chain homotopy equivalence* is a map of chain complexes $f : A \to B$ for which there is a chain map $g : B \to A$ such that $g \circ f$ and $f \circ g$ are chain homotopic to the identity maps on A and B respectively. The map g is called a chain homotopy inverse to f.

(a) Prove that the maps i_0 and i_1 in the preceding exercise are chain homotopy equivalences.

(b) Prove that two chain homotopy inverses to f are chain homotopic.

(c) Suppose that $f : A \to B$ and $g : B \to C$ are chain maps. Prove that if any two of the maps f, g and $g \circ f$ are chain homotopy equivalences, then so is the third.

3. Suppose that $f : A \to B$ is a chain complex map. Define the *mapping cylinder* $M(f)_*$ to be the quotient of the direct sum $M(B)_* \oplus A_*$, where $M(A)$ is the cylinder complex as above, modulo the subcomplex given by the image of the map $F : A_* \to M(A)_* \oplus B_*$ sending a to $\big(i_0(a), -f(a)\big)$. Let $j_t : A \to M(f)$ denote the canonical mappings induced by the inclusions $i_t : A \to M(A)$, let $J : B \to M(f)$ be induced by inclusion, and let $q : M(f) \to B$ denote the map induced by projections; it follows that all these maps are chain maps (verify this!).

(a) Explain why $q \circ J$ is the identity on B, and show that $J \circ q$ is chain homotopic to the identity on $M(f)$.

(b) Explain why f is chain homotopic to $q \circ j_0$. — In other words, up to chain homotopy equivalence, every chain map is homotopic to the composite of a chain complex inclusion and a chain homotopy equivalence. [*Hint:* Why is $f = q \circ j_1$?]

IV. Singular homology

These exercises are for all sections of the unit

1. Suppose that X is a nonempty space with the indiscrete topology, and let $H : X \times [0,1] \to X$ be defined by $H(x,t) = x$ if $t < 1$ and $H(x,1) = x_0$ for some $x_0 \in X$. Prove that H is continuous, and use this to show that the singular homology groups of a nonempty indiscrete space are isomorphic to the singular homology groups of a one point space.

2. (a) Suppose that F is a free abelian group on a set X, and let $p > 1$ be an integer. Explain why there is a short exact sequence

$$0 \longrightarrow F \xrightarrow{\ p\ } F \longrightarrow F \otimes \mathbb{Z}_p \longrightarrow 0$$

in which $F \otimes \mathbb{Z}_p$ is isomorphic to a vector space with formal basis X.

(b) If (C_*, d_*) is a chain complex, define the *mod p homology groups* $H_*(C, \mathbb{Z}_p)$ to be the homology groups of the $\mathbb{Z}_p$-vector space chain complex $(C_* \otimes \mathbb{Z}_p, d_* \otimes \mathbb{Z}_p)$. Given a topological space X, prove that there is an exact sequence of the form

$$H_{k+1}(X;\mathbb{Z}_p) \xrightarrow{\ \beta\ } H_k(X;\mathbb{Z}) \xrightarrow{\ p\ } H_k(X;\mathbb{Z}) \xrightarrow{\ j_*\ } H_k(X;\mathbb{Z}_p) \xrightarrow{\ \beta\ } H_{k-1}(X;\mathbb{Z})$$

which extends indefinitely to the left and right. The maps denoted by β are called *Bockstein maps* and the sequence is called the *Bockstein exact sequence* for the short exact sequence associated to the inclusion map $\mathbb{Z} \cong p\mathbb{Z} \subset \mathbb{Z}$; the map j_* is induced by the projection map from $\mathbb{Z}$ to $\mathbb{Z}_p$. — Explain the naturality properties of this with respect to a continuous mapping $f : X \to Y$.

3. Suppose that U is a nonempty bounded open subset of $\mathbb{R}^n$. Let $(P, \mathbf{K})$ be a large polyhedron in $\mathbb{R}^n$ which contains U (for example, a sufficiently large hypercube), and for each k let Q_k be the union of all simplices in the k^{th} barycentric subdivision of P which are entirely contained in U. Let δ denote the maximum diameter of a simplex in $\mathbf{K}$.

(a) Prove that if $x \in U - Q_k$, then the distance from x to $\mathbb{R}^n - U$ is at most $\delta \cdot (n/n+1)^k$. [*Hint:* Why does x lie on some simplex σ of the k^{th} barycentric subdivision, and why must this simplex contain some point y not in U? Why is the diameter of σ an upper estimate for the distance between x and y and hence for the distance between x and $\mathbb{R}^n - U$?]

(b) Prove that evey compact subset $C \subset U$ is contained in some Q_k. [*Hint:* Let $\eta > 0$ denote the distance between C and $\mathbb{R}^n - U$. Why is $C \subset Q_k$ if $\delta \cdot (n/n+1)^k < \eta$?]

(c) Use Theorems IV.1.6 and IV.2.11 to prove that each singular homology groups $H_q(U)$ is (at most) countable and that $H_q(U) = 0$ if $q > n$.

(d) Prove that every open subset in $\mathbb{R}^n$ is homeomorphic to a bounded open set (hence the conclusion in (c) holds for all open subsets of $\mathbb{R}^n$.

(e) Prove that if U is the complex plane with all nonnegative integers removed, then $H_1(U)$ is a free abelian group on a countably infinite set of generators. [*Hint:* Let U_n be the set of all points in U whose real part is less than k, and let V_k be the set of all points $x + yi$ such that $k - 1 < x < k$ but $x + yi \neq k$. Why is $U_k \cup V_k = U_{k+1}$? What are the homology groups of V_k and $V_k \cap U_k$? Using Mayer-Vietoris sequences, prove by induction that $H_1(U_k)$ is a free abelian group on k generators and that the inclusion map $U_k \subset U_{k+1}$ induces an injection from $H_1(U_1)$ to a direct summand of $H_1(U_{k+1})$. Why is every compact subset of U contained in some U_k? Combine this with the previous argument to conclude that $H_1(U)$ has the specified form.]

NOTE. One can strengthen (c) to show that $H_q(U) = 0$ if $q \geq n$ (see Lemma 6.1 on page 147 of Vick, *Homology Theory*).

4. (a) Let X be a nonempty topological space, and let $n \geq 0$. Prove by induction on n that $H_q(S^n \times X) \cong H_q(X) \oplus H_{q-n}(X)$ for all integers q. [*Hint:* Look at the Mayer-Vietoris sequence arising from $S^n \times X = U \times X \cup V \times X$, where $U = S^n - \{\mathbf{e}\}$ and $V = S^n - \{-\mathbf{e}\}$ for some unit vector $\mathbf{e}$. What do we know about the homotopy types of U, V and $U \cap V$, and how can we exploit this?]

(b) Let n_1 and n_2 be positive. Explain why $S^{n_1} \times S^{n_2}$ is not homeomorphic to $S^{n_1+n_2}$.

(c) Let n_1 and n_2 be as in the preceding part of this exercise. Prove that $S^{n_1} \times S^{n_2}$ is not homotopy equivalent to its suspension $\Sigma\,(\,S^{n_1} \times S^{n_2}\,)$. [*Hint:* Use Exercise 20 on page 132 of Hatcher.]

5. (a) Suppose that we are given simplicial complexes $(P, \mathbf{K})$ and $(Q, \mathbf{L})$ in $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and suppose that $\mathbf{x}$ and $\mathbf{y}$ are vertices of $\mathbf{K}$ and $\mathbf{L}$ respectively. Define the *wedge*

$$(P, \mathbf{K}; \mathbf{x}) \ \vee \ (Q, \mathbf{L}; \mathbf{y})$$

to be the simplicial complex whose underlying space is

$$P \times \{\mathbf{y}\} \ \cup \ \{\mathbf{x}\} \times Q \quad \subset \mathbf{R}^n \times \mathbf{R}^m \quad \cong \mathbf{R}^{n+m}$$

and whose simplices have one of the forms $A \times \{\mathbf{y}\}$ or $\{\mathbf{x}\} \times Q$, where A is a simplex of $\mathbf{K}$ or B is a simplex of $\mathbf{L}$. Find a formula for the Euler characteristic of the wedge in terms of $\chi(P)$ and $\chi(Q)$. [*Hint:* How many simplices are there in a fixed dimension k? There are two cases depending upon whether or not k is positive. Check your formula using some examples.]

(b) Given an arbitrary integer p, prove that there is a finite connected simplicial complex whose Euler characteristic whose Euler characteristic is equal to p. [*Hint:* There are obvious examples when p is 0, 1 or 2. How can one combine this with the result in the first part?]

(c) Suppose that we are given simplicial complexes $(P_i, \mathbf{K}_i)$ for $1 \leq i \leq n$ with vertices $\mathbf{x}_i \in P_i$, and define the wedge

$$\vee_{i=1}^{n} \, (P_i, \mathbf{x}_i)$$

as in part (a). Prove that if (Y, y) is a pointed space then there is a 1–1 correspondence of base point preserving homotopy classes

$$[\vee_{i=1}^{n} \, (P_i, \mathbf{x}_i), \, (Y, y)] \;\; \cong \;\; \prod_{i=1}^{n} \, [(P_i, \mathbf{x}_i), \, (Y, y)]$$

where the map from the domain to the i^{th} coordinate of the codomain is given by restriction to P_i. (d) In the preceding notation, assume that each P_i is isomorphic to $\partial\Delta_2$. Prove that $H_1\,(\vee_i \; P_i)$ is free abelian on n generators, and show that for every algebraic self-map φ of $H_1\,(\vee_i \; P_i)$ there is a continuous base point preserving self-map f of $\vee_i \; P_i$ whose induced map of homology is equal to φ. [*Hint:* First prove the analogous result in which the fundamental group replaces H_1.]

V. Geometric applications

These exercises are for all sections of the unit

1. Construct a continuous self-map $f : S^1 \to S^1$ such that f is onto but the degree of f is equal to zero. [*Hint:* Wind and rewind.]

2. Use the methods of Section V.2 to prove the following generalization of the Jordan Curve Theorem; this is Theorem 63.5 on page 392 of Munkres, *Topology*:

> *If C_1 and C_2 are compact subsets of S^2 such that each set $S^2 - C_i$ is connected and $C_1 \cap C_2 \cong S^0$, then the complement of $C_1 \cup C_2$ in S^2 has exactly two components.*

NOTES. In Munkres, this result is applied to prove that two specific 1-dimensional polyhedra (the gas-water-electricity network and the complete graph on five vertices) are not homeomorhic to subsets of $\mathbb{R}^2$. The relevant pages of Munkres are posted in this directory as the file `planar-graphs.pdf`.

Kuratowski's Theorem. At the end of Section 64, Munkres mentions a celebrated result of C. Kuratowski and L. S. Pontryagin, which states that every graph which cannot be realized as a subset of $\mathbf{R}^2$ must contain a homeomorphic copy of either the utilities network or the complete graph on five vertices. Here is an online reference for the proof:

`http://cs.princeton.edu/~ymakaryc/papers/kuratowski.pdf`

Further background and additional references are given at the following online sites:

`http://mathworld.wolfram.com/KuratowskiReductionTheorem.html`

`http://en.wikipedia.org/wiki/Planar_graph`

3. Let $\mathfrak{F}$ be a finite family of simple closed curves in $\mathbb{R}^2$. Prove that there is an innermost curve $\Gamma_0 \in \mathfrak{F}$ in the sense that no curve in $\mathfrak{F}$ lies in the bounded component of $\mathbb{R}^2 - \Gamma_0$. [*Hint:* Define a binary relation $\mathcal{R}$ on $\mathfrak{F}$ such that $\Gamma \mathrel{\mathcal{R}} \Gamma'$ if and only if Γ lies in the closure of the bounded component of $\mathbb{R}^2 - \Gamma$. Prove that ρ defines a partial ordering of $\mathfrak{F}$. Why must there be a minimal element?]

4. Suppose that $f : S^n \to S^n$ is a continuous self-map, and let $\rho : S^n \to S^n$ be an orthogonal reflection. Prove that either f has a fixed point or else $x \in S^n$ such that $f(x) = \rho(x)$. [*Hint:* Compute the Lefschetz numbers of f and $f \circ \rho$.]

5. (a) Suppose that $(P, \mathbf{K})$ is a simplicial complex and $f : P \to P$ is a continuous map which is homotopic to a constant. Prove that f has a fixed point. [*Hint:* What is its Lefschetz number?]

(b) Give an example to show that the naïve converse to the Lefschetz Fixed Point Theorem is false; namely, if P is given as above and $f : P \to P$ has Lefschetz number equal to zero, then it does not necessarily follow that f has no fixed points. [*Note:* However, in many cases it follows that if the Lefschetz number is zero then f is homotopic to a map without fixed points; one general reference is E. Fadell, *Recent advances in fixed point theory*, Bulletin of the American Mathematical Society **76** (1970), 1—29 — this paper is freely available online.]

6. Let P be a 1-dimensional polyhedron, and suppose that $x \in P$ is an n-fold branch point in the sense of Exercise I.2.3. Prove that $H_1(P, P - \{x\})$ is free abelian on n generators. [*Hint:* Let $\mathcal{A}$ be the set of all edges which contain x as a vertex; prove that the union of the open edges in $\mathcal{A}$, with the endpoints removed, together with $\{x\}$, is open — for example, show that its complement is the union of all edges which do not have x as a vertex and all points other than x which do not lie on any edge. Verify that the conclusion holds for $(U, U - \{x\})$ and use excision to derive the result as stated in the problem.]

7. If X is a topological space and $x \in X$, then the *local homology groups of X at x* are given by $H_*(X, X - \{x\})$.

(a) Prove that if U is an open neighborhood of x in X, then $H_*(U, U - \{x\}) \cong H_*(X, X - \{x\})$.

(b) Prove that if $f : X \to Y$ is a homeomorphism then for all values of a the local homology groups $H_q(X, X - \{x\})$ are isomorphic to the corresponding local homology groups $H_q(Y, Y - \{y\})$.

(c) Let $(X, \mathcal{E})$ denote the finite cell complex in $\mathbb{R}^3$ whose 2-cells are the upper and lower hemispheres $D^2_{\pm} \subset S^2$ (last coordinate nonnegative or nonpositive) whose 1-cells are the upper and lower semicircles of S^1 plus the line segment joining the points $(\pm 1, 0, 0)$, and whose 0-cells are the latter points. Compute the local homology groups of X for all $x \in X$; there are three cases, depending upon whether or not x lies on the extra 1-cell and whether of not x is a 0-cell.

(d) Prove that that the space X in the preceding part of this exercise is not a topological manifold. [*Hint:* The local homology groups for a topological n-manifold are described in the proof of Theorem IV.2.16.]

8. For most of the standard examples of closed curves in the plane, it is fairly easy to figure out which points lie in the bounded component of the curve's complement and which lie in the unbounded component. However, for more complicated examples like the one in `fishmaze.pdf`, it is not immediately clear which points lie in which component. This exercise provides a test for determining whether two points not on a planar closed curve are in the same component of the complement. As suggested by the drawing in `fishmaze2.pdf`, this rule can be used effectively to answer such questions.

(a) Suppose that Γ is a smooth regular simple closed curve in $\mathbb{R}^2$ with parametrization γ such that $\gamma(0) = p$, and let $\beta(t)$ be a regular smooth parametrized curve ($\beta' \neq \mathbf{0}$ always) such that $\beta(0) = \gamma(0)$ and the vectors $\beta'(0)$, $\gamma'(0)$ form a vasis for $\mathbb{R}^2$. Prove that there is some $\varepsilon > 0$ such that β maps $(-\varepsilon, 0)$ and $(0, \varepsilon)$ to different components of $\mathbb{R}^2 - \Gamma$. [*Hint:* Use the Inverse function Theorem to change coordinates so that near $t = 0$ the curve γ transforms into a horizontal line. Let W be a small neighborhood of the image point p such $W - \Gamma$ has two components, and using the change of coordinates explain why the restrictions of β to $(-\varepsilon, 0)$ and $(0, \varepsilon)$ lie in different components of $W - \Gamma$. Use the fact that Γ is the frontier of both components of $\mathbb{R}^2 - \Gamma$ to show that different components of $W - \Gamma$ lie in different components of $\mathbb{R}^2 - \Gamma$.]

(b) Let Γ and γ be as above, and let $\alpha : [0, 1] \to \mathbb{R}^2$ be a piecewise smooth curve in $\mathbb{R}^2$ joining two points in $\mathbb{R}^2 - \Gamma$. Suppose that α and γ have transverse intersections in the following sense: There are only finitely many common points, the curve α is smooth at these common points, and at each such point the vectors α' and γ' are linearly independent. — Prove that $\alpha(0)$ and $\alpha(1)$ lie in the same component of $\mathbb{R}^2 - \Gamma$ if the number of common points is even, and $\alpha(0)$ and $\alpha(1)$ lie in different components of $\mathbb{R}^2 - \Gamma$ if the number of common points is odd.

VI. Cohomology

These exercises are for all sections of the unit

1. Suppose that $i : A \to X$ is a retract, and let $r : X \to A$ be a one-sided inverse such that $r \circ i = \mathrm{id}_A$. Prove that for every field $\mathbb{F}$ and integer q the induced cohomology map $i^* : H^q(X; \mathbb{F}) \to H^q(A; \mathbb{F})$ is onto and its kernel is a direct summand.

2. Suppose that $(P, \mathbf{K})$ is a polyhedron in $\mathbb{R}^3$, let U be an open subset of $\mathbb{R}^3$ containing P, and let $\mathbf{F}$ and g be (respectively) a smooth vector field and smooth function on U (strictly speaking, $\mathbf{F}$ is a smooth 3-dimensional vector valued function on U). — Given a free generator $\alpha = \mathbf{v}_0 \ \cdots \ \mathbf{v}_q$ for $C_q(P, \mathbf{K})$ let $T(\alpha) : \Delta_q \to P$ be the usual affine q-simplex defined by

$$T(t_0, \ \cdots \ , t_q) \quad = \quad \sum_i \ t_i \, \mathbf{v}_i \ .$$

Now define cochains for the complex $C_*(P; \mathbf{K})$ as follows:

(a) $E_g \in C^0(P, \mathbf{K}; \mathbb{R})$ is given on free generators by evaluating a vertex $\mathbf{v} \in \mathbf{K}$ at $\mathbf{v}$.

(b) $L_{\mathbf{F}} \in C^1(P, \mathbf{K}; \mathbb{R})$ is given on free generators by taking a formal 1-simplex $\alpha = \mathbf{v}_0\mathbf{v}_1$ and forming the line integral $\int \mathbf{F} \cdot d\,\mathbf{s}$ along the curve $T(\alpha)$.

(c) $S_{\mathbf{F}} \in C^2(P, \mathbf{K}; \mathbb{R})$ is given on free generators by taking a formal 2-simplex $\alpha = \mathbf{v}_0\mathbf{v}_1\mathbf{v}_2$ and forming the surface (flux) integral $\int \mathbf{F} \cdot d\,\mathbf{S}$ along the parametrized surface $T(\alpha)$, with the preferred normal direction given by $(\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_2 - \mathbf{v}_0)$.

(*d*) $V_g \in C^3(P, \mathbf{K}; \mathbb{R})$ is given on free generators by taking a formal 3-simplex α and forming the triple (volume) integral $\int g \, dV$.

Using standard theorems from vector analysis, prove the following identities:

$$\delta E_g \ = \ L_{\nabla g} \ , \quad \delta L_{\mathbf{F}} \ = \ \mathbf{S}_{\nabla \times \mathbf{F}} \ , \quad \delta S_{\mathbf{F}} \ = \ V_{\nabla \cdot \mathbf{F}}$$

In particular, $L_{\mathbf{F}}$ is a cocycle if $\nabla \times \mathbf{F} = \mathbf{0}$, and $S_{\mathbf{F}}$ is a cocycle if $\nabla \cdot \mathbf{F} = 0$.

3. Given a space X, let $\{X_\alpha\}$ denote its arc or path components, and for each α let $i_\alpha : X_\alpha \to X$ be the inclusion map. Prove that for each field $\mathbb{F}$ there is an isomorphism

$$H^*(X; \mathbb{F}) \ \to \ \prod_\alpha H^*(X_\alpha; \mathbb{F})$$

whose projection onto the α factor is given by i_α^*. [*Hint:* The singular chain complex of X is the direct sum of the singular chain complexes for the subspaces X_α. This yields a product isomorphism on the cochain complex level. Why does this pass to an isomorphism in cohomology?]

4. The file `cellcpxRPn.pdf` gives a cellular chain complex for computing the singular homology of $\mathbb{RP}^n$.

(*a*) Given a finite cell complex $(X, \mathcal{E})$ with associated cellular chain complex $C_*(X, \mathcal{E})$, explain why $H^*(X; \mathbb{F})$ is isomorphic to $H^*(C(X, \mathcal{E}); \mathbb{F})$.

(*b*) Compute the dimensions of the vector spaces $H^i(\mathbb{RP}^n; \mathbb{F})$ where $\mathbb{F}$ runs through the fields $\mathbb{Q}$ or $\mathbb{Z}_p$, where p is prime (there will be two cases depending upon whether $p = 2$ or p is odd).

(*c*) Let $f : \mathbb{RP}^{2n} \to S^{2n}$ be the map which collapses the $(2n-1)$-skeleton $\mathbb{RP}^{2n-1}$ to a point. Explain why f defines an isomorphism from $H_{2n}(\mathbb{RP}^{2n}, \mathbb{RP}^{2n-1})$ to $H_{2n}(S^{2n}, \mathrm{pt.})$.

(*d*) To continue the discussion in (*c*), using cellular cochains explain why the map

$$H^{2n}(\mathbb{RP}^{2n}, \mathbb{RP}^{2n-1}; \mathbb{Z}_2) \to H^{2n}(\mathbb{RP}^{2n}; \mathbb{Z}_2)$$

is an isomorphism, so that f induces an isomorphism in $\mathbb{Z}_2$ cohomology. [*Hint:* Show that if X is an m-dimensional cell complex and $\mathbb{F}$ is a field, then $H^m(X; \mathbb{F})$ is isomorphic to the quotient of $H^m(X, X_{m-1}; bbF)$ modulo the image of the composite $H^{m-1}(X_{m-1}, X_{m-2}; \mathbb{F}) \to H^{m-1}(X_{m-1}; \mathbb{F}) \to H^m(X, X_{m-1}; \mathbb{F})$.

(*e*) To complement the discussion in (*c*) and (*d*), explain why f induces the zero map in $\mathbb{F}$ cohomology for all the other fields $\mathbb{F}$ listed above.