Preface

Perhaps the simplest motivation for algebraic topology is the following basic question:

If m and n are distinct positive integers, is $\mathbb{R}^m$ ever homeomorphic to $\mathbb{R}^n$?

Results from point set topology imply the answer is **NO** if one of m and n is equal to 1. If a homeomorphism $h : \mathbb{R}^m \to \mathbb{R}$ existed then for each $\mathbf{x} \in \mathbb{R}^m$ we could conclude that $\mathbb{R}^n - \{\mathbf{x}\}$ is homeomorphic to $\mathbb{R} - \{h(\mathbf{x})\}$. Since $\mathbb{R}^m - \{\mathbf{x}\}$ is connected for all $\mathbf{x} \in \mathbb{R}$ if $m > 1$ while $\mathbb{R} - \{t\}$ is not connected for any choice of $t \in \mathbb{R}$, it follows that $\mathbb{R}^m - \{\mathbf{x}\}$ is never homeomorphic to $\mathbb{R} - \{t\}$ if $m > 1$ and hence $\mathbb{R}^m$ cannot be homeomorphic to $\mathbb{R}$. Similarly, results on fundamental groups imply that for all relevant choices of $\mathbf{x}$ the set $\mathbb{R}^m - \{\mathbf{x}\}$ is simply connected if $m > 2$ while $\mathbb{R}^2 - \{\mathbf{x}\}$ has an infinite cyclic fundamental group, so we also know that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^2$ provided $m > 2$. One basic goal of an introductory course in algebraic topology is to show that $\mathbb{R}^m$ is never homeomorphic to $\mathbb{R}^n$ if $m \neq n$.

The idea behind proving such results is to define certain abelian groups which give an **algebraic picture** of a given topological space; in particular, if two topological spaces are homeomorphic, then their associated groups will be algebraically isomorphic. Unfortunately, the definitions for these **homology groups** are less straightforward than the definition of the fundamental group, and much of the work in this course involves the construction of such groups and the proofs that they have good formal properties.

In analogy with standard results for fundamental groups, the homology groups of two spaces will be isomorphic if the spaces satisfy a condition that is somewhat weaker than the existence of a homeomorphism between them; namely, an the groups are isomorphic if the two spaces have the same *homotopy type* as defined on page 363 of the book by Munkres cited below.

Since the constructions for the associated groups are somewhat complicated, it is natural to expect that they should be useful for more than simply answering the homeomorphism question for Euclidean spaces. In particular, one might ask if these groups (and a course in algebraic topology) can shed new light on some questions left open in undergraduate or beginning graduate courses in mathematics.

1. The material in introductory graduate level courses does not really give much insight into the popular characterization of topology as a "rubber sheet geometry." In other words, topology is generally viewed as the study of properties that do not change under various sorts of bending or stretching operations. Some aspects of this already appear in the study of fundamental groups, and one objective of this course is to develop these ideas much further.

2. As a refinement of the problem at the beginning of this preface, one can ask if there is some topological criterion which characterizes the algebraic notion of n-dimensionality, at least for spaces that are relatively well-behaved.

3. An algebraic topology course should also yield better insight into several issues that arise in undergraduate courses, including (*a*) the Fundamental Theorem of Algebra, (*b*) various facts about planar and nonplanar networks, (*c*) insides and outsides of plane curves and closed surfaces in 3-dimensional space, and (*d*) Euler's Formula for "nice" polyhedra in $\mathbb{R}^3$; namely, if P is a polyhedron bounding a convex body in $\mathbb{R}^3$, then the numbers V, E and F of vertices, edges and faces satisfy the equation $E + 2 = V + F$.

4. Some of the basic results on the topology of the edge-path graphs in Mathematics 205B should be placed into a broader context. In particular, the notion of Euler characteristic should be extended to a larger class of spaces.

5. If time permits, another goal will be to give a unified approach to certain results in multivariable calculus involving the ∇ operator, Green's Theorem, Stokes' Theorem and the Divergence Theorem, and to formulate analogs for higher dimensions.

Throughout the course we shall use the following text for the basic graduate topology courses as a reference for many topics and definitions:

J. R. Munkres. *Topology* (Second Edition), *Prentice-Hall, Saddle River NJ*, 2000. ISBN: 0–13–181629–2.

The official text for this course is the following book:

A. Hatcher. *Algebraic Topology* (Third Paperback Printing), *Cambridge University Press, New York NY*, 2002. ISBN: 0–521–79540–0.

This book can be legally downloaded from the Internet at no cost for personal use, and here is the link to the online version:

`www.math.cornell.edu/∼hatcher/AT/ATpage.html`

Here are four other references. The first is a book that has been used as a text in the past, the second is a fairly detailed history of the subject during its formative years, and the last two are classic (but not outdated) books; the first also has detailed historical notes.

J. W. Vick. *Homology Theory*. (Second Edition). *Springer–Verlag, New York etc.*, 1994. ISBN: 3–540–94126–6.

J. Dieudonné. *A History of Algebraic and Differential Topology* (1900 − 1960). *Birkhäuser Verlag, Zurich etc.*, 1989. ISBN: 0–817–63388–X.

S. Eilenberg and N. Steenrod. *Foundations of Algebraic Topology*. (Second Edition). *Princeton University Press, Princeton NJ*, 1952. ISBN: 0–691–07965–X.

E. H. Spanier. *Algebraic Topology*, *Springer–Verlag, New York etc.*, 1994.

The `amazon.com` sites for Hatcher's and Spanier's books also give numerous other texts in algebraic topology that may be useful. Finally, there are two other books by Munkres that we shall quote repeatedly throughout these notes. The first will be denoted by [MunkresEDT] and the second by [MunkresAT]; if we simply refer to "Munkres," it will be understood that we mean the previously cited book, *Topology* (Second Edition).

J. R. Munkres. *Elementary differential topology.* (Lectures given at Massachusetts Institute of Technology, Fall, 1961. Revised edition. Annals of Mathematics Studies, No. 54.) *Princeton University Press, Princeton, NJ*, 1966. ISBN: 0–691–09093–9.

J. R. Munkres. *Elements of Algebraic Topology*. *Addison-Wesley, Reading, MA*, 1984. (Reprinted by Westview Press, Boulder, CO, 1993.) ISBN: 0–201–62728–0.

One important feature of homology groups is that if $f : X \to Y$ is a continuous mapping of topological spaces, then there is an associated homomorphism f_* from the homology groups of X to the homology groups of Y; this is again similar to the situation for fundamental groups of pointed spaces, and it plays an important role in addressing the issues listed above. In fact, algebraic topology turns out to be an effective means for analyzing the following central problem:

> Given two "reasonably well-behaved" spaces X and Y, describe the homotopy classes of continuous mappings from X to Y.

In general, the descriptions of the homotopy classes can be quite complicated, and only a few cases of such problems can be handled using the methods of a first course, but we shall mention a few special cases at various points in the course.

Many of the basic properties of homology groups and homomorphisms are best stated using the formalisms of **Category Theory**, and many of the constructions and theorems in algebraic topology are best stated within the framework of **Homological Algebra**. We shall develop these subjects in the course to the extent that we need them.

Prerequisites

The name "algebraic topology" suggests that the subject uses input from both algebra and topology, and this is in fact the case; since topology began as a branch of geometry, it is also reasonable to expect that some geometric input is also required. Our purpose here is to summarize the main points from prerequisite courses that will be needed. Additional background material which is usually not covered explicitly in the prerequisites will be described in the first unit of these notes.

Set theory

Everything we shall need from set theory is contained in the following online directory:

`http://math.ucr.edu/∼res/math144`

In particular, a fairly complete treatment is contained in the documents `setsnotes`n`.pdf`, where $1 \leq n \leq 8$.

There are two features of the preceding that are somewhat nonstandard. The first is the definition of a function from a set A to another set B. Generally this is given formally by the graph, which is a subset $G \subset A \times B$ such that for each $a \in A$ there is a unique $b \in B$ such that $(a, b) \in G$. Our definition of function will be a **triple** $f = (A, G, B)$, where $G \subset A \times B$ satisfies the condition in the preceding sentence. The reason for this is that we must specify the target set or **codomain** of the function explicitly; in fact, the need to specify the codomain has already arisen at least implicitly in prerequisite graduate topology courses, specifically in the definition of the fundamental group. A second nonstandard feature is the concept of **disjoint union** or **sum** of an indexed family $\{ X_\alpha \}$ of sets. The important features of the disjoint sum, which is written $\amalg_\alpha X_\alpha$, are that it is a union of subsets Y_α which are canonically in 1–1 correspondence with the sets X_α and that $Y_\alpha \cap Y_\beta = \emptyset$ if $\alpha \neq \beta$. Another source of information on such objects is Unit V of the online notes for Mathematics 205A which are cited below.

Topology

This course assumes familiarity with the basic material in graduate level topology courses through the theory of fundamental groups and covering spaces (in other words, the material in Mathematics 205A and 205B). Everything we need from the first of these courses can be found in the following online directory:

`http://math.ucr.edu/∼res/math205A`

In particular, the files `gentopnotes2008.*` contain a fairly complete set of lecture notes for the course. This material is based upon the textbook by Munkres cited in the Preface. Two major differences between the notes and Munkres appear in Unit V. The discussion of quotient topologies is somewhat different from that of Munkres, and in analogy with the previously mentioned discussion of set-theoretic disjoint sums there is a corresponding construction of disjoint sum for an indexed family of topological spaces.

There is a similar directory for the second course, which deals with the basic notions of homotopy, fundamental groups and covering spaces:

`http://math.ucr.edu/~res/math205B`

There is no self-contained set of notes in this directory, but there are comments and (additional) exercises to supplement the coverage in Munkres and indicate the sections which will be needed for this course. In addition to the sections mentioned in these references, it might be worthwhile to look also at the supplementary exercises for Chapter 13.

Chapter 14 of Munkres is not a required part of the material covered in the Department's qualifying exam, but it is sometimes covered in 205B and at some points in this course the topics covered in Chapter 14 are relevant, and we shall mention them when it seems appropriate or useful.

Algebra

As in the later parts of Munkres, we shall assume some familiarity with certain topics in group theory. Nearly everything we need is in Sections 67 – 69 of Munkres, but we shall also need the following basic result:

STRUCTURE THEOREM FOR FINITELY GENERATED ABELIAN GROUPS. *Let G be a finitely generated abelian group (so every element can be written as a monomial in integral powers of some finite subset $S \subset G$). Then G is isomorphic to a direct sum*

$$(H_1 \oplus \cdots \oplus H_b) \oplus (K_1 \oplus \cdots \oplus K_s)$$

where each H_i is infinite cyclic and each K_j is finite of order t_j such that t_{j+1} divides t_j for all j. — For the sake of uniformity set $t_j = 1$ if $j > s$. Then two direct sums as above which are given by $(b; t_1, \cdots)$ and $(b'; t'_1, \cdots)$ are isomorphic if and only if $b = b'$ and $t_j = t'_j$ for all j.

A proof of this fundamental algebraic result may be found in Sections II.1 and II.2 of the following standard graduate algebra textbook:

> **T. Hungerford.** *Algebra.* (Reprint of the 1974 original edition, Graduate Texts in Mathematics, No. 73.) *Springer-Verlag, New York–Berlin–etc.*, 1980. ISBN: 0–387–90518–9.

Material from standard undergraduate linear algebra courses will also be used as needed.

Analysis

We shall assume the basic material from an upper division undergraduate course in real variables as well as material from a lower division undergraduate course in multivariable calculus through the theorems of Green and Stokes as well as the 3-dimensional Divergence Theorem. The classic text by W. Rudin (*Principles of Mathematical Analysis*, Third Edition) is an excellent reference for real variables, and the following multivariable calculus text contains more information on the that subject than one can usually find in the usual 1500 page calculus texts (the book is not perfect, but especially at the graduate level it is useful as a background reference).

> **J. E. Marsden and A. J. Tromba.** *Vector Calculus* (Fifth Edition), *W. H. Freeman & Co., New York NY*, 2003. ISBN: 0–7147–4992–0.

I. Foundational and Geometric Background

Aside from the formal prerequisites, algebraic topology relies on some background material from other subjects that is generally not covered in prerequisites. In particular, two concepts from the foundations of mathematics, namely **categories** and **functors**, play a central role in formulating the basic concepts of algebraic topology. Furthermore, since algebraic topology places heavy emphasis on spaces that can be constructed from certain fundamental building blocks, some relatively elementary but fairly detailed properties of the latter are indispensable. The purpose of this unit is to develop enough of category theory so that we can use it to formulate things efficiently and to describe the topological and geometric properties of a class of well-behaved spaces called **polyhedra** that will be needed in the course.

I.1 : Categories and functors

(Hatcher, § 2.3)

If mathematics is the study of abstract systems, then category theory may be viewed as an abstract formal setting for working with such systems. In fact, the theory was originally developed by S. Eilenberg (1919–1998) and S. MacLane (1909–2005) in the 1940s to provide an effective conceptual framework for handling various constructions and phenomena related to algebraic topology (including some from the theory of groups). The formal definition may be viewed as a generalization of familiar properties of ordinary set-theoretic functions. There is a great deal of overlap between the discussion here and the file `categories.pdf` in the `math205A` directory. There is a classic book by P. Freyd (1936–), *Abelian Categories: An Introduction to the Theory of Functors*, which is still an extremely readable introduction to category theory and its role in abstract algebra, and it is available online at the following site:

`http://www.emis.de/journals/TAC/reprints/articles/3/tr3/pdf`

Definition. A CATEGORY is a system $\mathbf{C}$ consisting of

(a) a class $\mathsf{Obj}\,(\mathbf{C})$ of sets called the **objects** of $\mathbf{C}$,

(b) for each ordered pair of objects X and Y an associated set $\mathsf{Morph}_{\mathbf{C}}(X,Y)$ called the **morphisms** from X to Y,

(c) for each ordered triple of objects X, Y and Z, an associated map called a *composition pairing* $\varphi : \mathsf{Morph}_{\mathbf{C}}(X,Y) \times \mathsf{Morph}_{\mathbf{C}}(Y,Z) \longrightarrow \mathsf{Morph}_{\mathbf{C}}(X,Z)$, whose value for (f,g) is generally written $g \circ f$, such that the following hold:

(1) The sets $\mathsf{Morph}_{\mathbf{C}}(X,Y)$ and $\mathsf{Morph}_{\mathbf{C}}(Z,W)$ are disjoint unless $X = Z$ and $Y = W$.

(2) For each object X there is a unique identity morphism $1_X = \mathrm{id}_X \in \mathsf{Morph}_{\mathbf{C}}(X,X)$ such that for each $f \in \mathsf{Morph}_{\mathbf{C}}(X,Y)$ and $g \in \mathsf{Morph}_{\mathbf{C}}(Z,X)$ we have $f \circ 1_X = f$ and $1_X \circ g = g$.

(3) The composition pairings satisfy an associative law; *i.e.*, if $f \in \mathsf{Morph}_{\mathbf{C}}(X,Y)$, $g \in \mathsf{Morph}_{\mathbf{C}}(Y,Z)$, and $h \in \mathsf{Morph}_{\mathbf{C}}(Z,W)$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

By the assumptions, for each $f \in \mathsf{Morph}_{\mathbf{C}}(X,Y)$ the objects X and Y are uniquely determined, and they are called the **domain** and **codomain** of f respectively. When working within a given category we generally use familiar notation like $f : X \to Y$ to indicate that $f \in \mathsf{Morph}_{\mathbf{C}}(X,Y)$.

As in set theory, at some points one must take care to avoid difficulties with classes that are "too large" to be sets (for example, we cannot discuss the set of all sets), but in practice it is always possible to circumvent such problems by careful choices of definitions and wordings (for example, using the theory of *Grothendieck universes*), so we shall generally not dwell on such points.

Examples of categories

By the remarks preceding the definition of a category, it is clear that we have a category **SETS** whose objects are given by all sets, whose morphisms are set-theoretic functions from one set to another (with the conventions mentioned in the Prerequisites!), and whose composition is merely ordinary composition of mappings. Here are some further examples:

1. Given a field $\mathbb{F}$, there is the category $\mathbf{VEC}_{\mathbb{F}}$ whose objects are vector spaces, whose morphisms are $\mathbb{F}$-linear transformations, and whose composition is ordinary composition. The important facts here are that the identity on a vector space is a linear transformation, and the composite of two linear transformations is a linear transformation.

2. There is also a category **GRP** whose objects are groups and whose morphisms are group homomorphisms (with the usual composition). Once again, the crucial properties needed to check the axioms for a category are that identity maps are homomorphisms and the composite of two homomorphisms is a homomorphism.

3. Within the preceding example, there is the subcategory **ABGRP** whose objects are abelian groups, with the same morphisms and compositions. In this category, the set of morphisms from one object to another has a natural abelian group structure given by pointwise addition of functions, and the resulting abelian group of homomorphisms is generally denoted by $\mathrm{Hom}(X,Y)$.

4. More generally, if **C** is a category, then a subcategory **A** is a collection of morphisms and objects which is closed under (i) taking domains and codomains of objects, (ii) taking identity morphisms of objects, (iii) taking composites of morphisms. It is said to be a **full subcategory** if for each pair of objects X and Y in **A** we have $\mathsf{Morph}_{\mathbf{A}}(X,Y) = \mathsf{Morph}_{C}(X,Y)$. It follows that **ABGRP** is a full subcategory of **GRP**. On the other hand, if we let $\mathbf{GRP_{1-1}}$ be the category whose objects are groups and and whose morphisms are *injective* homomorphisms, then $\mathbf{GRP_{1-1}}$ is a subcategory of **GRP** but it is not a full subcategory.

5. If P is a partially ordered set with ordering relation $\leq$, then one has an associated category whose objects are the elements of P and such that $\mathsf{Morph}\,(x,y)$ consists of a single point if $x \leq y$ and is empty otherwise. This is an example of a **small** category in which the class of objects is a set.

6. One can also use partially ordered sets to define a category **POSETS** whose objects are partially ordered sets and whose morphisms are monotonically nondecreasing functions from one partially ordered set to another; as in most other cases, composition has its usual meaning.

7. If G is a group, then G also defines a small category as follows: There is exactly one object, the morphisms of this object to itself are given by the elements of G, and composition is given by the multiplication in G.

8. There is a category **TOP** whose objects are topological spaces, whose morphisms are continuous maps between topological spaces, and whose composition is the usual notion. Again, the crucial properties needed to verify the axioms for a category are that identity maps are continuous and composites of continuous maps are also continuous.

9. There are also categories whose objects are topological spaces and whose morphisms are **open** maps or **closed** maps. The categories with various types of morphisms are distinct. Of course, it is also possible to take combinations of such conditions and obtain structures like the category of spaces with *continuous open mappings* as the morphisms.

10. More generally, given any class of continuous mappings that is closed under taking identities and compositions, one can define a category of topological spaces with such maps as the morphisms. Two examples are maps that are **proper** (inverse images of compact subsets are compact) or **light** (inverse images of points are discrete sets; see Exercise II.3.4 in `gentopexercises2008.pdf` from the `math205A` directory for more on the latter).

11. One also has a category **MET–UNIF** whose objects are metric spaces and whose morphisms are **uniformly continuous** mappings (with the usual composition).

12. Similarly, there is the category **MET–LIP** whose objects are metric spaces and whose morphisms are **Lipschitz** mappings: *i.e.*, there is a constant M such that

$$d\big(\, f(x_1),\, f(x_2)\,\big) \quad \leq \quad M \cdot d(x,y)$$

for all x and y in the domain (such an inequality is called a *Lipschitz condition*). Standard results of (abstract) multivariable calculus show that if K is a compact convex set and $f : K \to \mathbb{R}^m$ extends to a function on an open neighborhood W of K whose coordinates have continuous first partial derivatives, then f satisfies a Lipschitz condition.

13. Still further in the same direction, there is the category **MET–ISO** whose objects are metric spaces and whose morphisms are **isometries** (but not necessarily surjective).

14. (A fundamentally important general construction.) Given an arbitrary category **C**, one has the **dual** or **opposite** category $\mathbf{D} = \mathbf{C}^{\mathbf{OP}}$ with the same objects as **C**, but with $\mathsf{Morph}_{\mathbf{D}}(X,Y) = \mathsf{Morph}_{\mathbf{C}}(Y,X)$ (note the reversal!) and the composition pairing $*$ defined by $g * f = f \circ g$. Note that if $\mathbf{D} = \mathbf{C}^{\mathbf{OP}}$ then $\mathbf{C} = \mathbf{D}^{\mathbf{OP}}$.

In most of the preceding examples of categories, there is a fundamental notion of **isomorphism**, and in fact one can formulate this abstractly for an arbitrary category:

Definition. Let **C** be a category, and let X and Y be objects of **C**. A morphism $f : X \to Y$ is an *isomorphism* if there is a morphism $g : Y \to X$ (an inverse) such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

This generalizes notions like an invertible linear transformation, a group isomorphism, and a homeomorphism of topological spaces.

PROPOSITION 1. *Suppose that $f : X \to Y$ is an isomorphism in a category* **C** *and g and h are inverses to f. Then $h = g$.*

Proof. Consider the threefold composite $h \circ f \circ g$. Since $h \circ f = 1_X$, this is equal to g, and since $f \circ g = 1_Y$, it is also equal to h.■

The examples of categories illustrate a basic principle in modern mathematics: Whenever one defines a type of mathematical system, there is usually a corresponding type of morphism for such systems (and in some cases there are several reasonable choices for morphisms). Since a category is an example of a mathematical system, it is natural to ask whether there is a corresponding notion of morphisms in this case too. In fact, there are two concepts of morphism that turn out to be important and useful. We shall start with the simpler one.

Definition. Let **C** and **D** be categories. A *covariant functor* assigns (i) to each object X of **C** an object $T(X)$ of **D**, (ii) to each morphism $f : X \to Y$ in **C** a morphism $T(f) : T(X) \to T(Y)$ in **D** such that the following hold:

(1) For each object X in **C** we have $T(1_X) = 1_{T(X)}$.

(2) For each pair of morphisms f and g in **C** such that $g \circ f$ is defined, we have $T(g \circ f) = T(g) \circ T(f)$.

HISTORICAL TRIVIA. Eilenberg and MacLane "borrowed" the word **category** from the philosophical writings of the 18^{th} century German philosopher I. Kant (1724–1804) and the word **functor** from the philosophical writings of the 20^{th} century German-American philosopher R. Carnap (1891–1970), who was strongly influenced by Kant's writings on the philosophy of science.

Examples of covariant functors

Numerous constructions from undergraduate and elementary graduate courses can be interpreted as functors; in many cases this does not shed much additional light on the objects constructed, but in other cases the concept does turn out to be extremely useful.

1. Given a category **C**, there is the **identity functor** from **C** to itself, which takes all objects and morphisms to themselves.

2. Given a category **C** and a (possibly different) nonempty category **D**, for each object A of **D** there is a **constant functor** k_A from **C** to **D** which sends every object of **C** to A and every morphism to the identity morphism 1_A.

3. In categories where the objects are given by sets with some extra structure and the morphisms are ordinary functions with additional properties, there are **forgetful functors** which take objects to the underlying sets and morphisms to the underlying mappings of sets. For example, there are forgetful functors from $\mathbf{VEC}_{\mathbb{F}}$, **GRP**, **POSETS**, and **TOP** to **SETS**. Likewise, there is an obvious forgetful functor from **MET–UNIF** to **TOP** which takes a metric space to its underlying topological space and simply views a uniformly continuous mapping as a continuous mapping.

4. There is a **power set functor** P_* on the category **SETS** defined as follows: The set $P_*(X)$ is just the set of all subsets (also known as the power set), and if $f : X \to Y$ is a set-theoretic function, then $P_*(f) : P_*(X) \to P_*(Y)$ takes an element $A \in P(X)$ — which by definition is just a subset of X — to its image $f[A] \subset Y$. A short argument is needed to verify this construction actually defines a covariant functor, but it is elementary. First, we need to check that for every set X we have $P_*(1_X) = 1_{P(X)}$; this follows because $1_X[A] = A$

for all $A \subset X$. Next, we must check that $P_*(g \circ f) = P_*(g) \circ P_*(f)$ for all composable f and g. But this is a consequence of the elementary identity $g[\,f[A]\,] \;=\; g \circ f[A]$.

5. If we are given two partially ordered sets and a weakly order-preserving mapping f from the first to the second such that $u \leq v$ implies $f(u) \leq f(v)$, then f may be interpreted as a covariant functor on the associated categories.

6. If we are given two groups and a homomorphism f from the first to the second, then f may be interpreted as a covariant functor on the associated categories.

7. Finally, we shall give a more substantial example that played a central role in Mathematics 205B. Define a new category $\mathbf{TOP}_*$ of *pointed topological spaces* whose objects are pairs (X, y), where X is a topological space and $y \in X$; the point y is said to be the *basepoint* of the pointed space. A morphism $f : (X, y) \to (Z, w)$ in this category will be a continuous mapping from X to Z (usually also denoted by f) which maps y to w (*i.e.*, a **basepoint preserving** continuous mapping). The fundamental group $\pi_1(X, y)$ then has a natural interpretation as a covariant functor from $\mathbf{TOP}_*$ to **GRP**, for if f is a morphism of pointed spaces, then then one has an associated homomorphism f_* from $\pi_1(X, y)$ to $\pi_1(Z, w)$, and these have the required properties that $1_{(X,y)*}$ is the identity and $(g \circ f)_* = g_* \circ f_*$.

Contravariant functors and examples

Experience shows there are many instances in which it is useful to work with functors that **reverse** the order of function composition; such objects are called *contravariant functors*.

Definition. Let **C** and **D** be categories. A *contravariant functor* assigns (i) to each object X of **C** an object $U(X)$ of **D**, (ii) to each morphism $f : X \to Y$ in **C** a morphism $U(f) : U(Y) \to U(X)$ in **D** (note that the domain and codomain are the *opposites* of those in the covariant case!) such that the following hold:

(1) For each object X in **C** we have $U(1_X) = 1_{U(X)}$.

(2) For each pair of morphisms f and g in **C** such that $g \circ f$ is defined, we have $U(g \circ f) = U(f) \circ U(g)$.

The simplest examples of contravariant functors are given by the *pseudo-identity functors*, which map the objects and morphisms in the category **C** to their obvious counterparts in the opposite category $\mathbf{C}^{\mathbf{OP}}$. In fact, there is an obvious correspondence between contravariant functors from **C** to **D** and covariant functors from **C** to $\mathbf{D}^{\mathbf{OP}}$, or equivalently covariant functors from $\mathbf{C}^{\mathbf{OP}}$ to **D**. The best way to motivate the definition is to give some less trivial examples.

1. Let **C** be the category of all vector spaces over some field, and consider the construction which associates to each vector space its *dual space* V^* of linear mappings from V to the scalar field F. There is a simple way of defining a corresponding construction for morphisms; if $L : V \to W$ is a linear transformation, consider the linear transformation $L^* : W^* \to V^*$ whose value on a linear functional $h : W \to F$ is given by $L^*(h) = h \circ L$, which is a linear functional on V. Standard results in linear algebra show that L^* is a linear transformation, that L^* is an identity map if L is an identity map, and if L is a composite $L_1 \circ L_2$, then we have $L^* = L_2^* \circ L_1^*$.

2. There is a contravariant power set functor P^* on the category **SETS** defined as follows: The set $P^*(X)$ is just the set of all subsets, but now if $f : X \to Y$ is a set-theoretic function, then $P^*(f) : P^*(Y) \to P^*(X)$ takes an element $B \in P(Y)$ — which by definition is just a subset of Y — to its **inverse image** $f^{-1}[B] \subset X$. As in the case of P_*, a short elementary argument is needed to verify this construction actually defines a contravariant functor. The construction preserves identity maps because $1_X^{-1}[B] = B$ for all $B \subset X$, and the identity $P^*(g \circ f) = P^*(f) \circ P^*(g)$ is essentially a restatement of the elementary identity $f^{-1}[g^{-1}[B]] = (g \circ f)^{-1}[B]$.

3. Example 2 actually yields a little more. Define a *Boolean algebra* to be a set with two binary operations $\cap$ and $\cup$, a unary operation $x \to x'$, and special elements 0 and 1 such that the system satisfies the usual properties for unions, intersections, and complementation for the algebra $P(X)$ of subsets of a set X, where 0 corresponds to the empty set and 1 corresponds to X. One then has an associated category **BOOL–ALG** whose objects are Boolean algebras and whose morphisms preserve unions, intersection, complementation, and the special elements. Obviously each power set $P(X)$ is a Boolean algebra, and in fact P^* defines a contravariant functor from **SETS** to **BOOL–ALG**. — In contrast, the covariant functor P_* does NOT define such a functor because $P_*(f)$ does not preserves intersections even though it does preserve unions (for example, we can have $f[A] \cap f[B] \neq \emptyset$ when $A \cap B = \emptyset$).

4. The desirability of having both contravariant and covariant functors is illustrated by the following examples. Given a category $\mathbf{C}$, modulo foundational questions we can informally view the set $\mathsf{Morph}_{\mathbf{C}}(X, Y)$ of morphisms from X to Y as a function of two variables on $\mathbf{C}$. What happens if we hold one of these variables constant to get a single variable construction? — Suppose first that we hold X constant and set $A_X(Y) = \mathsf{Morph}_{\mathbf{C}}(X, Y)$. Then we can make A_X into a covariant functor aas follows: Given a morphism $g : Y \to Z$, let $A_X(g)$ take $f \in A_X(Y) = \mathsf{Morph}_{\mathbf{C}}(X, Y)$ to the composite $g \circ f$. The axioms for a category then imply that $A_X(1_Y)$ is the identity and that $A_X(h \circ g) = A_X(h) \circ A_X(g)$ if g and h are composable. — Now suppose that we hold Y constant and set $B_Y(X) = \mathsf{Morph}_{\mathbf{C}}(X, Y)$. Then we can make B_Y into a **contravariant** functor as follows: Given a morphism $k : W \to X$, let $B_Y(g)$ take $f \in B_Y(X) = \mathsf{Morph}_{\mathbf{C}}(X, Y)$ to the composite $f \circ k$. The axioms for a category then imply that $B_Y(1_X)$ is the identity and that $B_Y(k \circ h) = B_Y(h) \circ B_Y(k)$ if h and k are composable.

5. Given a topological space X, let $\mathbf{CComp}(X)$ and $\mathbf{AComp}(X)$ denote the sets of components and arc components of X respectively. These constructions extend to covariant functors from the category of topological spaces and continuous maps to the category of sets because a continuous map $f : X \to Y$ sends a component or arc component of X into a single component or arc component of Y. Similarly, there is a notion of *quasicomponents* for topological space of a space is contained in a quasicomponent. component or arc component of Y. Similarly, there is a notion of *quasicomponents* for topological space such that each connected component lies in a quasicomponent (but the converse might not hold). This notion is described in Exercise 10 on page 163 of Munkres (see also Exercise 4 on page 236). One can show that if $f : X \to Y$ is continuous, the image of a quasicomponent of X is contained in a quasicomponent of Y (prove this!), it follows that one has a third functor $\mathbf{QComp}(X)$.

6. In Example 4, suppose that $\mathbf{C}$ is the category of topological spaces and continuous mappings, and let Y be the real numbers or complex with the usual topology. In this case the contravariant functor B_Y has the algebraic structure of a commutative ring with unit given by pointwise multiplication of continuous real valued functions, and if $f : W \to X$ is continuous then $B_Y(f)$ is in fact a homomorphism of commutative rings with unit.

Therefore, if we define a category of continuous rings with unit (whose morphisms are unit preserving homomorphisms), it follows that B_Y defines a functor from topological spaces and continuous mappings to commutative rings with unit. — In contrast, there is no natural, comparable structure for the covariant functor A_X if X is the real numbers.

Properties of functors

One of the most important properties of functors is that they send isomorphic objects in one one category to isomorphic objects in the other.

PROPOSITION 2. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories, let* $T : \mathbf{C} \to \mathbf{D}$ *be a (covariant or contravariant) functor, and let* $f : X \to Y$ *be an isomorphism in* $\mathbf{C}$. *Then* $T(f)$ *is an isomorphism in* $\mathbf{D}$. *Furthermore, if* g *is the inverse to* f, *then* $T(g)$ *is the inverse to* $T(f)$.

Proof. CASE 1. Suppose the functors are covariant. Then we have

$$1_{T(X)} = T(1_X) = T(g \circ f) = T(g) \circ T(f)$$

$$1_{T(Y)} = T(1_Y) = T(f \circ g) = T(f) \circ T(g)$$

and hence $T(g)$ is inverse to $T(f)$.

CASE 2. Suppose that the functors are contravariant. Then we have

$$1_{T(X)} = T(1_X) = T(g \circ f) = T(f) \circ T(g)$$

$$1_{T(Y)} = T(1_Y) = T(f \circ g) = T(g) \circ T(f)$$

and hence $T(g)$ is inverse to $T(f)$.■

The next result states that a composite of two functors is also a functor.

PROPOSITION 3. *Suppose that* $\mathbf{C}$, $\mathbf{D}$ *and* $\mathbf{E}$ *are categories and that* $F : \mathbf{C} \to \mathbf{D}$ *and* $G : \mathbf{D} \to \mathbf{E}$ *are functors (in each case, the functor may be covariant or contravariant). Then the composite* $G \circ F$ *also defines a functor; this functor is* covariant *if* F *and* G *are both covariant or contravariant, and it is* contravariant *if one of* F, G *is covariant and the other is contravariant.*■

This result has a curious implication:

COROLLARY 4. *There is a "category of small categories"* **SMCAT** *whose objects are small categories and whose morphisms are covariant functors from one small category to another.*■

SEMANTIC TRIVIA. (For readers who are familiar with contravariant and covariant tensors.) In the applications of linear algebra to differential geometry and topology, one often sees objects called *contravariant tensors* and *covariant tensors*, and for finite-dimensional vector spaces these are given by finitely iterated tensor products $V \otimes \cdots \otimes V$ of V with itself in the contravariant case and similar objects involving V^* in the covariant case; for our purposes it will suffice to say that if U and W are vector spaces with bases $\{ \mathbf{u}_i \}$ and $\{ \mathbf{w}_j \}$ respectively, then their tensor product $U \otimes W$ is a vector space having a basis of the form $\{ \mathbf{u}_i \otimes \mathbf{w}_j \}$ where i and j are allowed to vary independently (hence the dimension of $U \otimes W$ is $[\dim U] \cdot [\dim W]$). Since the identity functor on the category of vector spaces is covariant and the dual space functor is covariant, at first it might seem that something is the opposite of what it should be. However, the classical tensor notation refers to the manner in which the **coordinates** transform; now coordinates for a vector space may be viewed linear functionals on that space, or equivalently as elements of the dual space, which is contravariant. Therefore individual coordinates on $V \otimes \cdots \otimes V$ correspond to elements of the *dual space* of the latter, and in fact the construction which associates the space $(V \otimes \cdots \otimes V)^*$ to V defines a *contravariant* functor on the category of finite-dimensional vector spaces over the given scalars; likewise, the construction which associates the space $(V^* \otimes \cdots \otimes V^*)^*$ to V defines a **covariant** functor on the category of finite-dimensional vector spaces over the given scalars.

Natural transformations

The final concept in category theory to be considered here is the notion of **natural transformation** from one functor to another. In fact, the motivation for category theory in the work of Eilenberg and MacLane was a need to discuss "natural mappings" in a mathematically precise manner. There are actually two definitions, depending whether both functors under consideration are covariant or contravariant.

Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and let F and G be covariant functors from $\mathbf{C}$ to $\mathbf{D}$. A *natural transformation* θ from F to G associates to each object X in $\mathbf{C}$ a morphism $\theta_X : F(X) \to G(X)$ such that for each morphism $f : X \to Y$ we have $\theta_Y \circ F(f) = G(f) \circ \theta_X$.

The morphism identity is often expressed graphically by saying the the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow \theta_X & & \downarrow \theta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

is a **commutative diagram**. The idea is that all paths of arrows from one object-vertex to another yield the same function.

The definition of a natural transformation of contravariant functors is similar.

Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and let T and U be contravariant functors from $\mathbf{C}$ to $\mathbf{D}$. A *natural transformation* θ from F to G associates to each object X in $\mathbf{C}$ a morphism $\theta_X : T(X) \to U(X)$ such that for each morphism $f : X \to Y$ we have $\theta_X \circ T(f) = U(f) \circ \theta_Y$.

Here is the corresponding commutative diagram:

$$\begin{array}{ccc} T(Y) & \xrightarrow{T(f)} & T(X) \\ \downarrow \theta_Y & & \downarrow \theta_X \\ U(Y) & \xrightarrow{U(f)} & U(X) \end{array}$$

Once again we need to give some decent examples

1. Given any functor $T : \mathbf{C} \to \mathbf{D}$, there is an obvious identity transformation j^T from T to itself; specifically, j^T_X is the identity map on $T(X)$.

2. Let $\mathbf{C}$ be one of the categories as above for which reasonable products and diagonal maps can be defined. Then there is a natural diagonal transformation Δ from the identity to the diagonal functor such that for each object X the mapping $\Delta_X : X \to X \times X$ is the diagonal map.

3. On the category of vector spaces over some field F, one can iterate the dual space functor to obtain a covariant *double dual space functor* $(V^*)^*$. There is a natural transformation

$e_V : V \to (V^*)^*$ defined as follows: For each $\mathbf{v} \in V$, let $e_V(\mathbf{v}) : V^* \to F$ be the linear function given by evaluation at v; in other words, the value of $e_V(\mathbf{v})$ on a linear functional f is given by $f(\mathbf{v})$. If V is finite-dimensional, this map is an isomorphism (a *natural isomorphism*).

Note that if V is finite-dimensional then V and its dual space V^* are isomorphic, but the isomorphism depends upon some additional data such as the choice of a basis, an inner product, or more generally a nondegenerate bilinear form. In contrast, *the natural isomorphism* e_V *does not depend upon any such choices*.

4. In the category of sets or topological spaces and continuous mappings, let A be an arbitrary object and define functors L_A and R_A such that $L_A(X) = A \times X$ and $R_A(X) = X \times A$. One can make these into covariant functors by sending the morphism $f : X \to Y$ to $L_A(X) = 1_A \times f$ and $R_A(f) = f \times 1_A$. There is an obvious natural transformation $t : L + A \to R_A$ such that $t_A(X) : A \times X \to X \times A$ sends (a, x) to (x, a) for all $a \in A$ and $x \in X$, and it is an elementary exercise to verify that this is a natural transformation such that each map $t_A(X)$ is an *isomorphism*; in other words, t_A is a natural isomorphism from the functor L_A to the functor R_A.

5. For the morphism examples A_X and B_Y discussed previously, if $h : W \to X$ is a morphism in the category, then it defines a natural transformation $h^* : A_X \to A_W$ which sends $f \in A_X(Y)$ to $f \circ h \in A_W(Y)$; the naturality condition follows from associativity of composition. Similarly, if $g : Y \to Z$ is a morphism then there is a natural transformation $g_* : B_Y \to B_Z$ sending f to $g \circ f$; once again, the key naturality condition follows from the associativity of composition. Furthermore, h^* is a natural isomorphism if h is an isomorphism and g_* is a natural isomorphism if g is an isomorphism,

6. For the arc component, connected component and quasicomponent functors described above, there are natural transformations θ : **AComp** $\to$ **CComp** reflecting the fact that every arc component of a topological space X is contained in a connected component and ψ : **CComp** $\to$ **QComp** reflecting the fact that every connected component of a space is contained in a quasicomponent.

A basic exercise in category theory is to prove the following:

PROPOSITION 5. *There are* $1-1$ *correspondences between natural transformations from* A_X *to* A_W *and morphisms from* W *to* X *and between natural transformations from* B_Y *to* B_Z *and morphisms from* Y *to* Z.

Sketch of proof. The main point is to retrieve the function from the natural transformation. Given $\theta : A_X \to A_W$, one does this by considering the image of 1_X, and given $\varphi : B_Y \to B_Z$, one does this by considering the image of 1_Y.■

Finally, we have the following result on natural isomorphisms (*i.e.*, natural transformations θ such that each map θ_X is an isomorphism):

PROPOSITION 6. *Let* $\theta : F \to G$ *be a natural transformation such that for each object* X *the map* θ_X *is an isomorphism. The there is a natural transformation* $\varphi_X : G \to F$ *such that for each* X *the map* φ_X *is inverse to* θ_X.

Proof. The main thing to check is that the relevant diagrams are commutative; we shall only do the case where F and G are covariant, leaving the other case to the reader. Since $\theta_X \circ \varphi_X$ is the identity on $G(X)$ and $\varphi_X \circ \theta_X$ is the identity on $F(X)$, we have

$$\theta_Y \circ \varphi_Y \circ G(f) \quad = \quad G(f) \quad = \quad g(f) \circ \theta_X \circ \varphi_X \quad = \quad \theta_Y \circ F(f) \circ \varphi_X$$

and if we compose with the inverse θ_X on the left of these expressions we obtain

$$\varphi_Y \circ G(f) \quad = \quad F(f) \circ \varphi_X$$

which is the naturality condition.■

We say that two functors are *naturally isomorphic* if there is a natural isomorphism from one to the other.

Equivalences of categories

One can obviously define an isomorphism of categories to be a covariant functor $T : \mathbf{C} \to \mathbf{D}$ for which there is an inverse covariant functor $U : \mathbf{D} \to \mathbf{C}$ such that the composites $T \circ U$ and $U \circ T$ are the identities on $\mathbf{C}$ and $\mathbf{D}$ respectively. However, for many purposes one needs a less rigid notion of category equivalence.

Definition. A covariant functor $T : \mathbf{C} \to \mathbf{D}$ is a *category equivalence* (or equivalence of categories) if there is a covariant functor $U : \mathbf{D} \to \mathbf{C}$ such that the composites $T \circ U$ and $U \circ T$ are *naturally isomorphic to* the identities on $\mathbf{C}$ and $\mathbf{D}$ respectively.

In particular, if T and U define an equivalence of categories, then every object in $\mathbf{D}$ is isomorphic to an object of the form $T(X)$, and conversely every object in $\mathbf{C}$ is isomorphic to an object of the form $U(A)$.

I.2 : Barycentric coordinates and polyhedra

(Hatcher, § 2.1)

Drawings to illustrate many of the concepts in this and other sections of the notes can be found in the following document(s):

`http://math.ucr.edu/~res/math246A/algtop1figures01w09.pdf`

A more leisurely and detailed discussion of barycentric coordinates, and more generally the use of linear algebra to study geometric problems, is contained in Section I.4 of the following online document, in which $*$ is one of the options in the preceding paragraph:

`http://math.ucr.edu/~res/math133/geometrynotes1.pdf`

The file `math133exercises1.pdf` in the same directory has further material on these topics, and pages 13–30 of

`http://math.ucr.edu/~res/progeom/pgnotes02.pdf`

go further into the geometric uses of barycentric coordinates. Another standard reference is Chapter I of the following book:

> **J. F. P. Hudson.** *Piecewise Linear Topology. W. A. Benjamin, New York*, 1969. (Online: `http://www.maths.ed.ac.uk/~aar/surgery/hudson.pdf`)

An extremely detailed study of the topics in this section appears in the following online book:

`http://www.cis.penn.edu/~jean/gbooks/convexpoly.html`

Finally, Eilenberg and Steenrod also covers the portions this material needed for algebraic and geometric topology in greater detail.

Affine independence and barycentric coordinates

The crucial algebraic information is contained in the following result.

PROPOSITION 1. *Suppose that the ordered set of vectors* $\mathbf{v}_0, \cdots, \mathbf{v}_n$ *lie in some vector space* V. *Then the vectors* $\mathbf{v}_1 - \mathbf{v_0}, \cdots, \mathbf{v}_n - \mathbf{v}_n$ *are linearly independent if and only if every vector* $\mathbf{x} \in V$ *has at most one expansion of the form* $t_0\mathbf{v}_0 + \cdots + t_n\mathbf{v}_n$ *such that* $t_0 + \cdots + t_n = 1$.

A finite ordered set of vectors satisfying either (hence both) conditions is said to be *affinely independent*. Note that since the second condition does not depend upon the choice of ordering, a set of vectors is affinely independent if and only if for some arbitrary j the vectors $\mathbf{v}_i - \mathbf{v}_j$ (where $i \neq j$) is linearly independent. A linear combination in which the coefficients add up to 1 is called an *affine combination*.

Sketch of proof. To show the first statement implies the second, use the fact that $\mathbf{x} - \mathbf{v}_0$ has at most one expansion as a linear combination of $\mathbf{v}_1 - \mathbf{v_0}, \cdots, \mathbf{v}_n - \mathbf{v}_n$. To prove the reverse implication, show that if $\mathbf{x} - \mathbf{v}_0$ has more than one expansion as a linear combination of $\mathbf{v}_1 - \mathbf{v_0}, \cdots, \mathbf{v}_n - \mathbf{v}_n$, then $\mathbf{x}$ has more than one expansion as an affine combination of $\mathbf{v}_0, \cdots, \mathbf{v}_n$.■

COROLLARY 2. *If* $S = \{\mathbf{v}_0, \cdots, \mathbf{v}_n\}$ *is affinely independent, then every nonempty subset of* S *is affinely independent.*

This follows immediately from the uniqueness of expansions of vectors as affine combinations of vectors in S.■

The coefficients t_i are called **barycentric coordinates**. If we put physical weights of t_i units at the respective vertices $\mathbf{v}_i$, then the center of gravity for the system will be at the point $t_0\mathbf{v}_0 + \cdots + t_n\mathbf{v}_n$. If, say, $n = 2$, then this center of gravity will be inside the triangle with the given three vertices if and only if each t_i is positive, and it will be on the triangle defined by these vertices if and only if each t_i is nonnegative and at least one is equal to zero. A discussion of this physical interpretation in the 2-dimensional case appears in the following online document:

`http://math.ucr.edu/~ res/math133/centroids.pdf`

We should note that the discussion in this online reference can be extended to arbitrary (finite) dimensions.

More generally, if $\mathbf{v}_0, \cdots, \mathbf{v}_n$ are affinely independent then the n**-simplex** with vertices $\mathbf{v}_0, \cdots, \mathbf{v}_n$ is the set of all points expressible as affine combinations such that each coefficient is nonnegative (*i.e.*, **convex combinations**).

Frequently the n-simplex described above will be denoted by $\mathbf{v}_0 \cdots \mathbf{v}_n$. Note that if $n = 0$, then a 0-simplex consists of a single point, while a 1-simplex is a closed line segment, a 2-simplex is given by a triangle and the points that lie "inside" the triangle (also called a *solid triangular region*), and a 3-simplex is given by a pyramid with a triangular base (*i.e.*, a *tetrahedron*) together with the points inside this pyramid (also called a *solid tetrahedral region*).

The following definition will also play an important role in our discussions.

Definition. If $\mathbf{v}_0, \cdots, \mathbf{v}_n$ form the vertices of a simplex $\mathbf{v}_0 \cdots \mathbf{v}_n$, then the **faces** of this simples are the simplices whose vertices are given by proper subsets of $\{\mathbf{v}_0, \cdots, \mathbf{v}_n\}$; note that such proper subsets are affinely independent by Corollary 2. If a proper subset $T \subset S$ has $k + 1$

elements, then we shall say that the simplex $\Delta(T)$ whose vertices are given by T is a **k-face** of the original n-simplex, which in this notation is equal to $\Delta(S)$.

Definition. The *standard n-simplex* Δ_n is the set of all points $(t_0, \cdots, t_n) \in \mathbb{R}^{n+1}$ such that $t_j \geq 0$ for all j and $\sum_j t_j = 1$. Note that the set of unit vectors $\{\mathbf{e}_0, \cdots, \mathbf{e}_n\}$ is affinely independent because the set $\{\mathbf{e}_1 - \mathbf{e}_0, \cdots, \mathbf{e}_{n+1} - \mathbf{e}_0\}$ is linearly independent.

Sets with simplicial decompositions

In calculus textbooks, the derivation of Green's Theorem is often completed only for special sorts of closed regions such as the simplex whose vertices are $(0,0)$, $(1,0)$ and $(1,1)$. One then finds discussions indicating how the general case can be retrieved from special cases by splitting a general region into pieces that are nicely homeomorphic to closed regions of the special type; in particular, there is one such discussion on page 523 of the text by Marsden and Tromba, and it is taken further in the online document with figures for these notes (see Figure I.2.8 in the document `algtop1figures01w09.pdf`).

Here are the formal descriptions.

Definition. A subset $P \subset \mathbb{R}^m$ is a *polyhedron* if

(i) P is a finite union of simplices $A_1, \cdots, A_q$,

(ii) For each pair of indices $i \neq j$, the intersection $A_i \cap A_j$ is a common face.

The simplices $A_1, \cdots, A_q$ are said to form a *simplicial decomposition* of P, and if $\mathbf{K}$ is the collection of simplices given by the A_j and all their faces, then the ordered pair $(P, \mathbf{K})$ is called a (finite) *simplicial complex*.

If X is an arbitrary topological space, then a (finite) *triangulation* of X consists of a simplicial complex $(P, \mathbf{K})$ and a homeomorphism $t : P \to X$.

With these definitions, we can say that Green's Theorem holds for "decent" closed plane regions because Such regions have nice triangulations.

SIMPLE EXAMPLE. Consider the solid rectangle in the plane given by $[a, b] \times [c, d]$, where $a < b$ and $c < d$. Everyday geometrical experience shows this can be split into two 2-simplices along a diagonal, and in fact it is the union of two 2-simplices, one with vertices (a, c), (a, d) and (b, d), and the other with vertices (a, c), (b, c) and (b, d). A point (x, y) which lies in the solid rectangle will be in the first simplex if and only if

$$(y - c)(b - a) \quad \leq \quad (d - c)(x - a)$$

and this point will be in the second simplex if and only if

$$(y - c)(b - a) \quad \geq \quad (d - c)(x - a)$$

Generalizations of this example will play an important role in the standard approach to algebraic topology.

If $(P, \mathbf{K})$ is a simplicial complex, then a subset $\mathbf{L} \subset \mathbf{K}$ is said to be a *subcomplex* if $\sigma \in \mathbf{L}$ implies that every face of σ also lies in $\mathbf{L}$. The union of the simplices in $\mathbf{L}$ is a closed subspace of P which is denoted by $|\mathbf{L}|$. With this notation we have $P = |\mathbf{K}|$.

LINEAR GRAPHS. The final chapter of Munkres studies 1-dimensional complexes (called *linear graphs* on p. 394) in considerable detail, and the `commentaries` file in the 205B directory contains some comments (see the discussion for Section 64 which begins at the bottom of page 29 and continues into page 30). One way of viewing this section and the next is to think of them as laying the foundations for effective study of similar objects in higher dimensions.

The study of 1-dimensional complexes is the subject called *graph theory*; it is significant for both its theory and applications, but all of this is well beyond the scope of this course. Here are some written and electronic references:

J. A. Bondy and U. S. R. Murty. *Graph Theory: An Advanced Course*. Springer-Verlag, New York-*etc.*, 2008. ISBN: 1-846-28969-6.

G. Chartrand. *Introductory Graph Theory* [UNABRIDGED]. Dover Publications, New York, 1984. ISBN: 0-486-24775-9.

`http://en.wikipedia.org/wiki/Graph_theory`

`http://www.utm.edu/departments/math/graph/`

`http://www.math.fau.edu/locke/GRAPHTHE.HTM`

`http://www.math.uni-hamburg.de/home/diestel/books/`[*continue*]
`graph.theory/GraphTheoryIII.counted.pdf`

SIMPLICIAL COMPLEXES AND Δ-COMPLEXES. Our definition of simplicial complex is more restrictive than Hatcher's definition; this is explained on page 107 of Hatcher (see the third paragraph following Example 2.5). Each concept has its advantages and disadvantages. However, terms like Δ-complex or Δ-set are often also used for other constructions, and one should not assume that the meanings in other publications are "obviously" equivalent to the meaning in Hatcher.

Decompositions of prisms

The rectangular example has the following important generalization:

PROPOSITION 3. *Suppose that* $A \subset \mathbb{R}^m$ *is a simplex with vertices* $\mathbf{v}_0, \cdots, \mathbf{v}_n$. *Then* $A \times [0,1] \subset \mathbb{R}^{m+1}$ *has a simplicial decomposition with exactly* $n+1$ *simplices of dimension* $n+1$.

Proof. For each i between 0 and n let $\mathbf{x}_i = (\mathbf{v}_i, 0)$ and $\mathbf{y}_i = (\mathbf{v}_i, 1)$. We claim that the vectors

$$\mathbf{x}_0, \cdots, \mathbf{x}_i, \mathbf{y}_i \cdots, \mathbf{y}_n$$

are affinely independent and the corresponding simplices

$$\mathbf{x}_0 \cdots \mathbf{x}_i \mathbf{y}_i \cdots \mathbf{y}_n$$

(where $0 \leq i \leq n$) form a simplicial decomposition of $A \times [0,1]$.

An illustration for the case $n = 2$ is given in Figure I.2.11 of `algtop1figures01w09.pdf`).

To prove affine independence, take a fixed value of i and suppose we have

$$\sum_{j<i} t_j \mathbf{x}_j + a \mathbf{x}_i + b \mathbf{y}_i + \sum_{j>i} t_j \mathbf{y}_j =$$

$$\sum_{j<i} t'_j\,\mathbf{x}_j \;\;+\;\; a'\,\mathbf{x}_i \;+\; b'\,\mathbf{y}_i \;\;+\;\; \sum_{j>i} t'_j\,\mathbf{y}_j$$

where the coefficients in each expression add up to 1; the summation will be taken to be zero if the limits reduce to $j < 0$ or $j > n$. If we view $\mathbb{R}^{m+1}$ as $\mathbb{R}^m \times \mathbb{R}$ and project down to $\mathbb{R}^m$ we obtain the equation

$$\sum_{j<i} t_j\,\mathbf{v}_j \;\;+\;\; (a+b)\,\mathbf{x}_i \;+\; \sum_{j>i} t_j\,\mathbf{v}_j \;\;=\;\; \sum_{j<i} t'_j\,\mathbf{v}_j \;\;+\;\; (a'+b')\,\mathbf{v}_i \;+\; \sum_{j>i} t'_j\,\mathbf{v}_j$$

and by the affine independence of the vectors $\mathbf{v}_k$ it follows that $t_j = t'_j$ if $j \neq i$ and also that $a + b = a' + b'$. On the other hand, if we project down to the second coordinate (the copy of $\mathbb{R}$), then we obtain

$$b \;+\; \sum_{j>i} t_j \;\;=\;\; b' \;+\; \sum_{j>i} t'_j$$

and since $t_j = t'_j$ for all j it follows that $b = b'$. Finally, since the sum of all the coefficients is equal to 1, the preceding observations imply that $1 - a = 1 - a'$, and therefore we also have $a = a'$. Therefore the vectors

$$\mathbf{x}_0,\; \cdots\; ,\mathbf{x}_i, \mathbf{y}_i\; \cdots\; ,\mathbf{y}_n$$

are affinely independent.

We shall next check that every point $(\mathbf{z}, u) \in A \times [0, 1]$ lies in one of the simplices

$$\mathbf{x}_0\; \cdots\; \mathbf{x}_i\mathbf{y}_i\; \cdots\; \mathbf{y}_n$$

listed above. Write $\mathbf{z} = \sum_j t_j\,\mathbf{v}_j$ where $t_j \geq 0$ for all j and $\sum t_j = 1$. It follows that $u \leq 1 = \sum_{j\geq 0} t_j$; let $i \leq n$ be the largest nonnegative integer such that $u \leq \sum_{j\geq i} t_j$. We claim that $(\mathbf{z}, u)$ lies in the simplex $\mathbf{x}_0\; \cdots\; \mathbf{x}_i\mathbf{y}_i\; \cdots\; \mathbf{y}_n$. Let $b = \sum_{j\geq i} (t_j - u)$, and let $a = u - \left(\sum_{j>i} t_j\right) = t_i - b$. Then we have $a,\; b \geq 0$, and

$$(\mathbf{z}, u) \;\;=\;\; \sum_{j<i} t_j\,\mathbf{x}_j \;\;+\;\; a\,\mathbf{x}_i \;+\; b\,\mathbf{y}_i \;\;+\;\; \sum_{j>i} t_j\,\mathbf{y}_j$$

where all the coefficients are nonnegative and add up to 1.

To conclude the proof, we need to show that the intersection of two simplices as above is a common face. Suppose that $k < i$ and

$$(\mathbf{z}, u) \;\;\in\;\; \left(\mathbf{x}_0\; \cdots\; \mathbf{x}_i\mathbf{y}_i\; \cdots\; \mathbf{y}_n\right) \;\cap\; \left(\mathbf{x}_0\; \cdots\; \mathbf{x}_k\mathbf{y}_k\; \cdots\; \mathbf{y}_n\right)\,.$$

Then we must have

$$\sum_{j\leq i} p_j\,\mathbf{x}_j \;\;+\;\; \sum_{j\geq i} q_j\,\mathbf{y}_j \;\;=\;\; \sum_{j\leq k} p'_j\,\mathbf{x}_j \;\;+\;\; \sum_{j\geq k} q'_j\,\mathbf{y}_j$$

where all the coefficients are nonnegative and the coefficients on each side of the equation add up to 1. If we project down to $\mathbb{R}^m$ we obtain $p_j + q_j = p'_j + q'_j$ for all j (by convention, we take a coefficient to be zero if it does not lie in the corresponding summation as above). It follows immediately that

$p_j = p'_j$ if $j < k$, while $p_j = q'_j$ if $k < j < i$ and $q_j = q'_j$ if $j > i$. Furthermore, if we project down to the last coordinate we see that

$$u \;=\; \sum_{j \geq i} q_j \;=\; \sum_{j \geq k} q'_k \ .$$

Since $q_j = q'_j$ if $j > i$, it follows that

$$q_i \;=\; \sum_{k \leq j \leq i} q'_j$$

and since all the coefficients are nonnegative, it follows that $q_i \geq q'_i$. On the other hand, we also have $q'_i = p'_i + q'_i = p_i + q_i$, and hence we conclude that $q_i = q'_i$ and $p_i = 0$. Applying the first of these, we see that

$$0 \;=\; \sum_{k \leq j < i} q'_j$$

and hence the nonnegativity of the coefficients implies that $q'_j = 0$ for all j such that $k \leq j < i$. We also know that $p'_j = 0$ for $j > k$, and therefore it follows that $p'_j + q'_j = 0$ when $k < j < i$ The equations $p_j + q_j = p'_j + q'_j$ and the nonnegativity of all terms now imply that $p_j = q_j = 0$ when $k < j < i$.

The conclusions of the preceding paragraph imply that the point $(\mathbf{z}, u)$ actually lies on the simplex

$$\mathbf{x}_0 \;\cdots\; \mathbf{x}_k \mathbf{y}_i \;\cdots\; \mathbf{y}_n$$

and since the latter is a common face of $\mathbf{x}_0 \;\cdots\; \mathbf{x}_i \mathbf{y}_i \;\cdots\; \mathbf{y}_n$ and $\mathbf{x}_0 \;\cdots\; \mathbf{x}_k \mathbf{y}_k \;\cdots\; \mathbf{y}_n$ it follows that the $(n+1)$-simplices

$$\mathbf{x}_0 \;\cdots\; \mathbf{x}_i \mathbf{y}_i \;\cdots\; \mathbf{y}_n$$

(where $0 \leq i \leq n$) form a simplicial decomposition of $A \times [0,1]$.■

COROLLARY 4. *If $P \subset \mathbb{R}^m$ is a polyhedron, then $A \times [0,1] \subset \mathbb{R}^{m+1}$ is also a polyhedron.*

Before discussing the proof of this we note one important special case.

COROLLARY 5. *For each positive integer m, the hypercube $[0,1]^m \subset \mathbb{R}^m$ is a polyhedron.*

Proof of Corollary 5 from Corollary 4. If $m = 1$ this follows because the unit interval is a 1-simplex; by Corollary 4, if the result is true for $m = k$ then it is also true for $m = k+1$. Therefore the result is true for all m by induction.■

Proof of Corollary 4. Let $\mathbf{K}$ be a simplicial decomposition for P, and let $\mathbf{K}^*$ be obtained from $\mathbf{K}$ by including all the faces of simplices in $\mathbf{K}$. Choose a linear ordering of the vertices in $\mathbf{K}^*$ (note that there are only finitely many). For each vertex $\mathbf{v}$ of $\mathbf{K}^*$, as before let $\mathbf{x} = (\mathbf{v}, 0)$ and $\mathbf{y} = (\mathbf{v}, 1)$. Then $P \times [0,1]$ is the union of all simplices of the form

$$\mathbf{x}_0 \;\cdots\; \mathbf{x}_i \mathbf{y}_i \;\cdots\; \mathbf{y}_n$$

where $\mathbf{v}_i < \mathbf{v}_{i+1}$ with respect to the given linear ordering of the vertices in $\mathbf{K}^*$, **and furthermore** the vertices $\mathbf{v}_i$ are the vertices of a simplex in $\mathbf{K}^*$. The set $P \times [0,1]$ is the union of these simplices by Proposition 3 and the fact that P is the union of the simplices $\mathbf{v}_0 \;\cdots\; \mathbf{v}_n$. The fact that these simplices form a simplicial decomposition will follow from the construction and the next result.

LEMMA 6. *Suppose that we have two polyhedra P_1 and P_2 in $\mathbb{R}^q$, and there exist simplicial decompositions $\mathbf{K}_1$ and $\mathbf{K}_2$ such that the following hold:*

(i) Both $\mathbf{K}_1$ and $\mathbf{K}_2$ are closed under taking faces of simplices.

(ii) The set $\mathbf{L}_1$ of all simplices in $\mathbf{K}_1$ contained in $P_1 \cap P_2$ equals the set $\mathbf{L}_2$ of all simplices in $\mathbf{K}_2$, and this collection determines a simplicial decomposition of $P_1 \cap P_2$.

Then $\mathbf{K}_1 \cup \mathbf{K}_2$ determines a simplicial decomposition of $P_1 \cup P_2$.

The hypothesis clearly applies to the construction in Proposition 3, so Corollary 4 indeed follows once we prove Lemma 6.■

Proof of Lemma 6. It follows immediately that $P_1 \cup P_2$ is the union of the points of the simplices in $\mathbf{K}_1 \cup \mathbf{K}_2$. Suppose now that we are given an intersection of two simplices in the latter. This intersection will be a common face if both simplices lie in either $\mathbf{K}_1$ or $\mathbf{K}_2$, so the only remaining cases are those where one simplex α lies in $\mathbf{K}_1$ and the other simplex β lies in $\mathbf{K}_2$.

We know that $\alpha \cap \beta$ is convex. Furthermore, by the hypotheses we know that $\alpha \cap \beta$ must be a union of simplices that are faces of both α and β. Therefore it follows that every point in $\alpha \cap \beta$ is a convex combination of the vertices which lie in $\alpha \cap \beta$, and consequently $\alpha \cap \beta$ is the common face determined by all vertices which lie in $\alpha \cap \beta$.■

GENERALIZATIONS — CONVEX LINEAR CELLS. [Also known as CONVEX POLYTOPES] These are closed bounded subsets of some $\mathbb{R}^n$ defined by a finite number of linear equations or inequalities. Note that sets defined by finite systems of this type are automatically convex. Prisms, simplices and cubes are obvious examples, but of course there are also many others. For every such object, there is a finite set E of *extreme points* such that the cell is the set of all convex combinations of the extreme points; in other words, for each $\mathbf{x}$ in the cell and each extreme point $\mathbf{e}$ there are scalars $t_{\mathbf{e}}$ such that $t_{\mathbf{e}} \geq 0$, $\sum_{\mathbf{e}} t_{\mathbf{e}} = 1$, and $x = \sum_{\mathbf{e}} t_{\mathbf{e}} \, \mathbf{e}$. A basic theorem states that every convex linear cell has a simplicial decomposition for which E is the set of vertices. Proofs of this statement appear in [MunkresEDT] and the book by Hudson; we shall discuss some additional facts about such objects later in these notes.

Some easily stated but challenging problems on convex polytopes in $\mathbb{R}^3$ are contained in the file `wswGeometrytest.pdf`, and solutions to these exercises using vector geometry are given in the file `wswvectorproofs.pdf`.

DEFAULT HYPOTHESIS. Unless specifically indicated otherwise, we shall assume that the set of simplices in a simplicial decomposition $\mathbf{K}$ is closed under taking faces. In order to justify this, we need to know that if $\mathbf{K}^*$ is obtained from $\mathbf{K}$ by adding all the faces of simplices in the latter, then the intersection of two simplices in $\mathbf{K}^*$ is a (possibly empty) common face. — To see this, suppose that α and β are simplices in $\mathbf{K}^*$, where α and β are faces of the simplices α' and β' in $\mathbf{K}$. If $\mathbf{x} \in \alpha \cap \beta$, then $\mathbf{x}$ is a convex combination of vertices in $\alpha' \cap \beta'$, and in fact these vertices must lie in both α and β. Since $\alpha \cap \beta$ is convex, it follows that $\alpha \cap \beta$ must be the simplex whose vertices lie in α and in β.■

I.3 : Subdivisions

(Hatcher, § 2.1)

For many purposes it is convenient or necessary to replace a simplicial decomposition $\mathbf{K}$ of a polyhedron P by another decomposition $\mathbf{L}$ with smaller simplices. More precisely, we would like the smaller simplices in $\mathbf{L}$ to determine simplicial decompositions for each of the simplices in $\mathbf{K}$.

Simple examples

1. If P is a 1-simplex with vertices $\mathbf{x}$ and $\mathbf{y}$, and $\mathbf{K}$ is the standard decomposition given by P and the endpoints, then there is a subdivision $\mathbf{L}$ given by trisecting P; specifically, the vertices are given by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} = \frac{2}{3}\mathbf{x} + \frac{1}{3}\mathbf{y}$, and $\mathbf{w} = \frac{1}{3}\mathbf{x} + \frac{2}{3}\mathbf{y}$, and the 1-simplices are $\mathbf{xw}$, $\mathbf{wz}$ and $\mathbf{zy}$. This is illustrated as Figure I.3.1 in the file `algtop1figures01w09.pdf`.

2. Similarly, if $[a, b]$ is a closed interval in the real line and we are given a finite sequence $a = t_0 < \cdots < t_m = b$, then these points and the intervals $[t_{j-1}, t_j]$, where $1 \leq j \leq n$, form a subdivision of the standard decomposition of $[a, b]$.

3. If P is the 2-simplex with vertices $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, and $\mathbf{K}$ is the standard decomposition given by P and its faces, then there is an obvious decomposition $\mathbf{L}$ which splits P into two simplices $\mathbf{xyz}$ and $\mathbf{xyw}$, where $\mathbf{w} = \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}$ is the midpoint of the 1-simplex $\mathbf{yz}$. Similar eamples exist if we take $\mathbf{z} = a\mathbf{y} + (1-a)\mathbf{z}$, where a is an arbitrary number such that $0 < a < 1$ (see Figure I.3.2 in the file `algtop1figures01w09.pdf`).

Definition of subdivisions

Each of the preceding examples is consistent with the following general concept.

Definition. Let $(P, \mathbf{K})$ be a simplicial complex, and let $\mathbf{L}$ be a simplicial decomposition of P. Then $\mathbf{L}$ is called a (linear) *subdivision* of $\mathbf{K}$ if every simplex of $\mathbf{L}$ is contained in a simplex of $\mathbf{K}$.

The following observation is very elementary, but we shall need it in the discussion below.

PROPOSITION 0. *Suppose P is a polyhedron with simplicial decompositions $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{M}$ such that $\mathbf{L}$ is a subdivision of $\mathbf{K}$ and $\mathbf{M}$ is a subdivision of $\mathbf{L}$. Then $\mathbf{M}$ is also a subdivision of $\mathbf{K}$.*■

Figure I.3.3 in `algtop1figures01w09.pdf` depicts two subdivisions of a 2-simplex that are different from the one in Example 3 above. As indicated by Figure I.3.4 in the same document, in general if we have two simplicial decompositions of a polyhedron then neither is a subdivision of the other. However, it is possible to prove the following:

> *If $\mathbf{K}$ and $\mathbf{L}$ are simplicial decompositions of the same polyhedron P, then there is a third decomposition which is a subdivision of both $\mathbf{K}$ and $\mathbf{L}$.*

Proving this requires more machinery than we need for other purposes, and since we shall not need the existence of such subdivisions in this course we shall simply note that one can prove this result using methods from the second part of [MunkresEDT]:

SUBDIVISION AND SUBCOMPLEXES. These two concepts are related by the following elementary results.

PROPOSITION 1. *Suppose that $(P, \mathbf{K})$ is a simplicial complex and that $(P_1, \mathbf{K}_1)$ is a subcomplex of $(P, \mathbf{K})$. If $\mathbf{L}$ is a subdivision of $\mathbf{K}$ and $\mathbf{L}_1$ is the set of all simplices in $\mathbf{L}$ which are contained in P_1, then $(P_1, \mathbf{L}_1)$ is a subcomplex of $(P, \mathbf{L})$.*■

Recall our Default Hypothesis (at the end of Section I.2) that all simplicial decompositions should be closed under taking faces unless specifically stated otherwise.

COROLLARY 2. *Let P, $\mathbf{K}$ and $\mathbf{L}$ be as above, and let $A \subset P$ be a simplex of $\mathbf{K}$. Then $\mathbf{L}$ determines a simplicial decomposition of A.*■

We are particularly interested in describing a systematic construction for subdivisions that works for all simplicial complexes and allows one to form decompositions for which the diameters of all the simplices are very small. This will generalize a standard method for partitioning an interval $[a, b]$ into small intervals by first splitting the interval in half at the midpoint, then splitting the two subintervals in half similarly, and so on. If this is done n times, the length of each interval in the subdivision is equal to $(b-a)/2^n$, and if $\varepsilon > 0$ is arbitrary then for sufficiently large values of n the lengths of the subintervales will all be less than ε.

The generalization of this to higher dimensions is called the **barycentric subdivision**.

Definition. Given an n-simplex $A \subset \mathbb{R}^m$ with vertices $\mathbf{v}_0, \cdots, \mathbf{v}_n$, the *barycenter* $\mathbf{b}_A$ of A is given by

$$\mathbf{b}_A \quad = \quad \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{v}_i \ .$$

If $n \leq m \leq 3$, this corresponds to the physical center of mass for A, assuming the density in A is uniform.

Definition. If $P \subset \mathbb{R}^m$ is a polyhedron and $(P, \mathbf{K})$ is a simplicial complex, then the *barycentric subdivision* $\mathbf{B(K)}$ consists of all simplices having the form $\mathbf{b}_0 \ \cdots \ \mathbf{b}_k$, where (i) each $\mathbf{b}_j$ is the barycenter of a simplex $A_j \in \mathbf{K}$, (ii) for each $j > 0$ the simplex A_{j-1} is a face of A_j.

In order to justify this definition, we need to prove the following result:

PROPOSITION 3. *Let A be an n-simplex, suppose that we are given simplices $A_j \subset A$ such that A_{j-1} is a face of A_j for each j, and let $\mathbf{b}_j$ be the barycenter of A_j. Then the set of vertices $\{\mathbf{b}_0, \cdots, \mathbf{b}_q\}$ is affinely independent.*

Proof. We can extend the sequence of simplices $\{A_j\}$ to obtain a new sequence $C_0 \subset \cdots \subset C_n = A$ such that each C_k is obtained from the preceding one C_{k-1} by adding a single vertex, and it suffices to prove the result for the corresponding sequence of barycenters. Therefore we shall assume henceforth in this proof that each A_j is obtained from its predecessor by adding a single vertex and that A is the last simplex in the list.

It suffices to show that the vectors $\mathbf{b}_j - \mathbf{b}_0$ are linearly independent. For each j let $\mathbf{v}_{j_i}$ be the vertex in A_j that is not in its predecessor. Then for each $j > 0$ we have

$$\mathbf{b}_j - \mathbf{b}_0 \quad = \quad \left(\frac{1}{j+1} \sum_{k \leq j} \mathbf{v}_{i_k} \right) \ - \ \mathbf{v}_0 \quad = \quad \frac{1}{j+1} \sum_{k \leq j} (\mathbf{v}_{i_k} - \mathbf{v}_{i_0}) \ .$$

which is a linear combination of the linearly independent vectors $\mathbf{v}_{i_1} - \mathbf{v}_{i_0}, \cdots, \mathbf{v}_{i_j} - \mathbf{v}_{i_0}$ such that the coefficient of the last vector in the set is nonzero.

If we let $\mathbf{u}_k = \mathbf{v}_{i_k} - \mathbf{v}_{i_0}$, then it follows that for all $k > 0$ we have $\mathbf{b}_k - \mathbf{b}_0 = a_k \mathbf{u}_k + \mathbf{y}_k$, where $\mathbf{y}_k$ is a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_{k-1}$ and $a_k \neq 0$. Since the vectors $\mathbf{u}_j$ are linearly independent, it follows that the vectors $\mathbf{b}_k - \mathbf{b}_0$ (where $0 < k \leq n$) are linearly independent and hence the vectors $\mathbf{b}_0, \cdots, \mathbf{b}_n$ are affinely independent.■

The simplest nontrivial examples of barycentric subdivisions are given by 2-simplices, and Figure I.3.6 in `algtopfigures` gives a typical example. We shall enumerate the simplices in such a barycentric subdivision using the definition. For the sake of definiteness, we shall call the simplex P and the vertices $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$.

(*i*) The 0-simplices are merely the barycenters $\mathbf{b}_A$, where A runs through all the nonempty faces of P and P itself. There are 7 such simplices and hence 7 vertices in $\mathbf{B}(\mathbf{K})$.

(*ii*) The 1-simplices have the form $\mathbf{b}_A\mathbf{b}_C$, where A is a face of C. There are three possible choices for the ordered pair $(\dim A, \dim C)$; namely, $(0,1)$, $(0,2)$ and $(1,2)$. The number of pairs $\{A,\ C\}$ for the case $(0,1)$ is equal to 6, the number for the case $(0,2)$ is equal to 3, and the number for the case $(0,1)$ is also equal to 3, so there are 12 different 1-simplices in $\mathbf{B}(\mathbf{K})$.

(*iii*) The 2-simplices have the form $\mathbf{b}_A\mathbf{b}_C\mathbf{b}_E$, where A is a face of C and C is a face of E. There are 6 possible choices for $\{A,\ C,\ E\}$.

Obviously one could carry out a similar analysis for a 3-simplex but the details would be more complicated.

Of course, it is absolutely essential to verify the that barycentric subdivision construction actually defines simplicial decompositions.

THEOREM 4. *If* $(P, \mathbf{K})$ *is a simplicial complex and* $\mathbf{B}(\mathbf{K})$ *is the barycentric subdivision of* $\mathbf{K}$*, then* $(P, \mathbf{B}(\mathbf{K})\,)$ *is also a simplicial complex (in other words, the collection* $\mathbf{B}(\mathbf{K})$ *determines a simplicial decomposition of* P*).*

Proof. We shall concentrate on the special case where P is a simplex. The general case can be recovered from the special case and Lemma I.2.6.

Suppose now that P is a simplex with vertices vertices $\mathbf{v}_0, \cdots, \mathbf{v}_n$. We first show that P is the union of the simplices in $\mathbf{B}(\mathbf{K})$. Given $\mathbf{x} \in P$, write $\mathbf{x}$ as a convex combination $\sum_j t_j\,\mathbf{v_j}$, and rearrange the scalars into a sequence

$$t_{k_0} \;\geq\; t_{k_1} \;\cdots\; \geq\; t_{k_n}$$

(this is not necessarily unique, and in particular it is not so if $t_u = t_v$ for $u \neq v$). For each i between 0 and n, let A_i be the simplex whose vertices are $\mathbf{v}_{k_0}, \cdots, \mathbf{v}_{k_i}$. We CLAIM that $x \in \mathbf{b}_0 \cdots \mathbf{b}_n$, where $\mathbf{b}_i$ is the barycenter of A_i.

Let $s_i = t_{k_i} - t_{k_{i+1}}$ for $0 \leq i \leq n-1$ and set $s_n = t_{k_n}$. Then $s_i \geq 0$ for all i, and it is elementary to verify that

$$\mathbf{x} \;=\; \sum_{i=0}^{n} (i+1)\, s_i\, \mathbf{b}_i\ , \qquad \text{where} \qquad \sum_{1=0}^{n} (i+1)\, s_i \;=\; \sum_{i=0}^{n} t_{k_i} \;=\; 1\ .$$

Therefore $\mathbf{x} \in \mathbf{b}_0 \cdots \mathbf{b}_n$, so that every point in A lies on one of the simplices in the barycentric subdivision.

To conclude the proof, we must show that the intersection of two simplices in $\mathbf{B}(\mathbf{K})$ is a common face. First of all, it suffices to show this for a pair of n-dimensional simplices; this follows from the argument following the Default Hypothesis at the end of Section I.2.

Suppose now that α and γ are n-simplices in $\mathbf{B}(\mathbf{K})$. Then the vertices of α are barycenters of simplices $A_0, \cdots, A_n$ where A_j has one more vertex than A_{j-1} for each j, and the vertices of γ are barycenters of simplices $C_0, \cdots, C_n$ where C_j has one more vertex than C_{j-1} for each j. Label the vertices of the original simplex as $\mathbf{v}_{i_0}, \cdots, \mathbf{v}_{i_n}$ where $A_j = \mathbf{v}_{i_0} \cdots \mathbf{v}_{i_j}$ and also as $\mathbf{v}_{k_0}, \cdots, \mathbf{v}_{k_n}$

where $C_j = \mathbf{v}_{k_0} \ \cdots \ \mathbf{v}_{k_j}$. The key point is to determine how $(i_0, \ \cdots \ , i_n)$ and $(k_0, \ \cdots \ , k_n)$ are related.

If $\mathbf{x}$ lies on the original simplex and $\mathbf{x}$ is written as a convex combination $\sum_j \ t_j \, \mathbf{v}_j$, then we have shown that $\mathbf{x} \in A$ if $t_{i_0} \leq \ \cdots \ \leq t_{i_n}$. In fact, we can reverse the steps in that argument to show that if $\mathbf{x} \in A$ then conversely we have $t_{i_0} \leq \ \cdots \ \leq t_{i_n}$. Similarly, if $\mathbf{x} \in C$ then $t_{k_0} \leq \ \cdots \ \leq t_{k_n}$. Therefore if $\mathbf{x} \in A \cap C$ then $t_{i_j} = t_{k_j}$ for all j. Choose $m_0, \ \cdots \ , m_q \in \{0, \ \cdots \ , n\}$ such that $t_{m_j} > t_{m_{j+1}}$, with the convention that $t_{n+1} = 0$, and split $\{0, \ \cdots \ ,_n \}$ into equivalence classes $\mathcal{M}_0, \ \cdots \ , \mathcal{M}_q$ such that $\mathcal{M}_j$ is the set of all u such that $t_u = t_{m_j}$. It follows that $\mathbf{x}$ lies on the simplex $\mathbf{z}_0 \cdots \mathbf{z}_q$, where $\mathbf{z}_j$ is the barycenter of the simplex whose vertices are $\mathcal{M}_0 \cup \ \cdots \ \cup \mathcal{M}_j$. The vertices of this simplex are vertices of both A and C. Since $A \cap C$ is convex, this implies that it is the simplex whose vertices are those which lie in $A \cap C$, and thus $A \cap C$ is a face of both A and C.■

Terminology. Frequently the complex $(P, \mathbf{B}(\mathbf{K}))$ is called the *derived complex* of $(P, \mathbf{K})$. The barycentric subdivision construction can be iterated, and thus one obtains a sequence of decompositions $\mathbf{B}^r(\mathbf{K})$. The latter is often called the r^{th} barycentric subdivision of $\mathbf{K}$ and $(P, \mathbf{B}^r(\mathbf{K}))$ is often called the r^{th} derived complex of $(P, \mathbf{K})$.

Diameters of barycentric subdivisions

Given a metric space $(X, \mathbf{d})$, its *diameter* is the least upper bound of the distances $\mathbf{d}(y, z)$, where $y, z \in X$; if the set of distances is unbounded, we shall follow standard usage and say that the diameter is infinite or equal to ∞.

PROPOSITION 5. *Let $A \subset \mathbb{R}^n$ be an n-simplex with vertices $\mathbf{v}_0, \ \cdots \ , \mathbf{v}_n$. Then the diameter of A is the maximum of the distances $|\mathbf{v}_i - \mathbf{v}_j|$, where $0 \leq i, j \leq n$.*

Proof. Let $\mathbf{x}, \mathbf{y} \in A$, and write these as convex combinations $\mathbf{x} = \sum_j \ t_j \, \mathbf{v}_j$ and $\mathbf{y} = \sum_j \ s_j \, \mathbf{v}_j$. Then

$$\mathbf{x} - \mathbf{y} \ = \ \left(\sum_i \ s_i \right) \mathbf{x} \ - \ \left(\sum_j \ t_j \right) \mathbf{y} \ = \ \sum_{i,j} \ s_i t_j \, \mathbf{v}_j \ - \ \sum_{i,j} \ s_i t_j \, \mathbf{v}_i \ .$$

Since $0 \leq s_i, t_j \leq 1$ for all i and j, we have $0 \leq s_i t_j \leq 1$ for all i and j, so that

$$\mathbf{d}(\mathbf{x}, \, \mathbf{y}) \ = \ |\mathbf{x} - \mathbf{y}| \ \leq \ \Big| \sum_{i,j} \ s_i t_j \, (\mathbf{x}_j - \mathbf{x}_i) \Big| \ \leq$$

$$\sum_{i,j} \ s_i t_j \, |\mathbf{v}_i - \mathbf{v}_j| \ \leq \ \sum_{i,j} \ s_i t_j \max |\mathbf{v}_k - \mathbf{v}_\ell| \ = \ \max |\mathbf{v}_k - \mathbf{v}_\ell|$$

as required.■

Definition. If $\mathbf{K}$ is a simplicial decomposition of a polyhedron P, then the *mesh* of $\mathbf{K}$, written $\mu(\mathbf{K})$, is the maximum diameter of the simplices in $\mathbf{K}$.

PROPOSITION 6. *In the preceding notation, the mesh of $\mathbf{K}$ is the maximum distance $|\mathbf{v} - \mathbf{w}|$, where $\mathbf{v}$ and $\mathbf{w}$ are vertices of some simplex in $\mathbf{K}$.*■

The main result in this discussion is a comparison of the mesh of $\mathbf{K}$ with the mesh of $\mathbf{B}(\mathbf{K})$.

PROPOSITION 7. *Suppose that $(P, \mathbf{K})$ be a simplicial complex and that all simplices of $\mathbf{K}$ have dimension $\leq n$. Then*

$$\mu(\, \mathbf{B}(\mathbf{K}) \,) \ \leq \ \frac{n}{n+1} \cdot \mu(\mathbf{K}) \ .$$

Before proving this result, we shall derive some of its consequences.

COROLLARY 8. *In the preceding notation, if $r \geq 1$ then*

$$\mu(\mathbf{B}^r(\mathbf{K})) \quad \leq \quad \left(\frac{n}{n+1}\right)^r \cdot \mu(\mathbf{K}) \;.\blacksquare$$

COROLLARY 9. *In the preceding notation, if $\varepsilon > 0$ then there exists an r_0 such that $r \geq r_0$ implies $\mu(\mathbf{B}^r(\mathbf{K})) \quad < \quad \varepsilon$.*

Corollary 9 follows from Corollary 8 and the fact that

$$\lim_{r\to\infty} \left(\frac{n}{n+1}\right)^r \quad = \quad 0 \;.\blacksquare$$

Proof of Proposition 7. By Proposition 5 and the definition of barycentric subdivision we know that $\mu(\mathbf{B}(\mathbf{K}))$ is the maximum of all distances $|\mathbf{b}_A - \mathbf{b}_C|$, where $\mathbf{b}_A$ and $\mathbf{b}_C$ are barycenters of simplices $A,\ C \in \mathbf{K}$ such that $A \subset C$. Suppose that A is an a-simplex and C is a c-simplex, so that $0 \leq a < c \leq n$. We then have

$$|\mathbf{b}_A - \mathbf{b}_C| \quad = \quad \left| \frac{1}{a+1}\sum_{\mathbf{v}\in A} \mathbf{v} \;-\; \frac{1}{c+1}\sum_{\mathbf{w}\in C} \mathbf{w} \right|$$

and as in the proof of Proposition 5 we have

$$\frac{1}{a+1}\sum_{\mathbf{v}\in A} \mathbf{v} \;-\; \frac{1}{c+1}\sum_{\mathbf{w}\in C} \mathbf{w} \quad = \quad \frac{1}{(a+1)(c+1)}\sum_{\mathbf{v},\mathbf{w}} (\mathbf{v}-\mathbf{w}) \;.$$

There are $(a+1)$ terms in this summation which vanish (namely, those for which $\mathbf{w} = \mathbf{v}$), and therefore we have

$$|\mathbf{b}_A - \mathbf{b}_C| \quad = \quad \left| \frac{1}{(a+1)(c+1)}\sum_{\mathbf{v}\neq\mathbf{w}} (\mathbf{v}-\mathbf{w}) \right| \quad \leq \quad \frac{1}{(a+1)(c+1)}\sum_{\mathbf{v}\neq\mathbf{w}} |\mathbf{v}-\mathbf{w}| \quad \leq$$

$$\frac{1}{(a+1)(c+1)} \cdot \left(\max_{\mathbf{v},\mathbf{w}}\right) |\mathbf{v}-\mathbf{w}| \cdot \Big[(a+1)(c+1) - (a+1) \Big] \quad =$$

$$\left(\max_{\mathbf{v},\mathbf{w}} |\mathbf{v}-\mathbf{w}|\right) \cdot \left(1 \;-\; \frac{1}{c+1}\right) \quad \leq \quad \left(1 \;-\; \frac{1}{n+1}\right) \;.$$

At the last step we use $c \leq n$ and the fact that the function $1 - (x/n)$ is an increasing function of x if $x > 0$. The inequality in the corollary follows directly from the precedng chain of inequalities.■

One further consequence of Proposition 7 will be important for our purposes.

COROLLARY 10. *Let $(P, \mathbf{K})$ be a simplicial complex, and let $\mathcal{W}$ be an open covering of P. Then there is a positive integer r_0 such that $r \geq r_0$ implies that every simplex of $\mu(\mathbf{B}^r(\mathbf{K}))$ is contained in an element of $\mathcal{W}$.*

Proof. By construction, P is a compact subset of a the metric space $\mathbb{R}^m$. Therefore the Lebesgue Covering Lemma implies the existence of a real number $\eta > 0$ such that every subset of diameter

$< \eta$ is contained in an element of $\mathcal{W}$. If we choose $r_0 > 0$ such that $r \geq r_0$ implies $\mu(\mathbf{B}^r(\mathbf{K})) < \eta$, then $\mathbf{B}^r(\mathbf{K})$ will have the required properties.■

I.4 : Cones and suspensions

(Hatcher, 0)

These two basic constructions are described on pages 8–9 of Hatcher. We shall say a little more about them and apply them to construct a homeomorphism from the standard n-disk and $(n-1)$-sphere to the standard n-simplex and its boundary.

The constructions and their properties

Definition. Let X be a topological space. The *cone on* X, usually written $\mathbf{C}(X)$, is the quotient of $X \times [0,1]$ modulo the equivalence relation whose equivalence classes are all one point subsets of the form $\{ (x,t) \}$, where $t \neq 0$, and the subset $X \times \{0\}$.

The first result explains the motivation for the name.

PROPOSITION 1. *If* X *is a compact subset of* $\mathbb{R}^n$*, then* $\mathbf{C}(X)$ *is homeomorphic to a subset of* $\mathbb{R}^{n+1}$ *so that the image of* $X \times \{1\}$ *in* $\mathbf{C}(X)$ *corresponds to* $X \times \{0\}$ *and every point of the image is on a closed line setment joining a point of the latter to the last unit vector* $(0, \cdots, 0, 1)$.

Proof. Define a continuous map g from $X \times [0,1]$ to $\mathbb{R}^{n+1}$ sending (x,t) to $(tx, 1-t)$. This passes to a continuous 1–1 mapping f from $\mathbf{C}(X)$ to $\mathbb{R}^{n+1}$ whose image is the set described in the statement of the result, and since $\mathbf{C}(X)$ is a (continuous image of a) compact space it follows that f maps the cone homeomorphically onto its image.■

Examples. The cone on S^n is canonically homeomorphic to D^{n+1}; specifically, the map $S^n \times [0,1] \to D^{n+1}$ which sends (x,t) to $(1-t)x$ passes to a map of quotients $\mathbf{C}(S^n) \to D^{n+1}$ which is a homeomorphism. Also, the cone on D^n is canonically homeomorphic to D^{n+1}. Perhaps the quickest way to see this is the following: The preceding argument shows that the cone on the upper hemisphere D^n_+ of S^n (where the last coordinate is nonnegative) is the set of points in D^{n+1} whose last coordinate is nonnegative (its "upper half"), so we have to show that the latter is homeomorphic to D^{n+1}. If we let $|x|_2$ and $|x|_\infty$ denote the appropriate norms on $\mathbb{R}^{n+1}$ (see the 205A notes), then the homeomorphism h of $\mathbb{R}^{n+1}$ to itself defined by

$$h(x) \quad = \quad \frac{|x|_\infty}{|x|_2} \cdot x \qquad \text{if} \quad x \neq 0$$

and $h(0) = 0$ (continuity here must be checked, but this is not difficult) will send the upper half of D^{n+1} to the subspace $[-1,1]^n \times [0,1] \subset \mathbb{R}^{n+1}$. Since this product of closed intervals is homeomorphic to $[-1,1]^n$ and the latter is homeomorphic to D^{n+1} by the inverse of the map h, the assertion about $\mathbf{C}(D^n)$ and D^{n+1} follows.■

The cone construction extends to a covariant functor as follows: If $f : X \to Y$ is continuous, then the map $f \times \mathrm{id}_{[0,1]} : X \times [0,1] \to Y \times [0,1]$ is also continuous, and if $q_W : W \times [0,1] \to \mathbf{C}(W)$

is the quotient projection for $W = X$ or Y, then passage to quotients defines a unique continuous mapping $\mathbf{C}(f) : \mathbf{C}(X) \to \mathbf{C}(Y)$ such that

$$\mathbf{C}(f) \circ q_X \quad = \quad q_Y \circ \left(f \times \mathrm{id}_{[0,1]} \right) \ .$$

It is a routine exercise to verify that this construction satisfies the covariant functor identities $\mathbf{C}(\mathrm{id}_{[0,1]}) \ = \ \mathrm{id}_{\mathbf{C}(X)}$ and $\mathbf{C}(g \circ f) \ = \ \mathbf{C}(g) \circ \mathbf{C}(f)$.

Definition. Let X be a topological space. The *(unreduced) suspension on* X, usually written $\mathbf{S}(X)$, is the quotient of $X \times [-1,1]$ modulo the equivalence relation whose equivalence classes are all one point subsets of the form $\{ \ (x,t) \ \}$, where $|t| < 1$, and the subsets $X \times \{\pm\}$.

The suspension of a circle is illustrated in the `figures` file. The name arises because the original space is effectively "suspended" between the north and south poles (the classes of $X \times \{\pm 1\}$ in the quotient), being held in place by the "cables" $\{x\} \times [-1,1]$.

We have the following analog of Propositions 1 for cones.

PROPOSITION 3. *If X is a compact subset of $\mathbb{R}^n$, then $\mathbf{S}(X)$ is homeomorphic to a subset of $\mathbb{R}^{n+1}$ so that the images of $X \times \{\pm 1\}$ in $\mathbf{S}(X)$ correspond to the point $(0, \ \cdots \ 0, \pm 1)$ and the homeomorphism is the inclusion on $X \times \{0\}$.*

Proof. This is very similar to the proof for cones. Define a continuous map g from $X \times [-1,1]$ to $\mathbb{R}^{n+1}$ sending (x,t) to $\big((1-|t|)x, t \big)$. This passes to a continuous 1–1 mapping f from $\mathbf{S}(X)$ to $\mathbb{R}^{n+1}$ whose image is the set described in the statement of the result, and since $\mathbf{C}(X)$ is a (continuous image of a) compact space it follows that f maps the suspension homeomorphically onto its image.■

Examples. The suspension on S^n is canonically homeomorphic to S^{n+1} by the map sending the class of $(x,t) \in S^n \times [0,1]$ to $\left(\sqrt{1-t^2} \cdot x, t \right) \in \mathbb{R}^{n+1}$. Similarly, the suspension of D^n is canonically homeomorphic to D^{n+1}, and this can be shown by adapting the previous argument which proved that the cone on D^n is homeomorphic to the upper half of D^{n+1} (the cone is just the upper half of the suspension; use symmetry considerations to define the homeomorphism on the lower halves of everything).■

The suspension construction extends to a covariant functor as follows: If $f : X \to Y$ is continuous, then the map $f \times \mathrm{id}_{[-1,1]} : X \times [0,1] \to Y \times [-1,1]$ is also continuous, and if $q_W : W \times [-1,1] \to \mathbf{S}(W)$ is the quotient projection for $W = X$ or Y, then passage to quotients defines a unique continuous mapping $\mathbf{S}(f) : \mathbf{S}(X) \to \mathbf{S}(Y)$ such that

$$\mathbf{S}(f) \circ q_X \quad = \quad q_Y \circ \left(f \times \mathrm{id}_{[0,1]} \right) \ .$$

It is a routine exercise to verify that this construction satisfies the covariant functor identities $\mathbf{S}(\mathrm{id}_{[0,1]}) \ = \ \mathrm{id}_{\mathbf{S}(X)}$ and $\mathbf{S}(g \circ f) \ = \ \mathbf{S}(g) \circ \mathbf{S}(f)$.

Observe that projection onto the second coordinate from $X \times [-1,1]$ to $[-1,1]$ passes to a continuous map from $\mathbf{S}(X)$, and we shall say that the value of this map on a point is the latter's *second coordinate* or *latitude* (the second term is suggested by the drawing in the `figures` file).

Definition. If X is a topological space, then the *upper and lower cones* $\mathbf{C}_{\pm}(X)$ are the subspaces of $\mathbf{S}(X)$ consisting of all classes of all poinst whose second coordinates are nonnegative and nonpositive respectively.

By construction, both the upper and lower cones on X are canonically homeomorphic to the cone on X; in fact, these concepts extend to subfunctors $\mathbf{C}_{\pm}$ of the suspension functor (in other words, the inclusions of the upper and lower cones are natural transformations).

Given a simplex $A \subset \mathbb{R}^k$ with vertex set $V = \{ \mathbf{v}_0, \ \cdots \ , \mathbf{v}_n \}$, its *boundary* ∂A is the union of the faces with vertex sets $V_i = V - \{\mathbf{x}_i\}$, where $i = 0, \ \cdot \ , n$. For each i, the simplex A_i with vertex set V_i is called the i^{th} *face* of A.

We shall use the concepts of cones and suspensions to prove the following result, which will be needed in subsequent units.

THEOREM 4. *For each $n \geq 0$ there is a homeomorphism from the n-simplex Δ_n to the n-disk D^n which maps $\partial \Delta_n$ onto S^{n-1} and sends the barycenter of Δ_n to the center $\mathbf{0}$ of D^n.*

This result is obvious if $n = 0$ because Δ_0 and D^0 each contain only one point. Let $\mathbf{A}_n$ be the statement of the theorem for a fixed nonnegative integer n, and let $\mathbf{B}_n$ be the following statement:

> *There is a homeomorphism from $\partial \Delta_n$ to S^{n-1} such that the lower half corresponds to the 0^{th} face and the upper half corresponds to the union of the other faces.*

We shall prove that $\mathbf{A}_n$ implies $\mathbf{B}_{n+1}$ for all $n \geq 0$ and $\mathbf{B}_n$ implies $\mathbf{A}_n$ for all $n \geq 1$. If we combine this with the validity of $\mathbf{A}_0$, we obtain Theorem 4.■

Proof that $\mathbf{B}_n$ implies $\mathbf{A}_n$ for all $n \geq 1$. Let $h : \partial \Delta_n \to S^{n-1}$ be the homeomorphism which exists by $\mathbf{B}_n$. Consider the maps $f_0 : S^{n-1} \times [0,1] \to D^n$ and $g_0 : \partial \Delta_n \times [0,1] \to \Delta_n$ defined by $f_0(\mathbf{x}, t) = t\mathbf{x}$ and $g_0(\mathbf{x}, t) = t\mathbf{x} + (1-t)\mathbf{b}$, where $\mathbf{b}$ is the barycenter of Δ_n. Since each of these maps is constant on the set of all points where $t = 0$, it follows that they pass to continuous maps on the cones of the domains, and we shall denote these maps by $f : \mathbf{C}(S^{n-1} \to D^n$ and $g : \mathbf{C}(\partial \Delta_n) \to \Delta_n$. Elementary considerations from linear algebra imply that f is bijective, and the basic results on barycentric subdivisions imply that g is also bijective; since all relevant spaces are compact Hausdorff, it follows that these maps are homeomorphisms and that the composite $f \circ \mathbf{C}(h) \circ g^{-1}$ defines a homeomorphism from Δ_n to D^n. By construction the maps f and g send the bases of the cones to S^{n-1} and $\partial \Delta_n$ respectively, and since the cone homeomorphism $\mathbf{C}(h)$ sends the base of one cone to the base of the other it follows that the composite homeomorphism sends the boundary ot the simplex to the unit sphere.■

We shall prove that $\mathbf{A}_n$ implies $\mathbf{B}_{n+1}$ for all $n \geq 0$ and $\mathbf{B}_n$ implies $\mathbf{A}_n$ for all $n \geq 1$. If we combine this with the validity of $\mathbf{A}_0$, we obtain Theorem 4.■

Proof that $\mathbf{A}_n$ implies $\mathbf{B}_{n+1}$ for all $n \geq 0$. The idea in this case is similar, but we shall use suspensions instead of cones.

As usual, let $\partial_{n+1} \Delta_{n+1}$ denote the face opposite the last vertex $\mathbf{e}_{n+1}$ (so the vertices of this face are $\mathbf{e}_i$ for $0 \leq i \leq n$). Then $\mathbf{A}_n$ implies the existence of a homeomorphism from $\Delta_n = \partial_{n+1} \Delta_n$ to $D^n \cong \mathbf{C}_-(S^{n-1})$. Let E denote the union of all the remaining faces of Δ_{n+1}, and let φ be the homeomorphism from $\partial \Delta_n$ to S^{n-1} which is given by $\mathbf{A}_n$ as above. Define a map k_0 from $\partial \Delta_n \times [0,1]$ to E which sends $\mathbf{x}$ to $(1-t)\mathbf{x} + t\mathbf{e}_{n+1}$. Since every point on E lies on a line segment joining $\mathbf{e}_{n+1}$ to a point on $\partial \Delta_n$, one can proceed as before to conclude that k passes to a homeomorphism k from $\mathbf{C}_+(\partial \Delta_n)$ to E, and its restriction to $\partial \Delta_n$ is the identity. If we piece together these two homeomorphisms, we obtain a homeomorphism from $\mathbf{S}(\partial \Delta_n)$ to $\partial \Delta_{n+1}$.

Since the suspension construction is functorial, we also know that the suspension of the mapping φ defines a homeomorphism from the suspension of $\partial \Delta_n$ to the suspension of S^{n-1}. To complete the proof, we need to construct a homeomorphism from $\mathbf{S}(S^{n-1})$ to S^n which sends the upper and lower cones to the upper and lower hemispheres respectively. The ideal behind this is given by the drawing in the `figures` file. Formally, the homeomorphism ψ is given by taking the continuous map $\psi_0 : S^{n-1} \times [-1,1] \to S^n$ sending $(\mathbf{x}, t)$ to $(\sqrt{1-t^2}\mathbf{x}, t)$ and verifying that it passes to a continuous bijective map ψ defined on $\mathbf{S}(S^{n-1})$.■

NOTE. Another proof of this result is given in an earlier version of these notes. The latter does not require the use of cones and suspensions, but the argument is considerably longer.

II. Homotopy and cell complexes

The notion of homotopy is introduced in Mathematics 205B, and it is central to both algebraic and geometric topology as well as many of the applications of topology to algebra and analysis. Part of the material is a review of topics from the second part of Munkres' book; some of the revies topics and most of the new material are also covered in Chapters 0 and 1 of Hatcher.

The new material covers two related topics. The first (in Section 3) is to describe generalizations of simplicial complexes called **cell complexes** that are more convenient for many purposes of algebraic topology, and the second (in Section 4) provides a fundamental illustration of the usefulness of such objects. One objective is an important result on the following central problem:

EXTENSION QUESTION. *Suppose that X and Y are topological spaces, that A is a subspace of X, and $g : A \to Y$ is continuous. Is there an extension of g to a continuous mapping $f : X \to Y$ (in other words, a continuous mapping f such that the restriction $f|A$ is equal to g)?*

One of the main results in Section 4 provides an extremely useful answer to this question in terms of the main concepts of this unit: *If X is a cell complex and A is a subcomplex, then g has a continuous extension to X if and only if some mapping homotopic to g has such an extension.*

This and subsequent units of the notes will be less self-contained than Unit I, and there will be numerous references to Munkres or Hatcher for details.

II.1 : Homotopic mappings

(Hatcher, Ch. 0; Munkres, §§ 51, 58)

The general notion of homotopy for (continuous) mappings is defined on page 323 of Munkres and page 3 of Hatcher. Following standard practice we shall write $f \simeq g$ to indicate that f is homotopic to g. We shall state some basic properties of homotopic mappings that are particularly important for our purposes.

PROPOSITION 1. (Munkres, Lemma 51.1, p. 324.) *The binary relation $\simeq$ of homotopy on the set of continuous mappings from one topological space X to a second topological space Y is an equivalence relation.*■

In the proposition above, we allow the possibility that $X = Y$. The set of homotopy classes of continuous mappings from X to Y is generally denoted by $[X, Y]$.

PROPOSITION 2. (Munkres, Exercise 1, p. 330.) *If we are given continuous maps $f_0 \simeq f_1 : X \to Y$ and $g_0 \simeq g_1 : Y \to Z$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$.*■

COROLLARY 3. *There is a category* **HTOP** *(the* homotopy category*) whose objects are topological spaces and whose morphisms are given by $[X, Y]$ such that if $u \in [X, Y]$ is represented by f and $v \in [X, Y]$ is represented by g, then $v \circ u = [g \circ f]$.*■

Not surprisingly, the identity morphism in $[X, X]$ is the homotopy class of the identity on X.

Given a continuous mapping $f : X \to Y$, then f represents an isomorphism in **HTOP** if and only if there is a mapping $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. A mapping f which satisfies these properties is said to be a *homotopy equivalence.* — Since every map is homotopic to itself, it follows immediately that *every homeomorphism is a homotopy equivalence.*

Definition. Two topological spaces X and Y are *homotopy equivalent* if there is a homotopy equivalence from X to Y (in which case there is also a homotopy equivalence from Y to X). Note that the relation "X is homotopy equivalent to Y" is reflexive, symmetric and transitive. Frequently one also says that *X and Y have the same homotopy type.*

Special types of homotopy equivalences

We shall begin with a homotopy between to basic types of continuous mappings.

Definition. A *contracting homotopy* of a topological space X is a mapping $H : X \times [0,1] \to X$ such that $H(x,0) = x$ for all $x \in X$ and $H|X \times \{1\}$ is a constant mapping.

We shall say that a topological space is **contractible** if it admits a contracting homotopy.

An arbitrary topological space X is not necessarily contractible, and in some sense most spaces are not. For example, if X is the circle S^1 this is not the case because in $[S^1, S^1] \cong \pi_1(S^1, 1)$ the identity map and the constant map determine different homotopy classes. In fact, one can manufacture many similar examples using the following lemma.

PROPOSITION 4. *If A, B and C are topological spaces, then there is an isomorphism*

$$\theta : [A, B \times C] \quad \cong \quad [A, B] \times [A, C]$$

sending a homotopy class $[f]$ to the ordered pair $([p_B \circ f], [p_C \circ f])$, where $p_B : B \times C \to B$ and $p_C : B \times C \to C$ are the coordinate projections.

Sketch of proof. The mapping θ is well-defined by the preceding two results. It is onto, for if we are given an ordered pair of homotopy classes $([g], [h])$, then this class is $\theta([f])$, where $f : A \to B \times C$ is the unique continuous mapping such that $p_B \circ f = g$ and $p_C \circ f = h$. To see it is also 1–1, suppose $\theta([f]) = \theta([f'])$. Then there are homotopies $K : p_B \circ f \simeq p_B \circ f'$ and $L : p_C \circ f \simeq p_C \circ f'$, and if we take the map H whose projections onto B and C are K and L respectively, then H defines a homotopy from f to f'.■

COROLLARY 5. *If X is a nonempty topological space, then $X \times S^1$ is* NOT *contractible.*

The proof of this result is relatively simple and formal, but it is important to understand it because the argument reflects the viewpoint underlying much of algebraic topology.

Proof. It will suffice to show that the identity map on $X \times S^1$ is not homotopic to a constant map. Let $q : X \times S^1$ to S^1 be projection onto the second coordinate, let $j : S^1 \to X \times S^1$ project to the constant map on the first factor and to the identity on the second, and let k be a constant map from $X \times S^1$ to itself. If the identity on $X \times S^1$ is homotopic to a constant map, then we have

$$[\mathrm{id}(S^1)] \quad = \quad [q \circ j] \quad = \quad [q] \circ [j] \quad = \quad [q] \circ [\mathrm{id}(X \times S^1)] \circ [j] \quad =$$

$$[q] \circ [k] \circ [j] \quad = \quad [q \circ k \circ j] \quad = \quad [\mathrm{constant}]$$

which contradicts the fact that the identity on S^1 is not homotopic to a constant. Therefore the identity on $X \times S^1$ cannot be homotopic to a constant map.■

One can clearly "leverage" this result to construct further examples; in particular, if T^k is the product of k copies of S^1, then an inductive argument combined with the preceding corollary shows that $X \times T^k$ is not contractible.■

Example. If K is a convex subset of $\mathbb{R}^n$, then K is contractible by a so-called *straight line homotopy*: Take an arbitrary point $\mathbf{y} \in K$ and set

$$H(\mathbf{x}, t) \quad = \quad (1-t)\mathbf{x} \; + \; t\mathbf{y}$$

so that H shrinks K down to $\{\mathbf{y}\}$ along the straight lines joining points $\mathbf{x} \in K$ to the chosen point $\mathbf{y}$.■

In the preceding example, the inclusion of $\{\mathbf{y}\}$ in K is a special case of the following general concept.

Definition. Let X be a topological space, and let $A \subset X$ with inclusion mapping i_A. Then A is said to be a *deformation retract* of X if there is a map $r : X \to A$ such that $r|A$ is the identity and $i_A \circ r_A$ is homotopic to the identity on X. — If there is a homotopy $H : i_a \circ r_A \simeq 1_X$ such that $H(a, t) = a$ for all $(a, t) \in A \times [0, 1]$ (*i.e.*, the homotopy is fixed on A), we say that A is a *strong deformation retract of* X.

More generally, in a category $\mathbf{C}$, a morphism $f : X \to Y$ is said to be a *retract* if there is a morphism $g : Y \to X$ such that $g \circ f = 1_X$, and a morphism $h : A \to B$ is said to be a *retraction* if there is a morphism $k : B \to A$ such that $k \circ h = 1_B$. — If A is a deformation retract of X, then the inclusion i_A is a retract and the mapping r is a retraction.

Example. The sphere S^n is a strong deformation retract of $\mathbb{R}^{n+1} - \{0\}$. The standard choice of r in this case is given by $r(\mathbf{x}) = |\mathbf{x}|^{-1} \cdot \mathbf{x}$ and $i \circ r$ is homotopic to the identity by the straight line homotopy sending $(\mathbf{x},\, t)$ to $t\mathbf{x} + (1-t)r(\mathbf{x})$.

Counting homotopy classes

We shall conclude this section by proving a result mentioned earlier.

THEOREM 6. *If K is a compact subset of $\mathbb{R}^n$ for some n and U is an open subset of $\mathbb{R}^m$ for some m, then $[K, U]$ is countable.*

One major step in the proof is the following result of independent interest:

LEMMA 7. *Let X and U be as above, and let $f : K \to U$ be continuous. Then there is some $\delta > 0$ such that if $g : K \to U$ is another continuous function satisfying $\mathbf{d}(\, f(\mathbf{x}),\, g(\mathbf{x})\,) < \delta$ for all $\mathbf{x}$, then g is homotopic to f as mappings from X to U.*

Sketch of proof of Lemma 7. We can define a continuous function $h : K \to \mathbb{R}$ by $h(\mathbf{x}) = \mathbf{d}(\, f(\mathbf{x}), \mathbb{R}^m - U\,)$. In fact, this function is positive valued because f maps K into U, and by the compactness of K it takes a minimum value δ. Therefore, if $\mathbf{x}$ is an arbitrary point in K and $\mathbf{d}(\, f(\mathbf{x}),\, \mathbf{v}\,) < \delta$, then the closed line segment joining $f(\mathbf{x})$ to $\mathbf{v}$ lies entirely in U. Consequently, if g satisfies the condition in the lemma for this choice of δ, the image of the straight line homotopy from f to g lies entirely in U.■

NOTE AND EXAMPLE. The preceding lemma reflects one reason for including the codomain as an extra piece of data in our definition of a function. Given any two functions f and g as above, they are always homotopic as maps into $\mathbb{R}^m$ by a straight line homotopy. The crucial point in the lemma is that the image of the homotopy is contained in U. — Without the constraint involving a positive constant δ, the result is false. To see this, let $K = S^1$ and $U = \mathbb{R}^2 - \{\mathbf{0}\}$, and take f to be the usual inclusion. Then f is not homotopic to a constant map, for if $r : U \to K$ is the retraction described above, then $r \circ f$ is not homotopic to a constant, but if f were homotopic to a constant map k, then we would have

$$\mathrm{id}(S^1) \quad \simeq \quad r \circ f \quad \simeq \quad r \circ k \quad = \quad \text{constant}$$

and we know this is not the case.

The observations of the previous paragraph have the following positive implication: *If $H : S^1 \times [0,1] \to \mathbb{R}^2$ is a homotopy from the inclusion map to the constant map, then there is some $(\mathbf{x}_0,\, t_0\,) \in S^1 \times [0,1]$ such that $H(\mathbf{x}_0,\, t_0\,) = \mathbf{0}$.*■

A major objective of the course is to develop tools that will yield generalizations of the preceding observation to mappings from $S^n \times [0,1] \to \mathbb{R}^{n+1}$.

Sketch of proof of Theorem 6. Suppose that $f : K \to U$ as above is continuous, and let $\delta > 0$ be given as in Lemma 7. Denote the coordinate projections of f by f_i, where $1 \leq i \leq m$.

By the Stone-Weierstrass Approximation Theorem, there are polynomial functions p_i on $K \subset \mathbb{R}^n$ such that

$$|\,(p_i|K) - f_i\,| \quad < \quad \frac{\delta}{2\sqrt{n}}$$

for each i, and in fact we can also find polynomials g_i with rational coefficients such that

$$|\,(p_i|K) - (g_i|K)\,| \quad < \quad \frac{\delta}{2\sqrt{n}}\ .$$

If we let $g : \mathbb{R}^n \to \mathbb{R}^n$ be the function whose coordinates are given by the polynomials g_i, it follows that $|\,f - (g|K)\,| < \delta$.

Standard set-theoretic computations show that there are only countably many polynomials in n variables with rational coefficients, and it follows that there are only countably many choices for g.

Combining the preceding two paragraphs with Lemma 7, we conclude that f is homotopic to one of the countable family of continuous functions whose coordinates are given by polynomials in n variables with rational coefficients, and therefore the set $[K, U]$ is countable.■

Using the fact that the inclusion of S^1 in $\mathbb{R}^2 - \{\mathbf{0}\}$ is a homotopy equivalence, one can show that

$$\mathbb{Z} \quad \cong \quad [S^1, S^1] \quad \cong \quad [S^1, \mathbb{R}^2 - \{\mathbf{0}\}]$$

(see the exercises for this section) and therefore the cardinality bound of $\aleph_0$ on $[K, U]$ is the best possible general result.■

Important standard notation

Unless stated otherwise, in the remainder of these notes the symbol I will denote the closed unit interval $[0, 1]$.

II.2 : The fundamental group

(Hatcher, §§ 1.1 – 1.3, 1.A – 1.B; Munkres, §§ 52, 54)

This subject was treated in Mathematics 205B, and it might be useful to review this material before proceeding.

Section 1.B of Hatcher is devoted to proving a fundamental result in topology which has numerous uses in geometry and complex variables:

THEOREM 1. *Let G be an arbitrary group. Then there is an arcwise connected, locally arcwise connected, and locally simply connected Hausdorff space BG such that $\pi_1(BG, \mathrm{pt.})$ is isomorphic to G and the universal covering space of G is contractible. Furthermore, if X and Y are spaces which have these properties, then X is homotopy equivalent to Y.*

The existence argument is contained in Example 1.B.7 of Hatcher, while the uniqueness up to homotopy type is stated as Theorem 1.B.8 and established by the argument in Proposition 1.B.9.

Definition. A topological space X is (strongly) **aspherical** if it is arcwise connected and it has a contractible covering space.

As noted in Hatcher, the torus T^k is aspherical because its universal covering space is $\mathbb{R}^k$, and the covering space projection is given by $p(x_1, \cdots, x_k) = \big(\exp(2\pi\, i\, x_1), \cdots, \exp(2\pi\, i\, x_k)\big)$. Also, as noted in Hatcher, all compact connected surfaces except S^2 and $\mathbb{RP}^2$ are aspherical.

Generalization. (For students who have taken Mathematics 205C or are familiar with the notion of sectional curvature in a riemannian manifold.) There is an important generalization of all these facts due to J. Hadamard (1865–1963): *If M is a compact smooth n-manifold which has a riemannian metric whose sectional curvature is everywhere nonpositive, then the universal covering of M is diffeomorphic to $\mathbb{R}^n$.* — We shall not use this result at any future point in the course.

II.3 : Abstract cell complexes

(Hatcher, Ch. 0)

One possible way to view a polyhedron is to think of it as an object that is constructible in a finite number of steps as follows:

- (0) Start with the finite set P_0 of vertices,
- (n) If P_{n-1} is the partial polyhedron constructed at Step $(n-1)$, at Step (n) one adds finitely many simplices S_j, identifying each face of each simplex S_j with a simplex in P_{n-1}.

In fact, one can do this in order of increasing dimension, attaching all 1-simplices to the vertices at Step 1, then attaching 2-simplices along the boundary faces at Step 2, and so on. It is often useful in topology to consider objects that are generalizations of this procedure that are more flexible in certain key respects. The objects used these days in algebraic topology are known as **cell complexes**.

One immediate difference between cell complexes and simplicial complexes is that the former use the closed unit disk $D^n \subset \mathbb{R}^n$ and its boundary S^{n-1} in place of an n-simplex Δ and its boundary $\partial\Delta_n$. Since the results of Section I.4 imply that D^n is homeomorphic to Δ_n such that S^{n-1} corresponds to $\partial\Delta_n$, it follows that one can view simplicial complexes as special cases of cell complexes.

Adjoining cells to a space

We shall now give the basic step in the construction of cell complexes. The discussion below relies heavily on the material in Unit V of the online Mathematics 205A notes that were previously cited.

Definition. Let X be a compact Hausdorff space and let A be a closed subset of X. If k is a nonnegative integer, we shall say that *the space X is obtained from A by adjoining finitely many k-cells* if there are continuous mappings $f_i : S^{k-1} \to A$ for $i = 1, \cdots, n$ such that X is homeomorphic to the quotient space of the topological disjoint union

$$A \amalg \left(\{1, \cdots, N\} \times D^k \right)$$

modulo the equivalence relation generated by identifying $(j, \mathbf{x}) \in \{j\} \times S^{k-1}$ with $f_j(\mathbf{x}) \in A$, where the homeomorphism maps $A \subset X$ to the image of A in the quotient by the canonical mapping.

By construction, there is a 1–1 correspondence of sets between X and

$$A \amalg \left(\{1, \cdots, N\} \times \mathbf{open}(D^k) \right)$$

where $\mathbf{open}(D^k) \subset D^k$ is the complement of the boundary sphere. The set $E_j \subset X$ corresponding to the image of $\{j\} \times D^k$ in the quotient is called a *(closed) k-cell*, and the subset $E_j^{\mathbf{O}}$ corresponding to the image of $\{j\} \times \mathbf{open}(D^k)$ in the quotient is called an *open k-cell*. One can then restate the observation in the first sentence of the paragraph to say that X is a union of A and the open k-cells, and these subsets are pairwise disjoint.

Before discussing some topological properties of a space obtained by adjoining k-cells, we shall consider some special cases.

Example 1. Let $(P, \mathbf{K})$ be a simplicial complex,let P_k be the union of all k-simplices in $\mathbf{K}$, and let P_{k-1} be defined similarly. Then the whole point of stating and proving Theorem 1 was to justify an assertion that P_k is obtained from P_{k-1} by attaching k-cells, one for each k-simplex in $\mathbf{K}$. Specifically, for each k-simplex A the map f_A is given by the composite of the homeomorphism $S^{k-1} \to \partial A$ with the inclusion $\partial A \subset P_{k-1}$. The homeomorphism from the quotient of the disjoint union to P_k is given by starting with the composite

$$P_{k-1} \; \left(\amalg \, \{1, \cdots, N\} \times D^k \right) \;\longrightarrow\; P_{k-1} \; \amalg_A \; A \;\longrightarrow\; P_k$$

where $\amalg_A$ runs over all the k-simplices of $\mathbf{K}$, the first map is a disjoint union of homeomorphisms on the pieces where the maps of Theorem 1 are used to define the homeomorphisms $\{j\} \times D^k \cong A$, and the second map is inclusion on each disjoint summand. This composite passes to a map of the quotient of the space on the left modulo the equivalence relation described above, and it is straightforward to show this map is 1–1 onto and hence a homeomorphism (all relevant spaces are compact Hausdorff).

Example 2. (GRAPHS) As in Section 64 of Munkres, one may define a finite (vertex-edge) graph to be a space obtained from a finite discrete space by adjoining 1-cells. Frequently there is an added condition that the attaching maps for the boundaries should be 1–1 (so that each

1-cell has two endpoints), and the weaker notion introduced here (and in Hatcher) is then called a *pseudograph.* The graph corresponds to a simplicial decomposition of a simplicial complex if and only if different 1-cells have different endpoints. The simplest example of a graph structure that is not a pseudograph and does not come from a simplicial complex is given by taking $X = S^1$ and $A = S^0$ with two 1-cells corresponding to the upper and lower semicircles $E^1_\pm$ in the complex plane. The attaching maps are defined to map the endpoints of $D^1 = [-1, 1]$ bijectively to $-1, 1$. — Another example that is historically noteworthy is the Königsberg Bridge Graph, in which the vertices correspond to four land masses in the city of Königsberg (now Kaliningrad, Russia) and the 1-cells (or *edges*) correspond to the bridges which joined pairs of land masses in the 18th century (see the `figures` file for drawing). This is another example of a graph that does not come from a simplicial complex but is not a pseudograph; if there are two bridges joining the same pairs of land masses, then the graph has two edges with the same boundary points. — In the next unit we shall see how Euler's analysis of this graph may be stated in terms of algebraic topology.

We shall encounter further examples after we define the main concept of this section. For the time being, we mention a few simple properties of spaces obtained by attaching k-cells for some k

PROPOSITION 2. *If X is obtained from A by attaching 0-cells, then X is homeomorphic to the disjoint union of A with a finite discrete space.*

This is true because the 0-disk D^0 has an empty unit sphere, so there are no attaching maps and the equivalence relation on the space $A \amalg \{1, \cdot, N\}$ is the equality relation.■

PROPOSITION 3. *If X is obtained from A by attaching k-cells, then each open cell $E_j^{\mathbf{O}}$ is an open subset of X, and each such open cell is homeomorphic to* $\mathbf{open}(D^k)$.

Proof. Each closed cell is compact because it is a continuous image of D^k, and hence each such subset is closed in X. By the set-theoretic description given above, the open cell $E_j^{\mathbf{O}}$ is just the complement of the closed set

$$A \cup \bigcup_{i \neq j} E_i$$

and hence it is open in X. Since the quotient space map from the disjoint union to X defines a 1–1 onto continuous mapping from $\mathbf{open}(D^k)$ to $E_j^{\mathbf{O}}$, it suffices to show that an open subset of $\mathbf{open}(D^k)$ is sent to an open subset of $E_j^{\mathbf{O}}$. Let

$$\varphi : A \amalg \left(\{1, \cdots, N\} \times D^k\right) \longrightarrow X$$

be the continuous onto quotient map corresponding to the cell attachments, and suppose that U is open in $\{j\} \times \mathbf{open}(D^k)$. By construction we then have

$$U = \varphi^{-1}\left[\varphi[U]\right]$$

and thus $\varphi[U]$ is open in X by the definition of the quotient topology.■

The last result in this subsection implies that the inclusion of A in X is homotopically well-behaved if X is obtained from A by adjoining k-cells.

PROPOSITION 4. *If X is obtained from A by attaching k-cells and U is an open subset of X containing A, then there is an open subset V such that*

$$A \subset V \subset \overline{V} \subset U$$

and A is a strong deformation retract of both V and $\overline{V}$.

The `figures` file contains an drawing for the case $N = 1$.

Proof. As in the preceding argument, take

$$\varphi : A \ \left(\amalg \{1, \ \cdots \ , N\}\right) \times D^k \ \longrightarrow \ X$$

to be the continuous onto map corresponding to the k-cell attachments.

Let $F = X - U$, and let $F_0 = \varphi^{-1}[F]$, so that F_0 corresponds to a disjoint union $\amalg_j \, F_j$, where each F_j is a compact subset of $\mathbf{open}(D^k)$; compactness follows because the image of each F_j in X is a closed subset of the compact k-cell E_j. Therefore we can find constants c_j such that $0 < c_j < 1$ and F_j is contained in the open disk of radius c_j about the origin in $\{j\} \times D^k$; let c be the maximum of the numbers c_j, and let $V \subset X$ be the image under φ of the set

$$W \quad = \quad A \ \amalg \quad \left(\bigcup_j \{j\} \times \{ \ \mathbf{x} \in D^k \ | \ c < |\mathbf{x}| \leq 1 \ \} \right) \ .$$

Then V is open because it is the complement of a compact set, and it follows that $\overline{V}$ is the image of

$$Y \quad = \quad A \ \left(\amalg \ \bigcup_j \{j\} \times \{ \ \mathbf{x} \in D^k \ | \ c \leq |\mathbf{x}| \leq 1 \ \} \right) \ .$$

Each of the sets W and Y is a strong deformation retract of

$$B \quad = \quad A \ \amalg \quad \left(\bigcup_j \{j\} \times S^{k-1} \right) \ .$$

Specifically, the homotopies deforming W and Y into B are the identity on A and map each of the sets $\{ \ c < |\mathbf{x}| \leq 1 \ \}$, $\{ \ c \leq |\mathbf{x}| \leq 1 \ \}$ to S^{k-1} by sending a (necessarily nonzero) vector $\mathbf{y}$ to $|\mathbf{y}|^{-1}\mathbf{y}$ and taking a staight line homotopy to join these two points. A direct check of the equivalence relation defining φ shows that the associated maps and homotopies $W \to B \to W$ and $Y \to B \to Y$ pass to the quotients $V \to A \to V$ and $\overline{V} \to A \to \overline{V}$, and these quotient maps display A as a strong deformation retract of both V and $\overline{V}$.■

Cell complex structures

By the preceding discussion, a simplicial complex $(P, \mathbf{K})$ has a finite, linearly ordered chain of closed subspaces

$$\emptyset \quad = \quad P_{-1} \quad \subset \quad P_0 \quad \subset \quad \cdots \quad \subset \quad P_m \quad = \quad P$$

such that for each k satisfying $0 \leq k \leq m$, the subspace P_k is obtained from P_{k-1} by attaching finitely many k-cells. We shall generalize this property into a definition for arbitrary cell complex structures.

Definition. Let X be a topological space. A *finite cell complex structure* (or finite CW structure) on X is a chain $\mathcal{E}$ of closed subspaces

$$\emptyset \quad = \quad X_{-1} \quad \subset \quad X_0 \quad \subset \quad \cdots \quad \subset \quad X_m \quad = \quad X$$

such that for each k satisfying $0 \leq k \leq m$, the subspace X_k is obtained from X_{k-1} by attaching finitely many k-cells. The subspace X_k is called the *k-skeleton* of X, or more correctly the k-skeleton of $(X, \mathcal{E})$

At this level of abstraction, the notion of cell complex structure is due to J. H. C. Whitehead (1904–1960); his definition extended to infinite cell complex structures and the letters CW were described as abbreviations for two properties of the infinite complexes that are explained in the Appendix of Hatcher's book, but one should also note that the letters also represent Whitehead's last two initials.

It follows immediately that simplicial complexes are examples of cell complexes. Numerous further examples appear on pages 5–8 of Hatcher. Furthermore, the Δ-complexes discussed on pages 102–104 are also examples of cell complexes. In analogy with (edge-vertex) graphs, the main difference between Δ-complexes and simplicial complexes is that two k-simplices in a Δ-complex may have the same faces, but two k-simplices in a simplicial complex have at most a single $(k-1)$-face in common.

Because of the following result, one often describes a cell complex structure as a *cellular decomposition* of X.

PROPOSITION 5. *If X is a space and $\mathcal{E}$ is a cell decomposition of X, then every point of X lies on exactly one open cell of X.*

Proof. Since $X = \cup_k \, (X_k - X_{k-1})$, it follows that every point $y \in X$ lies in a exactly subset of the form $X_k - X_{k-1}$. Therefore there is at most one value of k such that x can lie on an open k-cell. Furthermore, since $X_k - X_{k-1}$ is a union of the open k-cells and the latter are pairwise disjoint, it follows that x lies on exactly one of these open k-cells.■

NOTE. If a cell complex has an n-cell for some $n > 0$ and $0 < m < n$, the cell complex might not have any m-cells (in contrast to the situation for, say, simplicial complexes); see Example 0.3 on page 6 of Hatcher.

Finally, we shall give a slightly different definition of subcomplex than the one in Hatcher.

Definition. If $(X, \mathcal{E})$ is a cell complex, we say that a closed subspace $A \subset X$ determines a *cell subcomplex* if for each $k \geq 0$ the set $A_k = X_k \cap A$ is obtained from A_{k-1} by attaching k-cells such that the every k-cell for A is also a k-cell for X.

There is an simple relationship between this notion of cell subcomplex and the previous definition of subcomplex for a simplicial complex; the proof is straightforward.

PROPOSITION 6. *If $(P, \mathbf{K})$ is a simplicial complex and $(P_1, \mathbf{K}_1)$ is a simplicial subcomplex, then P_1 also determines a cell subcomplex.*■

Finally, here are two further observations regarding subcomplexes. Again, the proofs are straightforward.

PROPOSITION 7. *If X is a cell complex such that $A \subset X$ determines a subcomplex of X and $B \subset A$ determines a subcomplex of A, then B also determines a subcomplex of X. Likewise, if B determines a subcomplex of X then B determines a subcomplex of A.*■

PROPOSITION 8. *If X is a cell complex such that $A \subset X$ determines a subcomplex of X, then for each $k \geq 0$ the set $X_k \cup A$ determines a subcomplex of X.*■

II.4 : The Homotopy Extension Property

(Hatcher, Ch. 0, § 2.1)

In this section we shall bring together several concepts from the preceding sections. The basis is the following central Extension Question stated at the beginning of this unit, and our first result describes a condition under which this question always has an affirmative answer.

PROPSITION 1. *Suppose that X and Y are topological spaces, that $A \subset X$ is a retract, and that $g : A \to Y$ is continuous. Then there is an extension of g to a continuous mapping $f : X \to Y$.*

Proof. Let $r : X \to A$ be a continuous function such that $r|A$ is the identity, and define $f = g \circ r$. Then if $a \in A$ we have $f(a) = g \circ r(a) = g(\, r(a)\,)$, and the latter is equal to $g(a)$ because $r|A$ is the identity.■

The hypothesis of the proposition is fairly rigid, but the result itself is a key step in proving a general result on the Extension Question.

THEOREM 2. (HOMOTOPY EXTENSION PROPERTY) *Let $(X, \mathcal{E})$ be a cell complex, and suppose that A determines a subcomplex. Suppose that Y is a topological space, that $g : A \to Y$ is a continuous map, and $f : X \to Y$ is a continuous map such that $f|A$ is homotopic to g. Then there is a continuous map $G : X \to Y$ such that $G|A = g$.*

COROLLARY 3. *Suppose that X and A are as above and that $g : A \to Y$ is homotopic to a constant map. Then g extends to a continuous function from X to Y.*

COROLLARY 4. *Suppose that X and A are as above and that $g : A \to X$ is homotopic to the inclusion map. Then g extends to a continuous function from X to itself.*

Corollary 3 follows because every constant map from A to Y extends to the analogous constant map from X to Y, and Corollary 4 follows because the inclusion of A in X extends continuously to the identity map from X to itself.■

One important step in the proof of the Homotopy Extension Property relies upon the following result:

PROPOSITION 5. *For all $k > 0$ the set $D^k \times \{0\} \;\cup\; S^{k-1} \times [0,1]$ is a strong deformation retract of $D^k \times [0,1]$.*

Proof. This argument is outlined in Proposition 0.16 on page 15 of Hatcher, and there is a drawing to illustrate the proof in the `figures` document.

The retraction $r : D^k \times [0,1] \to D^k \times \{0\} \;\cup\; S^{k-1} \times [0,1]$ is defined by a radial projection with center $(\mathbf{0}, 2) \in D^k \times \mathbb{R}$. As indicated by the drawing, the formula for r depends upon whether $2|\mathbf{x}| + t \geq 2$ or $2|\mathbf{x}| + t \leq 2$. Specifically, if $2|\mathbf{x}| + t \geq 2$ then

$$r(\mathbf{x},\, t) \quad = \quad \frac{1}{|\mathbf{x}|}\; (\,\mathbf{x},\, 2|\mathbf{x}| + t - 2)$$

while if $2|\mathbf{x}| + t \leq 2$ then we have

$$r(\mathbf{x},\, t) \quad = \quad \frac{1}{2}\left(\,(2-t)\mathbf{x},\, 0\,\right)$$

and these are consistent when $2|\mathbf{x}| + t = 2$ then both formulas yield the value $|\mathbf{x}|^{-1}(\mathbf{x},\, 0)$. Elementary but slightly tedious calculation then implies that $r(\mathbf{x},\, t)$ always lies in $D^k \times [0,1]$, and likewise that r is the identity on $D^k \times \{0\} \,\cup\, S^{k-1} \times [0,1]$. The homotopy from inclusion$\circ r$ to the identity is then the straight line homotopy

$$H(\mathbf{x},\, t;\, s) \quad = \quad (1-s)\cdot r(\mathbf{x},\, t) \;+\; s \cdot (\mathbf{x},\, t)$$

and this completes the proof of the proposition.∎

Proof of Theorem 2. In the course of the proof we shall need the following basic fact: *If A and B are compact Hausdorff spaces and $\varphi : A \to B$ is a quotient map in the sense of Munkres' book, then for each compact Hausdorff space C the product map $\varphi \times 1_C : A \times C \to B \times C$ is also a quotient map.* — This follows because $\varphi \times 1_C$ is closed, continuous and surjective.

Since the homotopy relation on continuous functions is transitive, a standard recursive argument reduces the proof of the theorem to the special cases subcomplex inclusions

$$X_{k-1} \,\cup\, A \quad \subset \quad X_k \,\cup\, A\,.$$

In other words, *it will suffice to prove the theorem when X is obtained from A by attaching k-cells.*

We now assume the condition in the preceding sentence. Let $h : A \times [0,1] \to Y$ be the homotopy from f (when $t = 0$) to g (when $t = 1$). If we can show that the inclusion

$$A \times [0,1] \,\cup\, X \times \{0\} \quad \subset \quad X \times [0,1]$$

is a retract, then we can use Proposition 1 to find an extension of the map

$$\theta \quad = \quad \text{“}h \cup f\text{''} : A \times [0,1] \,\cup X \times \{0\} \quad \longrightarrow \quad Y$$

to $X \times [0,1]$, and the restriction of this extension to $X \times \{1\}$ will be a continuous extension of g. — In fact, we shall show that *the space $A \times [0,1] \cup X \times \{0\}$ is a strong deformation retract of $X \times [0,1]$.*

As in earlier discussions let

$$\varphi : A \;\amalg\; \big(\{1,\; \cdots\;, N\} \times D^k\big) \;\longrightarrow\; X$$

be the topological quotient map which exists by the definition of attaching k-cells. By Proposition 5 we know that the space

$$A \times [0,1] \;\amalg\; (\{1,\; \cdots\;, N\}) \times \big(S^{k-1} \times [0,1] \cup D^k \times \{0\}\big)$$

is a strong deformation retract of

$$\big(A \;\amalg\; \{1,\; \cdots\;, N\} \times D^k\;\big) \times [0,1]$$

because we can the mappings piecewise using the identity on $A \times [0,1]$ and the functions from Proposition 5 on each of the pieces $\{j\} \times D^k \times [0,1]$. Let

$$r' : \big(A \;\amalg\; \big(\{1,\; \cdots\;, N\} \times D^k\big)\big) \times [0,1] \;\longrightarrow$$

$$A \times [0,1] \;\amalg\; \big(\{1,\; \cdots\;, N\} \times \big(S^{k-1} \times [0,1] \cup D^k \times \{0\}\big)\big)$$

be the retraction obtained in this fashion, and let

$$H' : \Big(\big(A \ \amalg \ \{1, \ \cdots \ , N\} \times D^k \big) \times [0,1] \Big) \times [0,1] \ \longrightarrow \ \big(\ A \ \amalg \ \{1, \ \cdots \ , N\} \times D^k \ \big) \times [0,1]$$

be defined similarly. It will suffice to show that these pass to continuous mappings of quotient spaces; in other words, we want to show there are (continuous) mappings r and H such that the following diagrams are commutative, in which ψ is the mapping whose values are given by φ:

$$\begin{array}{ccc} (A \amalg \cdots) \times [0,1] & \xrightarrow{r'} & A \times [0,1] \amalg \big(\{1, ..., N\} \times [\cdots] \big) \\ \big\downarrow \varphi \times 1 & & \big\downarrow \psi \\ X \times [0,1] & \xrightarrow{r} & A \times [0,1] \cup X \times \{0\} \end{array}$$

$$\begin{array}{ccc} \Big((A \amalg \cdots) \times [0,1] \Big) \times [0,1] & \xrightarrow{H'} & (A \amalg \cdots) \times [0,1] \\ \big\downarrow \varphi \times 1 \times 1 & & \big\downarrow \phi \times 1 \\ \big(X \times [0,1] \big) \times [0,1] & \xrightarrow{H} & X \times [0,1] \end{array}$$

Standard results on factoring maps through quotient spaces imply that such commutative diagrams exist if and only if (i) if two points map to the same point under $\psi \circ r'$, then they map to the same point under $\varphi \times 1$, (ii) if two points map to the same point under $\phi \times 1 \circ H'$, then they map to the same point under $\varphi \times 1 \times 1$. It is a routine exercise to check both of these statements are true.■

COROLLARY 6. *Suppose that X and Y are as in the theorem and Y is contractible. Then every continuous mapping $f : X \to Y$ has a continuous extension to X.*

Proof. It will suffice to prove that an arbitrary continuous mapping $f : A \to Y$ is homotopic to a constant. We know that 1_Y is homotopic to a constant map k, and therefore $f = 1_Y \circ f$ is homotopic to the constant map $k \circ f$.■

III. Simplicial homology

The goal of this unit is to define a sequence of abelian groups associated to a simplicial complex $(P, \mathbf{K})$ which are called **homology groups** and denoted by $H_n(P, \mathbf{K})$, where n runs through all the integers but the groups are all zero if n is negative. These groups may be interpreted as furnishing an "algebraic picture" of the underlying topological space P. In order to develop the important properties of these groups it will be necessary to introduce some basic concepts and results from *homological algebra*, but efforts will be made to keep this to a minimum.

We have stated that the groups provide information about the underlying space P rather than the simplicial complex $(P, \mathbf{K})$ because *these groups turn out to depend only upon P itself*. This fact will drop out of the more general constructions in the next unit, where homology groups are defined for an arbitrary topological space and shown to agree with the groups of this unit if the space P has a simplicial decomposition.

Some motivation from vector analysis

Suppose that U is an open subset of $\mathbb{R}^3$ and Σ is some sort of compact oriented surface in U (for our purposes, it suffices to think of Σ as having a continuously defined unit normal vector at every point). Then the boundary of Σ is some union of closed curves Γ_i, where the sense of Γ_i is chosen such that for each point of such a curve the ordered triple of vectors given by

> the chosen unit normal vector to the surface at the point,
>
> the unit tangent vector to the curve at the point,
>
> the unit vector which is tangent to the surface at the point, but perpendicular to the curve's tangent vector and directed **into** the surface

will form a right handed triad (see the illustration in the `figures` document); we shall not try to make everything rigorous here because the goal is to provide some intuition. In such a situation one sometimes says that the formal sum $\sum_i \Gamma_i$ of the sensed curves Γ_i is homologous to zero in U, and by Stokes' Theorem we have the following:

If $\sum_i \Gamma_i$ is homologous to zero in U and $\mathbf{F}$ is a smooth vector field defined on U such that $\nabla \times \mathbf{F} = \mathbf{0}$, then

$$\sum_i \int_{\Gamma_i} \mathbf{F} \cdot d\mathbf{x} \quad = \quad 0 \,. \blacksquare$$

It is important to note that if V is an open subset of U and $\sum_i \Gamma_i$ is homologous to zero in U, then $\sum_i \Gamma_i$ *is not necessarily homologous to zero in V*. The standard example for this involves the ordinary unit circle Γ in $\mathbb{R}^2 \subset \mathbb{R}^3$ whose center is the origin and whose radius is 1. This curve is homologous to zero in $\mathbb{R}^3$ because it bounds the closed unit disk. To see it is not homologous to zero in $V = (\mathbb{R}^2 - \{\mathbf{0}\}$, consider the vector field given by

$$\mathbf{F}(u, v) \quad = \quad \left(\frac{v}{u^2 + v^2}, \frac{-u}{u^2 + v^2}, 0 \right)$$

and note that $\nabla \times \mathbf{F} = \mathbf{0}$ and the standard computation

$$\int_{\Gamma} \mathbf{F} \cdot d\mathbf{x} \quad = \quad 2\pi$$

imply that Γ cannot be homologous to zero in V.

Suppose now that we have a union of pairwise disjoint closed oriented surface Σ_j in our open set U; the term "closed" means that the surfaces have no boundary curves, just like the unit sphere defined by $u^2 + v^2 + w^2 = 1$. We shall say that the formal sum $\Sigma_1 + \cdots + \Sigma_j$ is homologous to zero in U if $\cup_j \Sigma_j$ bounds a region $W \subset U$ such that the closure of W is equal to the union of W and $\cup_j \Sigma_j$ and the normal directions to Σ are all outward pointing. — For example, the unit sphere is homologous to zero in $\mathbb{R}^3$ because it bounds a unit disk, and if Σ_r denotes the sphere of radius r in $\mathbb{R}^3$, then $\Sigma_1 \cup \Sigma_2$ is homologous to zero if we orient the pieces so that the normal vectors on Σ_2 point outward (away from the origin) and the normal vectors on Σ_1 point inward (towards the origin). The Divergence Theorem from vector analysis then has the following implication:

If $\Sigma_1 + \cdots + \Sigma_n$ is homologous to zero in U and $\mathbf{F}$ is a smooth vector field defined on U such that $\nabla \cdot \mathbf{F} = 0$, then

$$\sum_i \iint_{\Sigma_i} \mathbf{F} \cdot d\mathbf{\Sigma} \quad = \quad 0 \ . \blacksquare$$

We can now show that Σ_1 is not homologous to zero in $\mathbb{R}^3 - \{\mathbf{0}\}$ by an argument similar to the preceding one. Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3 - \{\mathbf{0}\}$ defined by $\mathbf{F}(\mathbf{x}) = |\mathbf{x}|^{-1}\mathbf{x}$; then it is a routine exercise to prove that $\nabla \cdot \mathbf{F} = 0$ but direct computation shows that

$$\iint_{\Sigma_1} \mathbf{F} \cdot d\mathbf{\Sigma} \quad = \quad 4\pi \ .$$

Homology theory provides an organized algebraic framework for studying such phenomena.

III.1 : Exact sequences and chain complexes

(Hatcher, § 2.1)

This section is basically algebraic, and at first the need for formally introducing the concepts may be unclear. However, the notions described here arise repeatedly in algebraic topology and other subjects.

Definition. Suppose we are given a diagram of the form

$$A \ \xrightarrow{f} \ B \ \xrightarrow{g} \ C$$

in which the objects are abelian groups (possibly with some additional structure) and the morphisms are abelian group homomorphisms (possibly preserving the extra structure). We shall say that the diagram is *exact* at B if the kernel of g is equal to the image of f.

More generally, if we are given a linear diagram such as

$$\cdots \longrightarrow Z \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow \cdots$$

we shall say that it is an *exact sequence* if it is exact at every object which is the domain of one morphism and the codomain of another.

Examples

There are many standard exact sequences in elementary algebra.

1. A *short exact sequence* is one having the form $0 \to A \to B \to C \to 0$. Exactness at A means that the kernel of $A \to B$ is the image of $0 \to A$, which is equivalent to saying that the map is injective. Similarly, exactness at C means that the kernel of $C \to 0$ is the image of $B \to C$, whic is equivalent to saying that the map is surjective. The short exact sequence property is then equivalent to saying that $A \to B$ is injective, and C is isomorphic to the quotient of B by the image of A.

2. The *cokernel* of a homomorphism $f : A \to B$ is defined to be the quotient group $B/f[A]$. Given an arbitrary homomorphism $f : A \to B$, one then has the following *kernel – cokernel exact sequence*:

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow A \longrightarrow B \longrightarrow \mathrm{Coker}(f) \longrightarrow 0$$

3. Let U be a connected open subset of $\mathbb{R}^2$, let $\mathbf{C}^\infty(U)$ denote the infinitely differentiable real valued functions on U, and let let $\mathbf{VF}(U)$ denote the infinitely differentiable (2-dimensional) vector fields on U in the sense of vector analysis. If we let $\mathbb{R} \to \mathbf{C}^\infty(U)$ denote the inclusion of the constant functions and take the gradient map from $\mathbf{C}^\infty(U)$ to $\mathbf{VF}(U)$, then it follows that the sequence $\mathbb{R} \to \mathbf{C}^\infty(U) \to \mathbf{VF}(U)$ is exact. Furthermore, if we take the map $\mathbf{VF}(U) \to \mathbf{C}^\infty(U)$ which sends a vector field $\mathbf{F} = (P, Q)$ to its "scalar curl" $Q_1 - P_2$, then the sequence $\mathbf{C}^\infty(U) \to \mathbf{VF}(U) \to \mathbf{C}^\infty(U)$ will be exact **provided** U is convex (or more generally star-shaped). — On the other hand, the second sequence is not exact if $U = \mathbb{R}^2 - \{\mathbf{0}\}$, for the previously described vector field on U with coordinate functions v/r and $-u/r$ has zero scalar curl but is not the gradient of any smooth function on U; this follows from Green's Theorem and the previous line integral calculation.

We can extend the preceding if U is a connected open set in $\mathbb{R}^3$ by considering the following sequence:

$$\mathbb{R} \xrightarrow{\text{constants}} \mathbf{C}^\infty(U) \xrightarrow{\text{grad}} \mathbf{VF}(U) \xrightarrow{\text{curl}} \mathbf{VF}(U) \xrightarrow{\text{div}} \mathbf{C}^\infty(U)$$

This is again exact at the left hand object $\mathbf{C}^\infty(U)$, and standard results in vector analysis imply that the kernel of the curl is contained in the image of the gradient, while the kernel of the divergence is contained in the image of the curl. If U is convex, then one can also show that the sequence is exact, but in general this is not true. Our previous examples give a vector field on $\mathbb{R}^2 - \{\mathbf{0}\}) \times \mathbb{R}$ whose curl is zero but cannot be expressed as a gradient over U, and a vector field on $\mathbb{R}^3 - \{\mathbf{0}\}$ whose divergence is zero but cannot be expressed as the curl of another vector field over U.

Graded objects

The next concept is simple but indispensable.

Definition. Let A be a set, and let $\mathbf{C}$ be a category. A *graded object* over $\mathbf{C}$ with grading set A is a function X from A to the objects of $\mathbf{C}$. The object corresponding to a is generally denoted by X_a.

For example, one can construct a graded vector space over the reals with grading set the integers $\mathbb{Z}$ by taking $V_n = \mathbb{R}^n$ for $n \geq 0$ and setting V_n equal to the zero space if $n < 0$.

Another example is obtainable from an algebra of polynomials $\mathbb{R}[x_1, \cdots, x_n]$ in finitely many indeterminates. Here we can take V_n to be the set of all homogeneous polynomials of degree n together with the zero polynomial.

In this course we shall mainly be interested in *nonnegatively graded objects*, where the indexing set is $\mathbb{Z}$ and the object X_n is a suitable zero object if $n < 0$. For the categories of abelian groups or modules over some associative ring with unit, the meaning of zero object is obvious, and these categories are the only ones to be considered here.

Definition. If X and Y are nonnegatively graded objects over a category $\mathbf{C}$, then a *graded morphism of degree zero* or *grade preserving morphism* is a function f which assigns to each $n \in \mathbb{Z}$ a morphism $f_n : X_n \to Y_n$ in the category $\mathbf{C}$.

In the polynomial example, one can define a grade preserving homomorphism by sending the homogeneous polynomial $p(x_1, x_2, \cdots x_n)$ to the homogeneous polynomial $q(x_1, x_2, \cdots x_n) = p(x_1, x_1 + x_2, \cdots x_n)$. Obviously there are many other maps of this type.

The following observation is immediate:

PROPOSITION 1. *Given a category* $\mathbf{C}$*, the* $\mathbb{Z}$*-graded objects over* $\mathbf{C}$ *and graded morphisms of degree zero form a category.*■

In fact, this category has many structural properties that are direct analogs of properties that hold for $\mathbf{C}$ (for example, subobjects, quotient objects, direct products, and so on).

Chain complexes

The following concept is absolutely fundamental.

Definition. Let $\mathbf{C}$ be the category of abelian groups and homomorphisms or a category of unital modules over an associative ring with unit R. A **chain complex** over $\mathbf{C}$ is a pair (A, d) consisting of a graded object A over $\mathbf{C}$ indexed by the integers together with morphisms $d_j : A_j \to A_{j-1}$ such that $d_{j-1} \circ d_j = 0$ for all j.

Here are a few simple examples.

1. Given an arbitrary graded module A, one can make it into a chain complex by taking $d_j = 0$ for all j. More generally, given a sequence of homomorphisms $f_{2j} : A_{2j} \to A_{2j-1}$, one can define a chain complex whose graded module is A with $d_{2j} = f_{2j}$ and $d_{2j-1} = 0$.
2. Suppose we are given three modules B, H, and B'. The we can define a chain complex with $A_2 = B$, $A_1 = B \oplus H \oplus B'$, and $A_0 = B'$ and $A_j = 0$ otherwise such that d_2 is injection into the first summand, d_1 is projection onto the third summand, and all other maps d_j must be zero (since either their domain or codomain is zero).
3. If U is open in $\mathbb{R}^2$, then one can obtain a chain complex from the previous sequence involving $\mathbf{C}^\infty(U)$ and $\mathbf{VF}(U)$, if one takes A_3 to be the reals, A_2 and A_0 to be the smooth functions, A_0 to be the vector fields, with morphisms given by inclusion of constants from A_3 to A_2, gradient from A_2 to A_1, scalar curl from A_1 to A_0, and with all other real vector spaces and morphisms equal to zero. Similarly, if U is open in $\mathbb{R}^3$ one has a system with A_4 equal to the reals, A_3 and A_0 equal to the smooth functions, A_2 and A_1 equal to the vector fields, with morphisms given by inclusion of constants from A_4 to A_3, gradient from A_3 to A_2, curl from A_2 to A_1, divergence from A_1 to A_0, and with all other real vector spaces and morphisms equal to zero.

The mapping d is often called a *differential*; the motivation is related to the preceding examples where the maps are given by some form of differentiation.

Definition. Given two chain complexes (A, d) and (B, e) a **chain map** $f : A \to B$ is a graded morphism such that for all integers j we have $e_j \circ f_j = f_{j-1} \circ d_j$. In other words, the following diagram is commutative:

$$\begin{array}{ccc} A_j & \xrightarrow{f_j} & B_j \\ \big\downarrow d_j & & \big\downarrow e_j \\ A_{j-1} & \xrightarrow{f_{j-1}} & B_{j-1} \end{array}$$

If the differential in a chain complex (A, d) is unambiguous from the context we shall frequently write A instead of (A, d).

The following consequences of the definitions are elementary but important.

PROPOSITION 2. *Given a category* **C**, *the chain complexes over over* **C** *and chain complex morphisms form a category.*

PROPOSITION 3. *If (A, d) and (B, e) are chain complexes over* **C** *and $f : (A, d) \to (B, e)$ is a morphism of chain complex such that the mappings f_j are all isomorphisms, then the map f^{-1} of graded modules defined by $(f^{-1})_j = f_j^{-1}$ is also a chain map.*

Proof. To simplify the formulas let $g_j = f_j^{-1}$. The conclusion of the proposition is equivalent to the identities $d_j \circ g_j = g_{j-1} \circ e_j$ as maps from B_j to A_{j-1}.

Let $b \in B_j$ be arbitrary. Since f_{j-1} is injective, it follows that $d_j \circ g_j(b) = g_{j-1} \circ e_j(b)$ if and only if $f_{j-1} \circ d_j \circ g_j(b) = f_{j-1} \circ g_{j-1} \circ e_j(b)$. The left hand side is equal to

$$f_{j-1} \circ d_j \circ g_j(b) \quad = \quad e_j \circ f_j \circ g_j(b) \quad = \quad e_j(b)$$

by the defining identity for chain maps and the fact that g is inverse to f, and the latter fact also implies that the right hand side is equal to $e_j(b)$. Therefore it follows that the maps g_j satisfies the defining conditions for a chain map.■

As before, the category of chain complexes over **C** has many structural properties that are direct analogs of properties that hold for **C** and the category of graded objects over **C** (such as subobjects, quotient objects, direct products).

A few additional remarks about subcomplexes and quotient complexes seem worthwhile. If (A, d') is a chain subcomplex of (B, d), then it follows that $A_j \subset B_j$ for all j and that d_j maps A_j to A_{j-1} via d'_j. The quotient complex has a differential d'' such that $d''_j[x] = [d_j x]$, where "$[\cdots]$" denotes the equivalence class in the appropriate quotient module. There is a well-defined map of this sort because d_j maps A_j into A_{j-1}.

ONE MORE EXAMPLE. Let Δ be a 2-simplex with vertices $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, let C_0 be the free abelian group generated by these vertices, let C_1 be the free abelian group generated by the three edges $\mathbf{yz}$, $\mathbf{xz}$ and $\mathbf{xy}$, let C_2 be the free abelian group generated by the element Δ, and define maps $d_2 : C_2 \to C_1$ and $d_1 : C_1 \to C_0$ by

$$\begin{aligned} d_2(A) \quad &= \quad \mathbf{yz} - \mathbf{xy} + \mathbf{xz} \\ d_1(\mathbf{xy}) \quad &= \quad \mathbf{y} - \mathbf{x}, \;\; d_1(\mathbf{yz}) \quad = \quad \mathbf{z} - \mathbf{y} \text{ and } \;\; d_1(\mathbf{xz}) \quad = \quad \mathbf{z} - \mathbf{x}. \end{aligned}$$

We set all other groups C_j equal to zero, and it follows that all remaining homomorphisms must also be zero. Direct examination shows that the kernel of d_1 is the set of all multiples of $d_2(\Delta)$.

Geometrically, $d_2(\Delta)$ represents the boundary of the simplex A with the edges oriented so that they correspond to a simple closed curve. More generally, if (A, d) is a chain complex then elements in the kernel of d_j are frequently called *cycles*, while elements in the image of d_{j+1} are frequently called *boundaries*, and the homomorphisms d_j are frequently called *boundary homomorphisms*.

III.2 : Homology groups

(Hatcher, § 2.1)

If (A, d) is a chain complex, then the condition $d_j \circ d_{j+1}$ implies that the kernel of d_j (the submodule of cycles) contains the image of d_{j+1} (the submodule of boundaries). The sequence determined by the chain complex is exact at A_j if and only if these two submodules are equal. One can view homology groups as measuring the extent to which a chain complex is not an exact sequence.

Definition. Let (A, d) be a chain complex. The j^{th} *homology group* $H_j(A) = H_j(A, d)$ is equal to the quotient module

$$(\text{Kernel } d_j)/(\text{Image } d_{j+1}) \ .$$

By the definitions, the sequence of morphisms determined by a chain complex (A, d) is exact at A_j if and only if $H_j(A) = 0$.

Computation of the homology groups for the examples in Section III.1 is fairly straightforward.

1. If we take an arbitrary graded module A and make it into a chain complex by taking $d_j = 0$ for all j, then $H_j(A, 0) = A_j$. If we are given a sequence of homomorphisms $f_{2j} : A_{2j} \to A_{2j-1}$ and define a chain complex whose graded module is A with $d_{2j} = f_{2j}$ and $d_{2j-1} = 0$, then $H_{2j}(A) = \text{Kernel } d_{2j}$ and $H_{2j-1}(A) = A_{2j-1}/\text{Image } d_{2j}$.

2. In Example 2 from the previous section, the homology is zero if U is a convex open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$.

3. In "ONE MORE EXAMPLE" from the previous section, we have $H_j(C) = 0$ if $j \neq 0$, while $H_0(C)$ is infinite cyclic, with the generator represented by the class of $\mathbf{x}$ (and the same generator also turns out to be represented by $\mathbf{y}$ and $\mathbf{z}$).

The next result is fairly simple to prove but absolutely fundamental.

THEOREM 1. *If $f : (A, d^A) \to (B, d^B)$ is a map of chain complexes, then there are unique homomorphisms $f_* : H_k(A) \to H_k(B)$ such that if $u \in H_k(A)$ is represented by $z \in A_q$, then $f_*(u)$ is represented by $f_q(z)$. Furthermore, if f is an identity chain map then f_* is also the identity, and if $g : (B, d^B) \to (C, d^C)$ is another chain map, then $(g \circ f)_* = g_* \circ f_*$.*

The second sentence of the theorem implies that the construction sending f to f_* defines a covariant functor from chain complexes to graded modules. Thus the following is immediate.

COROLLARY 2. *In the setting above, if f is an isomorphism then so is f_*.*■

Proof of Theorem 1. The condition in the first sentence of the theorem implies uniqueness, and the formula for f_* immediately yields the functoriality properties in the second sentence. Thus everything reduces to showing that there is indeed a homomorphism f_* with the asserted property.

First of all, we must check that $f_q(z)$ is a cycle if z is a cycle. To see this note that

$$d_q^B \circ f_q(z) \quad = \quad f_{q-1} \circ d_q^A(z) \quad = \quad f_{q-1}(0) \quad = \quad 0$$

so there is no problem here. Next, we need to check that if z and w represent the same class in A_q, then $f_q(z)$ and $f_q(w)$ represent the same class in B_q. However, it z and w represent the same class, then $z - w = d_{q+1}(y)$, and hence we have

$$f_q(z) - f_q(w) \quad = \quad f_q(z-w) \quad = \quad f_q \circ d_{q+1}^A(y) \quad = \quad d_q^B \circ f_{q+1}(y)$$

so that the images of z and w represent the same class in $H_q(B)$. The identities $f_*(u_1 + u_2) = f_*(u_1) + f_*(u_2)$ and $f_*(r \cdot u) = r \cdot f_*(u)$ now follow immediately from the definition of f_* and the standard choices of representatives for $u_1 + u_2$ and $r \cdot u$.■

III.3 : Homology and simplicial complexes

(Hatcher, § 2.1)

In this section we shall take the first step towards defining homology groups for topological spaces. At this stage we can only handle special classes of spaces with additional geometrical structures, and our definitions will also depend upon the extra structure. In fact, we shall give three different definitions of homology here, and a major objective of the rest of this unit will be to show that they are naturally equivalent. The following citation from a set of online lecture notes by J. W. Morgan (previously posted as

`http://www.math.columbia.edu/∼jm/algtop.ps`

but no longer on the Internet) summarizes the situation quite well.

> The main trouble with algebraic topology is that there are many different approaches to defining the basic ... homology ... groups. Each approach brings with it a fair amount of required technical baggage ... one must pay a fairly high price ... as one slogs through the basic constructions and proves the basic results. Furthermore, possibly the most striking feature of the subject, the interrelatedness (and often equality) of the theories ... requires even more machinery.

Some other passages from the same notes describe some reasons why such a "large and complicated array of tools" has proven to be worth knowing:

> The subject has turned out to have a vast ... range of applicability ... The power of algebraic topology is the generality of its application. The tools apply in situations so disparate as seemingly to have nothing to do with each other, yet the common thread linking them is algebraic topology. One of the most impressive arguments by analogy of twentieth century mathematics id the work of the French school of algebraic geometry, mainly [André] Weil [approximate pronunciation "VAY," 1906–1998], [Jean-Pierre] Serre [1926–], [Alexandre] Grothendieck [1929–] and [Pierre] Deligne [approximate pronunciation "de-LEEN," 1944–], to apply the machinery of algebraic topology to projective varieties defined over finite fields in order to prove the Weil Conjectures. On the face of it these conjectures, which dealt with counting the number of solutions over finite fields of polynomial equations, have nothing to do with usual topological spaces and algebraic topology. The powerful insight ... was to recognize that in fact there was a relationship and then to establish the vast array of technical results in algebraic geometry over finite fields necessary to implement this relationship. ... A quote from [Solomon] Lefschetz [1884–1972] seems appropriate to capture the spirit of the subject; after a long and complicated study ... he said, "we have succeeded in planting the harpoon of algebraic topology in the whale of algebraic geometry."

Further examples of the uses of ideas from algebraic topology are noted in the following passage:

> Let me list some of the contexts where algebraic topology is an integral part. It is related by deRham's theorem to differential forms on a manifold, by Poincaré duality to the study of intersection of cycles on manifolds, and by the Hodge theorem to periods of holomorphic differentials on complex algebraic manifolds. Algebraic topology is used to compute the infinitesimal version of the space of deformations of a complex analytic manifold (and in particular, the dimension of this space). Similarly, it is used to compute the infinitesimal space of deformations of a linear representation of a finitely presented group. In another context, it is used to compute the space of sections of a holomorphic vector bundle. In a more classical vein, it is used to compute the number of handles on a Riemann surface, estimate the number of critical points of a real-valued function on a manifold, estimate the number of fixed points of a self-mapping of a manifold, and to measure how much a vector bundle is twisted. In more algebraic contexts, algebraic topology allows one to understand short exact sequences of groups and modules over a ring, and more generally longer extensions. Lastly, algebraic topology can be used to define the cohomology groups of groups and Lie algebras, providing important invariants of these algebraic objects.

Three definitions of simplicial homology groups

One central feature of algebraic topology is that there are usually several different chain complexes which yield the same homology groups, each of which has its own advantages and disadvantages. We shall start with a definition involving a relatively small chain complex.

First Definition. Suppose that $(P, \mathbf{K})$ is a simplicial complex, and choose a linear ordering L for the vertices of $\mathbf{K}$; we shall use the usual notation $\mathbf{v} < \mathbf{w}$ to indicate that one vertex precedes another. For each integer k, the *k-dimensional ordered simplicial chain group* of $(P, \mathbf{K})$, written $C_k^{\text{ordered}}(P, \mathbf{K})$ is a free abelian group on all objects $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$, where $\mathbf{v}_0 < \ \cdots \ < \mathbf{v}_k$. By construction, it follows that $C_k^{\text{ordered}}(P, \mathbf{K}) = 0$ if $k < 0$ or $k > \dim \mathbf{K}$. The boundary homomorphism

$$d_k : C_k^{\text{ordered}}(P, \mathbf{K}) \ \longrightarrow \ C_{k-1}^{\text{ordered}}(P, \mathbf{K})$$

is defined on free generators by the formula

$$d_k(\, \mathbf{v}_0 \ \cdots \ \mathbf{v}_k \,) \quad = \quad \sum_{j=0}^{n} (-1)^j \, \mathbf{v}_0 \ \cdots \ \widehat{\mathbf{v}}_i \ \cdots \ \mathbf{v}_k$$

where $\widehat{\mathbf{v}}_i$ means that $\mathbf{v}_i$ is omitted; by the definition of free generators, it follows that there is a unique extension to the group $C_k^{\text{ordered}}(P, \mathbf{K})$.

Since our purpose is to define homology groups, presumably we want to verify that the preceding data define a chain complex. For this purpose it will be helpful to introduce some additional definitions.

If $k > 0$ and $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$ is as above, then the i^{th} **face operator** $\partial_i^{[k]}(\, \mathbf{v}_0 \ \cdots \ \mathbf{v}_k \,)$ is given by

$$\mathbf{v}_0 \ \cdots \ \widehat{\mathbf{v}}_i \ \cdots \ \mathbf{v}_k \ .$$

Frequently we shall suppress the superscript $[k]$ to simplify notation. The following identity for iterated faces is elementary but fundamentally important:

LEMMA 1. *If $k - 1 \geq j \geq i$, then $\partial_j^{[k-1]} \circ \partial_i^{[k]} = \partial_i^{[k-1]} \circ \partial_{j+1}^{[k]}$.*

The identity is true because the result of applying both composites to $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$ is given by deleting $\mathbf{v}_i$ and $\mathbf{v}_{j+1}$.■

With Lemma 1, it is fairly easy to prove that the boundary maps d_k define a chain complex.

THEOREM 2. *In the setting above we have $d_{k-1} \circ d_k = 0$.*

The proof of this result is given in Lemma 2.1 on pages 105–106 of Hatcher.■

We now define the k-dimensional simplicial homology group of $(P, \mathbf{K})$ for ordered simplicial chains, also called the k-dimensional ordered simplicial homology group and denoted by

$$H_k^{\text{ordered}}(P, \mathbf{K})$$

to be the k-dimensional homology of the chain complex $C_*^{\text{ordered}}(P, \mathbf{K})$, where the differential or boundary is given as above.

The preceding definition depends not only upon the choice of a simplicial decomposition but also upon choosing a linear ordering of the vertices. Ultimately we want to show the homology groups depend only upon the underlying space P, and as a first step we would like to prove the groups do not depend upon the choice of linear ordering. Our approach to doing will involve finding other definitions of homology that do not depend upon the choice of a vertex orderings and showing that the new definitions yield the same homology groups.

Second Definition. Given $(P, \mathbf{K})$ as above, the *unordered simplicial chain group* $C_k(P, \mathbf{K})$ is the free abelian group on all symbols $\mathbf{u}_0 \ \cdots \ \mathbf{u}_k$, where the $\mathbf{u}_j$ are all vertices of some simplex in

$\mathbf{K}$ and repetitions of vertices are allowed. A family of differential or boundary homomorphisms d_k is defined as before, and the k-dimensional simplicial homology $H_k(P, \mathbf{K})$ is defined to be the k-dimensional homology of this chain complex.

The unordered simplicial chain complex $C_*(P, \mathbf{K})$ contains the ordered simplicial chain complex $C_*^{\text{ordered}}(P, \mathbf{K})$ as a chain subcomplex, and we shall let i denote the resulting inclusion map of chain complexes. If we can show that the associated homology maps i_* are isomorphisms, then it will follow that the homology groups for the ordered simplicial chain complex agree with the corresponding groups for the unordered simplicial chain complex.

One major difference between the unordered and ordered simplicial chain groups is that the latter are nontrivial in every positive dimension. In particular, if $\mathbf{v}$ is a vertex of $\mathbf{K}$, then the free generator $\mathbf{v} \ \cdots \ \mathbf{v} = \mathbf{u}_0 \ \cdots \ \mathbf{u}_k$, with $\mathbf{u}_j = \mathbf{v}$ for all j, represents a nonzero element of $C_k(P, \mathbf{K})$. On the other hand, the ordered simplicial chain groups are nonzero for only finitely many values of k.

In order to analyze the mappings i_*, we shall introduce yet another definition of homology groups.

Third Definition. In the setting above, define the subgroup $C'_k(P, \mathbf{K})$ of *degenerate* simplicial k-chains to be the subgroup generated by

(a) all elements $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$ such that $\mathbf{v}_i = \mathbf{v}_{i+1}$ for some (at least one) i,

(b) all sums $\quad \mathbf{v}_0 \ \cdots \ \mathbf{v}_i \mathbf{v}_{i+1} \ \cdots \ \mathbf{v}_k \ + \ \mathbf{v}_0 \ \cdots \ \mathbf{v}_{i+1} \mathbf{v}_i \ \cdots \ \mathbf{v}_k, \qquad$ where $0 \leq i < k$.

We claim these subgroups define a chain subcomplex, and to show this we need to verify the following.

LEMMA 3. *The boundary homomorphism d_k sends elements of $C'_k(P, \mathbf{K})$ to $C'_{k-1}(P, \mathbf{K})$.*

It suffices to prove that the boundary map sends the previously described generators into degenerate chains, and checking this is essentially a routine calculation.∎

We now define the complex of *alternating simplicial chains* $C_*^{\text{alt}}(P, \mathbf{K})$ to be the quotient complex $C_*(P, \mathbf{K}) / C'_*(P, \mathbf{K})$ with the associated differential or boundary map.

PROPOSITION 4. *The composite $\varphi : C_*^{\text{ordered}}(P, \mathbf{K}) \to C_*(P, \mathbf{K}) \to C_*^{\text{alt}}(P, \mathbf{K})$ is an isomorphism of chain complexes.*

COROLLARY 5. *The morphism $i_* : H_*^{\text{ordered}}(P, \mathbf{K}) \to H_*(P, \mathbf{K})$ is injection onto a direct summand.*

Proof that Proposition 4 implies Corollary 5. Let q be the projection map from unordered to alternating chains, so that $\varphi_* = q_* \circ i_*$. General considerations imply that φ_* is an isomorphism.

Suppose now that $i_*(a) = i_*(b)$. Applying q_* to each side we obtain

$$\varphi_*(a) \quad = \quad q_* \circ i_*(a) \quad = \quad q_* \circ i_*(b) \quad = \quad \varphi_*(b)$$

and since φ_* is bijective it follows that $a = b$.

Now let B_* be the kernel of q_*. We shall prove that every element of $H_*(P, \mathbf{K})$ has a unique expression as $i_*(a) + c$, where $c \in B_*$. Given $u \in H_*(P, \mathbf{K})$, direct computation implies that

$$u \ - \ i_*(\varphi_*)^{-1} q_*(u) \quad \in \quad B_*$$

and thus yields existence. Suppose now that $u = i_*(a) + c$, where $c \in B_*$. It then follows from the definitions that

$$i_*(a) \quad = \quad i_*(\varphi_*)^{-1} q_*(u)$$

and hence we also have

$$c \quad = \quad u \ - \ i_*(a) \quad = \quad u \ - \ i_*(\varphi_*)^{-1} q_*(u)$$

which proves uniqueness.■

Proof of Proposition 4. Analogs of standard arguments for determinants yield the following observations:

(1) *The generator* $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k \in C_k(P, \mathbf{K})$ *lies in the subgroup of degenerate chains if two vertices are equal.*

(2) *If* σ *is a permutation of* $\{0, \ \cdots \ , k\}$, *then* $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k - (-1)^{\mathrm{sgn}(\sigma)} \mathbf{v}_{\sigma(0)} \ \cdots \ \mathbf{v}_{\sigma(k)}$ *is a degenerate chain.*

Define a map of graded abelian groups Ψ from $C_*(P, \mathbf{K})$ to $C_*^{\mathrm{ordered}}(P, \mathbf{K})$ which sends $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$ to zero if there are repeated vertices and sends $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$ to $(-1)^{\mathrm{sgn}(\sigma)} \mathbf{v}_{\sigma(0)} \ \cdots \ \mathbf{v}_{\sigma(k)}$ if the vertices are distinct and σ is the unique permutation which puts the vertices in the proper order:

$$\mathbf{v}_{\sigma(0)} \quad < \quad \cdots \quad < \quad \mathbf{v}_{\sigma(k)}$$

It follows that Ψ passes to a map ψ of quotients from $C_*^{\mathrm{alt}}(P, \mathbf{K})$ to $C_*^{\mathrm{ordered}}(P, \mathbf{K})$ such that $\psi \circ \varphi$ is the identity. In particular, it follows that φ is injective. To prove it is surjective, note that (1) and (2) imply that $C_k^{\mathrm{alt}}(P, \mathbf{K})$ is generated by the image of φ and hence φ is also surjective. It follows that φ determines an isomorphism of chain complexes as required.■

Acyclic complexes

Definition. An *augmented chain complex* over a ring R consists of a chain complex (C_*, d) and a homomorphism $\varepsilon : C_0 \to R$ (the augmentation map) such that ε is onto and $\varepsilon \circ d_1 = 0$.

All of the simplicial chain complexes defined above have canonical augmentations given by sending expressions of the form $\sum \ n_{\mathbf{v}} \, \mathbf{v}$ to the corresponding integers $\sum \ n_{\mathbf{v}}$.

Definition. A simplicial complex is said to be *acyclic* ("has no nontrivial cycles") if $H_j(P, \mathbf{K}) = 0$ for $j \neq 0$ and $H_0 \cong \mathbb{Z}$, with the generator in homology represented by an arbitrary free generator of $C_0(P, \mathbf{K})$.

There is a simple geometric criterion for a simplicial chain complexe to be acyclic.

Definition. A simplicial complex $(P, \mathbf{K})$ is said to be *star shaped* with respect to some vertex $\mathbf{v}$ in $\mathbf{K}$ if for each simplex A in $\mathbf{K}$ either $\mathbf{v}$ is a vertex of A or else there is a simplex $\mathbf{B}$ in $\mathbf{K}$ such that $\mathbf{A}$ is a face of $\mathbf{B}$ and $\mathbf{v}$ is a vertex of $\mathbf{B}$.

Some examples are described in the `figures` document. One particularly important example for the time being is the standard simplex Δ_n with its standard decomposition.

PROPOSITION 6. *If the simplicial complex* $(P, \mathbf{K})$ *is star shaped with respect to some vertex, then it is acyclic, and the map* $i_* : H_*^{\mathrm{ordered}}(P, \mathbf{K}) \to H_*(P, \mathbf{K})$ *is an isomorphism.*

Proof. Define a map of graded abelian groups $\eta : C_*(P, \mathbf{K}) \to C_*(P, \mathbf{K})$ such that $\eta_q : C_q(P, \mathbf{K}) \to C_q(P, \mathbf{K})$ is zero if $q \neq 0$ and η_0 sends a chain y to $\varepsilon(y) \, \mathbf{v}$. Then η is a chain map because $\varepsilon \circ d_1 = 0$.

We next define homomorphisms $D_q : C_q(P, \mathbf{K}) \to C_{q+1}(P, \mathbf{K})$ such that

$$d_{q+1} \circ D_q \quad = \quad \text{identity} \; - \; d_q \circ D_{q-1}$$

if q is positive and

$$d_1 \circ D_0 \quad = \quad \text{identity} \; - \; \eta_0$$

on C_0. We do this by setting $D_q(\mathbf{x}_0 \; \cdots \; \mathbf{x}_q) = \mathbf{v}\mathbf{x}_0 \; \cdots \; \mathbf{x}_q$ and taking the unique extension which exists since the classes $\mathbf{x}_0 \; \cdots \; \mathbf{x}_q$ are free generators for C_q. Elementary calculations show that the mappings D_q satisfy the conditions given above.

To see that $H_q(P, \mathbf{K}) = 0$ if $q > 0$, suppose that $d_q(z) = 0$. Then the first formula implies that $z = d_{q+1} \circ D_q(z)$. Therefore $H_q = 0$ if $q > 0$. On the other hand, if $z \in C_0$, then the second formula implies that $d_1 \circ D_0(z) = z - \varepsilon(z)\, \mathbf{v}$. Furthermore, since $\varepsilon \circ d_1 = 0$ and $d_0 = 0$, it follows that

(i) the map ε passes to a homomorphism from H_0 to $\mathbb{Z}$,

(ii) since $\varepsilon(\mathbf{v}) = 1$ this homomorphism is onto,

(iii) the multiples of the class $[\mathbf{v}]$ give all the classes in H_0.

Taken together, these imply that $H_0(P, \mathbf{K}) \cong \mathbb{Z}$, and it is generated by $[\mathbf{v}]$. This completes the computation of $H_*(P, \mathbf{K})$.

By Corollary 5 we know that $H_q^{\text{ordered}}(P, \mathbf{K})$ is isomorphic to a direct summand of $H_q(P, \mathbf{K})$ and since the latter is zero if $q > 0$ it follows that the former is also zero if $q > 0$. Similarly, we know that $H_0^{\text{ordered}}(P, \mathbf{K})$ is isomorphic to a direct summand of $H_0(P, \mathbf{K}) \cong \mathbb{Z}$. By construction we know that the generating class $[\mathbf{v}]$ for the latter lies in the image of i_*, and therefore it follows that the map from $H_0^{\text{ordered}}(P, \mathbf{K})$ to $H_0(P, \mathbf{K})$ must also be an isomorphism.■

COROLLARY 6. *If Δ is a simplex with the standard simplicial decomposition, then*

$$H_q^{\text{ordered}}(P, \mathbf{K}) \quad \cong \quad H_q(P, \mathbf{K})$$

is trivial if $q \neq 0$ and infinite cyclic if $q = 0$.■

Clearly we would like to “leverage” this result into a proof for an arbitrary simplicial complex $(P, \mathbf{K})$. This will require some additional algebraic tools, and it will be done in the next section. We shall conclude this section by using simplicial chains to solve the problem which is often viewed as the beginning of algebraic topology.

The Königsberg Bridge Problem

In this problem one has four masses of land joined by various bridges. This can be modeled by a 1-dimensional cell complex with vertices $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ representing the land masses and edges representing one bridge each from $\mathbf{w}$ to $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ along with two bridges joining $\mathbf{y}$ to each of $\mathbf{x}$ and $\mathbf{z}$. This configuation is homotopic to a simplicial comples if we add extra vertices $\mathbf{u}_1$ and $\mathbf{u}_2$ on each of the bridges joining $\mathbf{y}$ to $\mathbf{x}$ and $\mathbf{v}_1$ and $\mathbf{v}_2$ on each of the bridges joining $\mathbf{y}$ to $\mathbf{z}$. This will be our simplicial complex $(P, \mathbf{K})$, and we shall let C_* denote the ordered chain complex associated to some ordering of the vertices.

The problem is to determine whether there is a path on this complex in which each bridge is crossed exactly once, and the first step is to formulate this in terms of the chain complex C_*. What we want is a 1-chain $\sum_{\mathbf{E}} \theta(\mathbf{E})\, \mathbf{E}$, where the sum runs over all free generators of C_1 and $\theta_{\mathbf{E}} \in \{\pm 1\}$ for all $\mathbf{E}$, such that the boundary of this 1-chain has the form $\mathbf{p} - \mathbf{q}$ for two vertices in C_0 (the case $\mathbf{p} = \mathbf{q}$ is allowed). The problem is then to determine if such a 1-chain exists.

Euler's crucial insight into the problem can be stated as follows:

PROPOSITION 7. *Let $(P, \mathbf{K})$ be a 1-dimensional simplicial complex, let $\gamma \in C_1^{\text{ordered}}(P, \mathbf{K})$ be a 1-chain $\sum_{\mathbf{E}} \theta(\mathbf{E})\,\mathbf{E}$, where the sum runs over all free generators of C_1 and $\theta_{\mathbf{E}} \in \{\pm 1\}$ for all ordered edges $\mathbf{E}$, and write $d(\gamma) = \sum_{\mathbf{v}} n(\mathbf{v})\,\mathbf{v}$ for suitable integers $n(\mathbf{v})$, where the sum runs over all vertices of $(P, \mathbf{K})$. Then $n(\mathbf{v})$ is congruent modulo 2 to the number $m(\mathbf{v})$ of 1-simplices $\mathbf{E}$ that have $\mathbf{v}$ as one of their endpoints.*

Proof. The integer $n_{\mathbf{v}}$ is equal to $\sum_{\mathbf{F}} e(\mathbf{F})$, where the sum runs over all edges containing $\mathbf{v}$ as a vertex and $e(\mathbf{F}) \in \{\pm 1\}$. Since $e(\mathbf{F}) - 1$ is either equal to 0 or ± 2 for each $\mathbf{F}$, it follows that the sum of these differences, which is merely $n(\mathbf{v}) - m(\mathbf{v})$, must be a multiple of 2.■

COROLLARY 8. *In the preceding setting, if there is a 1-chain γ such that $d(\gamma) = \mathbf{p} - \mathbf{q}$, then $m_{\mathbf{v}}$ must be even if $\mathbf{v} \neq \mathbf{p}, \mathbf{q}$.*■

The impossibility of finding a suitable 1-chain for our Königsberg bridge network now follows by observing that $m = 3$ for $\mathbf{w}$, $\mathbf{x}$ and $\mathbf{z}$, while $m = 5$ for $\mathbf{y}$. In particular, if γ is a chain as in the statement of the theorem, then in $d(\gamma)$ the coefficients of all four of these vertices must be nonzero.■

It is left as an exercise for the reader to show that the homology groups of this simplicial complex are given by $H_1 \cong \mathbb{Z}^4$ and $H_0 \cong \mathbb{Z}$. This is essentially an exerise in linear algebra (however, the scalars here are integers rather than elements of some field).

III.4 : Comparison principles

(Hatcher, §§ 2.1 – 2.2)

We have already stated the goal of proving that the mappings i_* define isomorphisms from $H_*^{\mathrm{ordered}}(P, \mathbf{K})$ to $H_*(P, \mathbf{K})$ for every finite simplicial complex $(P, \mathbf{K})$. The proof of this requires some purely algebraic theorems involving large commutative diagrams, and the results involve a technique known as *diagram chasing*. We shall begin with a simple observation and a related question.

PROPOSITION 1. (Effaceability Property) *If (A, d) is a chain complex and $u \in H_k(A)$ for some k, then there is a chain complex (B, d') containing (A, d) as a chain subcomplex and the inclusion map $i : A \to B$ satisfies $i_*(u) = 0$.*

Proof. Define $B_q = A_q$ if $q \neq k+1$, set $B_{k+1} = A_{k+1} \oplus R$, where R is the ring for the underlying category of modules, and define $d'_q = d_q$ if $q \neq k+1$ with $d_{k+1}(a, r) = d_{k+1}(a) + rz$, where z is a cycle representing u. There is an obvious inclusion of chain complexes which is the identity in degrees $\neq k+1$ and is given in the remaining case by $i_{k+1}(a) = (a, 0)$, It is then straightforward to verify that the conclusion of the proposition is true.■

The preceding result leads naturally to the following question:

> *If $i : A \to B$ defines an inclusion of chain complexes, how can we analyze the kernel and cokernel of i_* in a relatively effective manner?*

As in many other instances, the answer to this question involves some additional constructions. Let $A \subset B$ be a chain complex inclusion, and consider the quotient complex B/A; let $i : A \to B$ denote the inclusion map, and let $j : B \to A/B$ denote the projection. We then have the following result:

PROPOSITION 2. *Let $i : A \to B$ and $j : A \to A/B$ be injection and projection maps of chain complexes as above. Then for each k there is a homomorphism $\partial : H_k(B/A) \to H_{k-1}(A)$ defined as follows: If $u \in H_k(B/A)$ and $x \in B_k$ is such that $j(x)$ represents u, then $\partial(u)$ is represented by $y \in A_{k-1}$ such that $i(y) = d(x)$. Furthermore, if we are given a second pair $i' : A' \to B'$ and $j' : B' \to B'/A'$ as above and a chain map $f : B \to B'$ such that f maps A to A' by a chain map g and $h : B/A \to B'/A'$ is the map given by passage to quotients, then the corresponding homomorphisms ∂ and ∂' satisfy $g_* \circ \partial = \partial' \circ h_*$.*

Proof. First of all, we should check that the definition makes sense. The first step in doing so is to verify that if we are given x there is always a suitable choice of y. In general the class x need not be a cycle, but we know that $j(x)$ is a cycle representing u, and therefore $0 = d \circ j(x) = j \circ d(x)$, which means that $d(x) = i(a)$ for some a. This element is a cycle; we know that $d(a) = 0$ if and only if $i \circ d(a) = 0$, and since $i \circ d(a) = d \circ i(a) = d \circ d(x) = 0$, it does follow that $d(a) = 0$ as required.

Next, we need to check that the construction is well defined when one passes to homology. Suppose that $j(x)$ and $j(x')$ represent the same class in $H_k(B/A)$. It then follows that $j(x - x')$ is a boundary, which means there is some $w \in B_{k+1}$ such that $d(w) - (x - x')$ lies in A, which is the image of i. Express the difference element as $i(z)$; then we have

$$i(dz) \quad = \quad d(iz) \quad = \quad d\big(\, d(w) - (x - x')\,\big) \quad = \quad d(x') - d(x)$$

so that $d(x) = i(a)$ and $d(x') = i(a')$ imply that $a' - a = d(z)$.

Next, we need to check that ∂ is a module homomorphism. Given classes u and u' represented by x and x', it follows that $x + x'$ represents $u + u'$, while $d(x) = i(a)$ and $d(x') = i(a')$ imply $d(x + x') = i(a + a')$. Thus $a + a'$ represents $u + u'$, showing that ∂ is additive. If $r \in R$, then similar considerations show that $\partial(r \cdot u)$ is represented by $r \cdot a$, and therefore ∂ is compatible with scalar multiplication.

Finally, suppose we have chain maps as described in the proposition, let $u \in H_k(B/A)$, and let $x \in B_k$ be such that $j(x)$ represents u. Then a representative for $g_*\partial(u)$ is given by $g(a)$, where $ia = dx$, while a representative for $\partial' h_*(u)$ is given by z such that $i'(z) = d'f(x)$. The right hand side equals $f \circ d(x) = f \circ i(a) = i' \circ g(a)$, and therefore we see that $z = g(a)$, which means that $g_*\partial(u) = \partial' h_*(u)$ as desired.■

We may now state and prove the following basic result:

THEOREM 3. (Long Exact Homology Sequence Theorem — Algebraic Version). *Let $i : A \to B$ and $j : A \to A/B$ be injection and projection maps of chain complexes as above. Then there is a long exact sequence of homology groups as follows:*

$$\cdots \quad H_{k+1}(B/A) \xrightarrow{\partial} H_k(A) \xrightarrow{i_*} H_k(B) \xrightarrow{j_*} H_k(B/A) \xrightarrow{\partial} H_{k-1}(A) \quad \cdots$$

This sequence extends indefinitely to the left and right. Furthermore, if we are given chain maps f, g and h as in Proposition 2, then we have the following commutative diagram in which the two rows are exact:

$$\begin{array}{ccccccccccc}
\cdots & H_{k+1}(B/A) & \xrightarrow{\partial} & H_k(A) & \xrightarrow{i_*} & H_k(B) & \xrightarrow{j_*} & H_k(B/A) & \xrightarrow{\partial} & H_{k-1}(A) & \cdots \\
\cdots & \downarrow h_* & & \downarrow g_* & & \downarrow f_* & & \downarrow h_* & & \downarrow g_* & \cdots \\
\cdots & H_{k+1}(B'/A') & \xrightarrow{\partial'} & H_k(A') & \xrightarrow{i'_*} & H_k(B') & \xrightarrow{j'_*} & H_k(B'/A') & \xrightarrow{\partial'} & H_{k-1}(A') & \cdots
\end{array}$$

A proof of this theorem appears on page 117 of Hatcher.■

Application to simplicial complexes

In order to apply the preceding algebraic results, we need to define *relative homology groups* associated to a *simplicial complex pair*

$$\big(\ (P, \mathbf{K}),\ (Q, \mathbf{L})\ \big)$$

consisting of a simplicial complex $(P, \mathbf{K})$ and a subcomplex $(Q, \mathbf{L})$. To simplify notation, we shall usually denote such a pair by $(\mathbf{K}, \mathbf{L})$.

Definition. In the setting above the *relative simplicial chain groups*, denoted by $C_*^{\mathrm{ordered}}(\mathbf{K}, \mathbf{L})$ and $C_*(\mathbf{K}, \mathbf{L})$, are respectively given by the corresponding quotient complexes

$$C_*^{\mathrm{ordered}}(\mathbf{K})/C_*^{\mathrm{ordered}}(\mathbf{L}) \quad \text{and} \quad C_*(\mathbf{K})/C_*(\mathbf{L})\ .$$

Since the chain complex mappings from ordered to unordered chains send ordered chains on $\mathbf{L}$ to unordered chains on $\mathbf{L}$, it follows that there are canonical homomorphisms

$$\varphi : C_*^{\mathrm{ordered}}(\mathbf{K})/C_*^{\mathrm{ordered}}(\mathbf{L}) \longrightarrow C_*(\mathbf{K})/C_*(\mathbf{L})$$

defined by passage to quotients. The *relative simplicial homology groups*, denoted by $H_*^{\text{ordered}}(\mathbf{K},\mathbf{L})$ and $H_*(\mathbf{K},\mathbf{L})$ respectively, are the homlogy groups of the associated chain complexes; by the preceding sentence, we have canonical homomorphisms from the relative homology groups for ordered chains to the relative homology groups for unordered chains. We should also note that the previously defined absolute chain groups may be viewed as special cases of this definition where $\mathbf{L} = \emptyset$.

By Theorem 3 above, we have the following result:

THEOREM 4. (Long Exact Homology Sequence Theorem — Simplicial Version). *Let $i : \mathbf{L} \to \mathbf{K}$ denote a simplicial subcomplex inclusion. Then there are long exact sequences of homology groups, and they fit into the following commutative diagram, in which the rows are exact and the horizontal arrows represent the canonical maps from ordered to unordered chains:*

$$\begin{array}{cccccccccccc}
\cdots & H_{k+1}^{\text{ord}}(\mathbf{K},\mathbf{L}) & \stackrel{\partial}{\longrightarrow} & H_k^{\text{ord}}(\mathbf{L}) & \stackrel{i_*}{\longrightarrow} & H_k^{\text{ord}}(\mathbf{K}) & \stackrel{j_*}{\longrightarrow} & H_k^{\text{ord}}(\mathbf{K},\mathbf{L}) & \stackrel{\partial}{\longrightarrow} & H_{k-1}^{\text{ord}}(\mathbf{L}) & \cdots \\
\cdots & \Big\downarrow \varphi_* & & \Big\downarrow \varphi_* & & \Big\downarrow \varphi_* & & \Big\downarrow \varphi_* & & \Big\downarrow \varphi_* & \\
\cdots & H_{k+1}(\mathbf{K},\mathbf{L}) & \stackrel{\partial}{\longrightarrow} & H_k(\mathbf{L}) & \stackrel{i_*}{\longrightarrow} & H_k(\mathbf{K}) & \stackrel{j_*}{\longrightarrow} & H_k(\mathbf{K},\mathbf{L}) & \stackrel{\partial}{\longrightarrow} & H_{k-1}(\mathbf{L}) & \cdots
\end{array}$$

This follows immediately from the definitions and Theorem 3.■

The Five Lemma

Theorem 4 provides one fundamental piece of algebraic input which is needed to show that ordered simplicial chains and unordered simplicial chains define isomorphic homology groups. Another is given by the following result:

PROPOSITION 5. *Suppose we are given a commutative diagram of modules as below in which the rows are exact and the horizontal maps a, b, d and e are isomorphisms. Then the mapping c is also an isomorphism:*

$$\begin{array}{ccccccccccc}
\cdots & A & \stackrel{f_*}{\longrightarrow} & B & \stackrel{g_*}{\longrightarrow} & C & \stackrel{j_*}{\longrightarrow} & D & \stackrel{k_*}{\longrightarrow} & E \\
 & \Big\downarrow a & & \Big\downarrow b & & \Big\downarrow c & & \Big\downarrow d & & \Big\downarrow e \\
 & A' & \stackrel{f'_*}{\longrightarrow} & B' & \stackrel{g'_*}{\longrightarrow} & C' & \stackrel{h'_*}{\longrightarrow} & D' & \stackrel{k'_*}{\longrightarrow} & E'
\end{array}$$

A proof of this theorem appears on page 129 of Hatcher.■

The isomorphism theorem

Here is the result that has been our main objective:

THEOREM 6. *If $(\mathbf{K},\mathbf{L})$ is a simplicial complex pair, then the canonical map*

$$\varphi_* : H_*^{\text{ordered}}(\mathbf{K},\mathbf{L}) \to H_*(\mathbf{K},\mathbf{L})$$

is an isomorphism.

Proof. Consider the following statements:

> $(\mathbf{X}_n)$ *The map φ above is an isomorphism for all simplicial complex pairs $(\mathbf{K},\mathbf{L})$ such that* $\dim \mathbf{K} \leq n$.

($\mathbf{Y}_{n+1}$) *The map* φ *above is an isomorphism for all* $(\mathbf{K},\mathbf{L})$ *such that* $\dim \mathbf{K} \leq n$ *and also for* $(\Delta_{n+1}, \partial\Delta_{n+1})$.

($\mathbf{W}_{n+1,m}$) *The map* φ *above is an isomorphism for all* $(\mathbf{K},\mathbf{L})$ *such that* $\dim \mathbf{K} \leq n$ *and also for all* $(\mathbf{K},\mathbf{L})$ *such that* $\dim \mathbf{K} \leq n+1$ *and* $\mathbf{K}$ *has at most* m *simplices of dimension equal to* $n+1$.

The theorem is then established by the following double inductive argument:

[F] The statement ($\mathbf{X}_0$) and the equivalent statement ($\mathbf{W}_{1,0}$) are true.

[G] For all nonnegative integers n, the statement ($\mathbf{X}_n$) implies ($\mathbf{Y}_{n+1}$).

[K] For all nonnegative integers n and m, the statements ($\mathbf{W}_{n+1,m}$) and ($\mathbf{Y}_{n+1}$) imply ($\mathbf{W}_{n+1,m+1}$).

Since statement ($\mathbf{X}_n$) is true if and only if ($\mathbf{W}_{n,m}$) is true for all m, and the latter are all true if and only if ($\mathbf{W}_{n+1,0}$) is true, we also have the following:

[L] For all n the statements ($\mathbf{X}_n$) $\Longleftrightarrow$ ($\mathbf{W}_{n+1,0}$) and ($\mathbf{Y}_{n+1}$) imply ($\mathbf{W}_{n+1,m}$) for all m, and hence ($\mathbf{X}_n$) implies ($\mathbf{X}_{n+1}$).

Therefore ($\mathbf{X}_n$) is true for all n, and this is the conclusion of the theorem.

Proof of [F]. By the Five Lemma it suffices to prove the result when $\mathbf{L}$ is empty. Since the 0-dimensional complex determined by $\mathbf{K}$ is merely a finite set of vertices, write these vertices as $\mathbf{w}_1, \cdots \mathbf{w}_m$. We then have canonical chain complex isomorphisms

$$\bigoplus_{j=1}^{m} C_*^{\text{ordered}}(\{\mathbf{w}_j\}) \longrightarrow C_*^{\text{ordered}}(\mathbf{K}) \, , \qquad \bigoplus_{j=1}^{m} C_*(\{\mathbf{w}_j\}) \longrightarrow C_*(\mathbf{K})$$

and these pass to homology isomorphisms

$$\bigoplus_{j=1}^{m} H_*^{\text{ordered}}(\{\mathbf{w}_j\}) \longrightarrow H_*^{\text{ordered}}(\mathbf{K}) \, , \qquad \bigoplus_{j=1}^{m} H_*(\{\mathbf{w}_j\}) \longrightarrow H_*(\mathbf{K}) \, .$$

These maps commute with the homomorphisms φ_* sending ordered to unordered chains. and since the maps φ_* are isomorphisms for one point complexes (= 0-simplices), it follows that φ defines an isomorphism from $H_*^{\text{ordered}}(\mathbf{K})$ to $H_*(\mathbf{K})$. The completes the proof of ($\mathbf{X}_0$).

Proof of [G]. By ($\mathbf{X}_n$) we know that φ_* is an isomorphism for the complex $\partial\Delta_{n+1}$. Since φ_* is also an isomorphism for Δ_{n+1} by Corollary III.3.6. Therefore the Five Lemma implies that φ_* is an isomorphism for $(\Delta_{n+1}, \partial\Delta_{n+1})$.

Proof of [K]. This is the crucial step. Let $\mathbf{K}$ be an $(n+1)$-dimensional complex, and let $\mathbf{M}$ be a subcomplex obtained by removing exactly one $(n+1)$-simplex from $\mathbf{K}$, so that φ_* is an isomorphism for $\mathbf{M}$ by the inductive hypothesis. If we can show that φ_* is an isomorphism for $(\mathbf{K},\mathbf{M})$, then it will follow that φ_* is an isomorphism for $\mathbf{K}$, and the relative case will the follow from the Five Lemma.

Let $\mathbf{S}$ be the extra simplex of $\mathbf{K}$ and let $\partial\mathbf{S}$ be its boundary. Then there are canonical isomorphism from the chain groups of Δ_{n+1}, $\partial\Delta_{n+1}$ and $(\Delta_n, \partial\Delta_{n+1})$ to the chain groups of $\mathbf{S}$, $\partial\mathbf{S}$

and $(\mathbf{S}, \partial\mathbf{S})$. We then have the following commutative diagram, in which the morphisms α and β are determined by subcomplex inclusions:

$$\begin{array}{ccc} C_*^{\text{ordered}}(\mathbf{S}, \partial\mathbf{S}) & \xrightarrow{\alpha} & C_*^{\text{ordered}}(\mathbf{S}, \partial\mathbf{S}) \\ \downarrow \varphi(\mathbf{S}, \partial\mathbf{S}) & & \downarrow \varphi(\mathbf{K}, \mathbf{M}) \\ C_*^{\text{ordered}}(\mathbf{S}, \partial\mathbf{S}) & \xrightarrow{\beta} & C_*^{\text{ordered}}(\mathbf{S}, \partial\mathbf{S}) \end{array}$$

We CLAIM that α and β are isomorphisms of chain complexes. For the mapping α, this follows because the relative ordered chain groups of a pair $(\mathbf{T}, \mathbf{T}_0)$ are free abelian groups on the simplices in $\mathbf{T} - \mathbf{T}_0$, and each of the sets $\mathbf{S} - \partial\mathbf{S}$ and $\mathbf{K} - \mathbf{M}$ is given by the same $(n+1)$-simplex. For the mapping β, this follows because the relative unordered chain groups of a pair $(\mathbf{T}, \mathbf{T}_0)$ are free abelian groups on the generators $\mathbf{v}_0 \ \cdots \ \mathbf{v}_k$, where the $\mathbf{v}_j$ are vertices of a simplex that is in $\mathbf{T}$ but not in $\mathbf{T}_0$ (with repetitions allowed as usual), and once again these free generators are identical for te pairs $(\mathbf{S}, \partial\mathbf{S})$ and $(\mathbf{K}, \mathbf{M})$ because $\mathbf{S} - \partial\mathbf{S}$ and $\mathbf{K} - \mathbf{M}$ are the same.

By $(\mathbf{Y}_{n+1})$ we know that $\varphi(\mathbf{S}, \partial\mathbf{S})$ defines an isomorphism in homology, and therefore it follows that the homology map

$$\varphi(\mathbf{K}, \mathbf{M})_* \quad = \quad \beta_* \circ \varphi(\mathbf{S}, \partial\mathbf{S})_* \circ \alpha_*^{-1}$$

also defines an isomorphism in homology. We can now use the Five Lemma and $(\mathbf{W}_{n+1,m})$ to conclude that the map $\varphi(\mathbf{K})$ defines an isomorphism in homology, and finally we can use the Five Lemma once more to see that the statement $(\mathbf{W}_{n+1,m+1})$ is true. This completes the proof of [K], and as noted above it also yields [L] and the theorem.■

The preceding result can be reformulated in an abstract setting that will be needed later. We begin by defining a category **SCPairs** whose objects are pairs of simplicial complexes $(\mathbf{K}, \mathbf{K}_0)$ and whose morphisms are given by subcomplex inclusions $(\mathbf{L}, \mathbf{L}_0) \subset (\mathbf{K}, \mathbf{K}_0)$; in other words, $\mathbf{L}_0$ is a subcomplex of both $\mathbf{L}$ and $\mathbf{K}_0$ while $\mathbf{L}$ is also a subcomplex of $\mathbf{K}$. A *homology theory* on this category is a covariant functor h_* valued in some category of modules together with a natural transformation

$$\partial(\mathbf{K}, \mathbf{L}) : h_*(\mathbf{K}, \mathbf{L}) \ \longrightarrow \ h_{*-1}(\mathbf{L})$$

such that

- (a) one has long exact homology sequences,
- (b) if $\mathbf{K}$ is a simplex and $\mathbf{v}$ is a vertex of $\mathbf{K}$ then $h_*(\{\mathbf{v}\}) \to h_*(\mathbf{K})$ is an isomorphism,
- (c) if $\mathbf{K}$ is 0-dimensional with vertices $\mathbf{v}_j$ then the associated map from $\oplus_j \ h_j(\{\mathbf{v}_j\})$ to $h_*(\mathbf{K})$ is an isomorphism,
- (d) if $\mathbf{K}$ is obtained from $\mathbf{M}$ by adding a single simplex $\mathbf{S}$, then $h_*(\mathbf{S}, \partial\mathbf{S}) \to h_*(\mathbf{M}, \mathbf{K})$ is an isomorphism,
- (d) if $\mathbf{K}$ is complex consisting only of a single vertex then $h_0(\mathbf{K})$ is the underlying ring R and $h_j(\mathbf{K}) = 0$ if $j \neq 0$.

A *natural transformation* from one such theory (h_*, ∂) to another (h'_*, ∂') is a natural transformation of θ of functors that is compatible with the mappings ∂ and ∂'; specifically, we want

$$\theta(\mathbf{L}) \circ \partial \quad = \quad \partial' \circ \theta(\mathbf{K}, \mathbf{L}) \ .$$

These conditions imply the existence of a commutative ladder diagram as in Theorem 4, where the rows are the long exact sequences determined by the two abstract homology theories. The definition is set up so that the proof of the next result is formally parallel to the proof of Theorem 6:

THEOREM 7. *Suppose we are given a natural transformation of homology theories θ as above such that $\theta(\mathbf{K})$ is an isomorphism if $\mathbf{K}$ consists of just a single vertex. Then $\theta(\mathbf{K}, \mathbf{L})$ is an isomorphism for all pairs $(\mathbf{K}, \mathbf{L})$.*■

Application to barycentric subdivisions

We shall now use the preceding results to show that the homology groups of a barycentric subdivision $B(\mathbf{K})$ are isomorphic to the homology groups of the original complex $\mathbf{K}$. In this case the homology theories will be $H_*^{\text{ordered}}(\mathbf{K}, \mathbf{L})$ and $H_*^{\text{ordered}}\big(B(\mathbf{K}), B(\mathbf{L})\big)$, and the natural transformation will be associated to maps defined on the chain level. It will suffice to define these chain maps for a simplex and to extend to arbitrary complexes and pairs by putting things together in an obvious manner.

PROPOSITION 8. *Given a nonnegative integer n, let $\partial_j : \Delta_{n-1} \to \Delta_n$ be the order preserving affine map sending Δ_{n-1} to the face of Δ_n opposite the j^{th} vertex, and let $(\delta_j)_\#$ generically denote an associated chain map. Then there are classes $\beta_n \in C_n^{\text{ordered}}(\Delta_n)$ such that β_0 is just the standard generator and if $n > 0$ then*

$$d_n(\beta_n) \;=\; \sum_{j=0}^{n} (-1)^j (\partial_j)_\#(\beta_{n-1}) \ .$$

Proof. Since Δ_n is acyclic, it suffices to show that the right hand side lies in the kernel of d_{n-1} if $n > 1$ and in the kernel of ε if $n = 1$. Both of these are routine (but tedious) calculations.■

Using the chains β_n one can piece together chain maps

$$C_*^{\text{ordered}}(\mathbf{K}, \mathbf{L}) \;\longrightarrow\; C_*^{\text{ordered}}\big(B(\mathbf{K}), B(\mathbf{L})\big) \ .$$

We claim these define a natural transformation of homology theories, but in order to do this we must first show that $H_*^{\text{ordered}}\big(B(\mathbf{K}), B(\mathbf{L})\big)$ actually defines a homology theory. Properties (a), (c) and (e) follow directly from the construction. Property (b) follows because $B(\Delta_n)$ is star shaped with respect to the vertex $\mathbf{b}$ given by the barycenter of Δ_n. Thus it only remains to verify property (d); in fact, direct inspection similar to an argument in the proof of Theorem 6 shows that the map on the chain level is an isomorphism.

By Theorem 7, it suffices to check that the natural transformation of homology theories is an isomorphism for a simplicial complex consisting of a single vertex; in fact, for such complexes the map is already an isomorphism on the chain level. Therefore the barycentric subdivision chain maps determine isomorphism of homology groups as asserted in the proposition.■

III.5 : Chain homotopies

(Hatcher, § 2.1)

In this section we shall generalize a key step in the proof of Proposition III.3.6. Recall that the latter gives the homology $H_*(\mathbf{K})$ if $\mathbf{K}$ is star shaped with respect to some vertex $\mathbf{v}$, and it does so by constructing an algebraic analog of the straight line contracting homotopy from the identity to the constant map whose value is $\mathbf{v}$.

Definition. Let (A, d) and (B, e) be chain complexes, and let f and g be chain maps from A to B. A *chain homotopy* from f to g is a sequence of mappings $d_k : A_k \to B_{k+1}$ satisfying the following condition for all integers k:

$$d^B_{k+1} \circ D_k \ + \ D_{k-1} \circ d^A_k \ = \ g_k \ - \ f_k$$

Two chain mappings f, g from A to B are said to be *chain homotopic* if there is a chain homotopy from the first to the second, and this is often written $f \simeq g$.

The proof of the following result is an elementary exercise:

PROPOSITION 1. *The relation $\simeq$ is an equivalence relation on chain maps from one chain complex (A, d) to another (B, e). Furthermore, if f and g are chain homotopic chain maps from (A, d) to (B, e), and h and k are chain homotopic chain maps from (B, e) to (C, θ), then the composites $h \circ f$ and $k \circ g$ are also chain homotopic. Finally, if f, g, h, k are chain maps from A to B and $r \in R$, then $f \simeq g$ and $h \simeq k$ imply $f + h \simeq g + k$ and $rf \simeq rg$.*

Proof. For the first part of the proof let f, g and h be chain maps from (A, d) to (B, e). The zero homomorphisms define a chain homotopy from f to itself. If D is a chain homotopy from f to g then $-D$ is a chain homotopy from g to f. Finally, if D is a chain homotopy from f to g and E is a chain homotopy from g to h, then $D + E$ is a chain homotopy from f to h.

To prove the assertion in the second sentence, let D be a chain homotopy from f to g and let E be a chain homotopy from g to h. Then one can check directly that

$$h \circ D \ + \ E \circ g$$

defines a chain homotopy from $h \circ f$ to $k \circ g$.

The proof of the final assertion is also elementary and is left to the reader.■

Chain homotopies are useful and important because of the following result:

PROPOSITION 2. *If f and g are chain homotopic chain maps from one chain complex (A, d) to another complex (B, e), then the associated homology mappings f_* and g_* are equal.*

Proof. Suppose that $u \in H_k(A)$ and $x \in A_k$ is a cycle representing u, so that $d_k(a) = 0$. If D is a chain homotopy from f to gh, then by definition we have

$$d^B_{k+1} \circ D_k(x) \ + \ D_{k-1} \circ d^A_k \ (x) \ = \ g_k(x) \ - \ f_k(x)$$

and since $d^A_k(x) = 0$ it follows that the expression above is a boundary. Therefore $g_*(u) - f_*(u)$ must be the zero element of $H_k(B)$.■

An important example

The following basic construction gives an explicit connection between the topological notion of homotopy and the algebraic notion of chain homotopy. Let $n \geq 0$, and let $\mathbf{P}_{n+1}$ denote the *standard* $(n+1)$*-dimensional prism* $\Delta_n \times [0, 1]$ with the simplicial decomposition given in Unit II. As in that unit, label the vertices of this prism decomposition by $\mathbf{x}_j = (\mathbf{e}_j, 0)$ and $\mathbf{y}_j = (\mathbf{e}_j, 1)$.

PROPOSITION 3. *The simplicial chain complexes $C^{\text{ordered}}_*(\mathbf{P}_{n+1})$ and $C_*(\mathbf{P}_{n+1})$ are acyclic.*

Proof. These follow from the isomorphism theorem and the fact that $\mathbf{P}_{n+1}$ is star shaped with respect to $\mathbf{y}_n$.■

For each integer j satisfying $0 \leq j \leq n$, let $\partial_j : \Delta_{n-1} \to \Delta_n$ be the affine map which sends Δ_{n-1} to the face opposite the vertex $\mathbf{e}_j$ and is order preserving on the vertices, and let $\partial_j \times \mathbf{I}$ denote the product of the map ∂j with the identity on $[0, 1]$. It then follows immediately that we have associated injections of simplicial chain groups

$$(\partial_j)_\# : C_j(\Delta_{n-1}) \longrightarrow C_j(\Delta_n) \ , \qquad (\partial_j \times \mathbf{I})_\# : C_*(\mathbf{P}_{n-1}) \longrightarrow C_*(\mathbf{P}_n)$$

and these are chain maps. Furthermore, these chain maps send ordered chains to ordered chains.

Similarly, for $t = 0, 1$ we also have injections of simplicial chain groups

$$(i_t)_\# : C_*(\Delta_n) \longrightarrow C_*(\mathbf{P}_n)$$

which send a free generator $\mathbf{v}_0 \ \cdots \ \mathbf{v}_q$ to $i_t(\mathbf{v}_0) \ \cdots \ i_t(\mathbf{v}_q)$, where $i_t(\mathbf{v}) = (\mathbf{v}, t)$.

We then have the following result:

THEOREM 4. *For all $n \geq 0$ there are chains $P_{n+1} \in C_{n+1}^{\text{ordered}}(\mathbf{P}_n)$ such that*

$$d_{n+1}(P_{n+1}) \ = \ \mathbf{y}_0 \ \cdots \ \mathbf{y}_n \ - \ \mathbf{x}_0 \ \cdots \ \mathbf{x}_n \ - \ \sum_j \ (-1)^j \ (\partial_j \times \mathbf{I})_\#(P_{n-1}) \ .$$

Sketch of proof. Not surprisingly, the construction is inductive, with $P_0 = 0$. Suppose we have constructed the chains P_j for $j \leq n$. There is a chain P_{n+1} with the required properties if and only if the expression on the right hand side of the equation is a cycle, so we need to show that the right hand side vanishes if we apply d_n. This is a straightforward but messy calculation like several previous ones. Some key details are worked out in the bottom half of page 112 of Hatcher.■

The preceding result implies that the inclusion mappings i_t, which are topologically homotopic, determine algebraic chain maps that are chain homotopic. Specifically, if we are given a free generator $\mathbf{v}_0 \ \cdots \ \mathbf{v}_q$ of $C_q(\Delta_n)$ then we may form a chain

$$D_q(\mathbf{v}_0 \ \cdots \ \mathbf{v}_q) \ \in \ C_{q+1}(\Delta_n \times \mathbf{I})$$

by substituting $i_0(\mathbf{v})$ for $\mathbf{x}$ and $i_1(\mathbf{v})$ for $\mathbf{y}$. In fact, one can carry out all of this for an arbitrary simplicial complex $(P, \mathbf{K})$, and one has the following conclusion.

PROPOSITION 5. *In the setting above the maps $(i_0)_\#$ and $(i_1)_\#$ from $C_*(\mathbf{K})$ to $C_*(\mathbf{K} \times \mathbf{I})$ are chain homotopic, and hence the associated homology maps*

$$(i_0)_*, \ (i_1)_* \ : \ H_*(\mathbf{K}) \ \longrightarrow \ H_*(\mathbf{K} \times \mathbf{I})$$

are equal.■

IV. Singular homology

In Section III.4 we showed that the homology groups of a simplicial complex are the same up to isomorphism if one replaces a given simplicial decomposition with its barycentric subdivision. Of course, one can iterate this, and if one considers further examples it becomes natural to ask whether the homology groups only depend upon the underlying topological space. The results of this unit yield a very strong affirmative answer to this question. In particular, we shall define analogs of simplicial chain complexes and homology groups for arbitrary topological spaces in a manner that only involves the spaces themselves. It took about a half century for mathematicians to come up with the formulation that is now standard, starting with Poincaré's initial papers on topology (which he called *analysis situs*) at the end of the 19^{th} century and culminating with the definition of *singular homology* by S. Eilenberg and N. Steenrod in the nineteen forties (with many important contributions by others along the way).

Some books start directly with singular homology and do not bother to develop simplicial homology. The reason for considering the latter here is that it is in some sense a "toy model" of singular homology for which many basic ideas appear in a more simplified framework.

IV.1 : Definitions

(Hatcher, § 2.1)

As before, let Δ_q be the standard q-simplex in $\mathbb{R}^{q+1}$ whose vertices are the standard unit vectors $\mathbf{e}_0, \cdots, \mathbf{e}_q$. If $(P, \mathbf{K})$ is a simplicial complex, then for each free generator $\mathbf{v}_0 \cdots \mathbf{v}_q$ of $C_q(P, \mathbf{K})$ there is a unique affine (hence continuous) map $T : \Delta_q \to P$ which sends a point $(t_0, \cdots, t_q) \in \Delta_{q+1}$ to $\sum_j t_j \mathbf{v}_j \in P$. One can think of these as linear simplices in P. The idea of singular homology is to consider more general continuous mappings from Δ_q to a space X, viewing them as simplices with possible singularities or *singular simplices* in the space.

Definition. Let X be a topological space. A *singular q-simplex* in X is a continuous mapping $T : \Delta_q \to X$, and the abelian group of *singular q-chains* $S_q(X)$ is defined to be the free abelian group on the set of singular q-simplices.

If we let $\partial_j : \Delta_{q-1} \to \Delta_q$ be the affine map which sends Δ_{q-1} to the face opposite the vertex $\mathbf{e}_j$ and is order preserving on the vertices, then as in the case of simplicial chains we have boundary homomorphisms $d_q : S_q(X) \to S_{q-1}(X)$ given on generators by the standard formula:

$$d_q(T) \quad = \quad \sum_{j=0}^{n} (-1)^i \partial_i(T) \quad = \quad \sum_{j=0}^{n} (-1)^i T \circ \partial_i$$

Likewise, there are augmentation maps $\varepsilon : S_0(X) \to \mathbb{Z}$ which send each free generator $T : \Delta_0 \to X$ to $1 \in \mathbb{Z}$.

We then have the following result:

PROPOSITION 1. *The homomorphisms d_q make $S_*(X)$ into a chain complex, and if $(P, \mathbf{K})$ is a simplicial complex, then the affine map construction makes $C_*(P, \mathbf{K})$ into a chain subcomplex* *of $S_q(P)$, and the inclusion is augmentation preserving. Furthermore, if A is a subset of X, then $S_*(A)$ is canonically identified with a subcomplex of $S_*(X)$ by the map taking $T : \Delta_q \to X$ into $i \circ T : \Delta_q \to X$, where $i : A \to X$ is the inclusion mapping.*■

Definition. If X is a topological space, then the *singular homology groups* $H_*(X)$ are the corresponding homology groups of the chain complex defined by $S_*(X)$. More generally, if A is a subset of X, then the *relative chain complex* $S_*(X, A)$ is defined to be $S_*(X)/S_*(A)$, and the *relative singular homology groups* $H_*(X, A)$ are the corresponding homology groups of that quotient complex. Note that if $(\mathbf{K}, \mathbf{L})$ is a pair consisting of a simplicial complex and a subcomplex with underlying space pair (P, Q), then Proposition 1 generalizes to yield a chain map from $C_*(\mathbf{K}, \mathbf{L})$ to $S_*(P, Q)$. — Note that the relative groups do not have augmentation homomorphisms (provided $A \neq \emptyset$).

It is not difficult to show that the singular homology groups of homeomorphic spaces are isomorphic, and in fact it is an immediate consequence of the following results:

PROPOSITION 2. *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous map. Then there is a chain map $f_\#$ from $S_*(X)$ to $S_*(Y)$ such that for each singular q-simplex T the value $f_\#(T)$ is given by $f \circ T$. This construction transforms the singular chain complex construction into a covariant functor from topological spaces and continuous maps to chain complexes (and chain maps).*

This is essentially an elementary verification, and probably the most noteworthy part is the need to verify that $f_\#$ is a chain map. Details are left to the reader.■

COROLLARY 3. *If X and Y are topological spaces and $f : X \to Y$ is a homeomorphism, then the associated homomorphism of graded homology groups $f_* : H_*(X) \to H_*(Y)$ is an isomorphism.*■

By Corollary 3, the simplicial homology groups of homeomorphic polyhedra will be isomorphic if we can give an affirmative answer to the following question for all simplicial complexes $(P, \mathbf{K})$:

PROBLEM. If $(P, \mathbf{K})$ is a simplicial complex and $\lambda : C_*(\mathbf{K}) \to S_*(P)$ is the associated chain map, does $\lambda_* : H_*(\mathbf{K}) \to H_*(P)$ define an isomorphism of homology groups?

We shall prove this later. For the time being we note that *the map λ is a chain level isomorphism if $\mathbf{K}$ is given by a single vertex* (in this case each of the groups $S_q(X)$ is cyclic, and it is generated by the constant map from Δ_q to X).

Some simple properties of homology groups

If X is a topological space and $T : \Delta_q \to X$ is a singular simplex, then the image of T lies entirely in a single path component of X. Therefore the following result is immediate.

PROPOSITION 4. *If X is a topological space and its path components are the subspaces X_α, then the maps $S_*(X_\alpha)$ to $S_*(X)$ induced by inclusion define an isomorphism of chain complexes $\bigoplus S_*(X_\alpha) \to S_*(X)$ and hence also a homology isomorphism from $\bigoplus H_*(X_\alpha)$ to $H_*(X)$.*■

COROLLARY 5. *In the setting above, $H_0(X)$ is isomorphic to the free abelian group on the set of path components of X.*

A proof of this result is given on pages 109 – 110 of Hatcher.■

One immediate consequence of the preceding observations is that the map from $C_*(\mathbf{K})$ to $S_*(P)$ is an isomorphism if $(P, \mathbf{K})$ is 0-dimensional.

Our next result is often summarized with the phrase, *singular homology is compactly supported.*

THEOREM 6. *Let X be a topological space, and let $u \in H_q(X)$. Then there is a compact subspace $A \subset X$ such that u lies in the image of the associated map from $H_q(A)$ to $H_q(X)$. Furthermore, if A is a compact subset of X and $u, v \in H_q(A)$ are two classes with the same image in $H_q(X)$, then there is a compact subset B satisfying $A \subset B \subset X$ such that the images of u and v are equal in $H_q(B)$.*

Proof. If c is a singular q-chain and

$$c \;=\; \sum_j n_j \, T_j$$

define the *support* of c, written $\mathsf{Supp}\,(c)$, to be the compact set $\cup_j \; T_j(\Delta_q)$. Note that this subset is compact.

If $u \in H_q(X)$ is represented by the chain z and if $A = \mathsf{Supp}\,(z)$, then since $S_*(A) \to S_*(X)$ is 1–1 it follows that z represents a cycle in A and hence u lies in the image of $H_q(A) \to H_q(X)$.

Suppose now that A is a compact subset of X and $u, v \in H_q(A)$ are two classes with the same image in $H_q(X)$. Let z and w be chains in $S_q(A)$ representing u and v respectively, and let $b \in S_{q+1}(X)$ be such that $d(b) = i_\#(z) - i_\#(w)$. If we set $B = A \cup \mathsf{Supp}\,(b)$, then it follows that the images of $z - w$ bounds in $S_q(B)$, and therefore it follows that u and v have the same image in $H_q(B)$.■

IV.2 : Eilenberg-Steenrod properties

(Hatcher, §§ 2.1, 2.3)

For many purposes, the explicit construction of singular homology is secondary in importance to a list of formal properties that essentially characterize the singular homology groups. These properties played an important role in the work of Eilenberg and Steenrod, and they have been extremely influential in topology and numerous related subjects. The first of these properties was already mentioned informally in the preceding section, but for the sakd of completeness we shall restate it formally.

PROPOSITION 1. (The "Dimension Axiom") *If $X = \{x\}$ consists of a single point, then $H_q(X) = 0$ if $q \neq 0$, and $H_0(X) \cong \mathbb{Z}$ with the isomorphism given by the augmentation map.*

Proof. Suppose first that $x \in \mathbb{R}^n$ for some n, so that $\{x\}$ is naturally a 0-dimensional polyhedron. We have already noted that the simplicial and singular chains on X are isomorphic. Since the conclusion of the proposition holds for simplicial chains by the results of the preceding unit, it follows that the same holds for singular chains. To prove the general case, note that if $\{x\}$ is an arbitrary space consisting of a single point and $\mathbf{0} \in \mathbb{R}^n$, then $\{\mathbf{0}\}$ is homeomorphic to $\{x\}$ and in this case the conclusion follows from the special case because homeomorphic spaces have isomorphic homology groups.■

The second Eilenberg-Steenrod property is also straightforward to prove with the algebraic machinery developed thus far in the course.

THEOREM 2. (Long Exact Homology Sequence Theorem — Singular Homology Version). *Let (X, A) be a pair of topological spaces where A is a subspace of X. Then there is a long exact sequence of homology groups as follows:*

$$\cdots \quad H_{k+1}(X,A) \xrightarrow{\partial} H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X,A) \xrightarrow{\partial} H_{k-1}(A) \quad \cdots$$

This sequence extends indefinitely to the left and right. Furthermore, if we are given another pair of spaces (Y, B) and a continuous map of pairs $f : (X, A) \to (Y, B)$ such that $f : X \to Y$ is continuous and $f[A] \subset B$, then we have the following commutative diagram in which the two rows are exact:

$$\begin{array}{ccccccccccccc}
\cdots & H_{k+1}(X,A) & \xrightarrow{\partial} & H_k(A) & \xrightarrow{i_*} & H_k(X) & \xrightarrow{j_*} & H_k(X,A) & \xrightarrow{\partial} & H_{k-1}(A) & \cdots \\
\cdots & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\
\cdots & H_{k+1}(Y,B) & \xrightarrow{\partial'} & H_k(B) & \xrightarrow{i'_*} & H_k(Y) & \xrightarrow{j'_*} & H_k(Y,B) & \xrightarrow{\partial'} & H_{k-1}(B) & \cdots
\end{array}$$

This follows immediately from the algebraic theorem on long exact homology sequences.■

There is also a map of long exact sequences relating simplicial and singular homology for simplicial complexes. This is not one of the Eilenberg-Steenrod properties, but logically it fits naturally into the discussion here.

THEOREM 3. *Let $(X, \mathbf{K})$ be a simplicial complex, and let $(A, \mathbf{L})$ determine a subcomplex. Then there is a commutative ladder as below in which the horizontal lines represent the long exact homology sequences of pairs and the vertical maps are the natural transformations from simplicial to singular homology.*

$$\begin{array}{ccccccccccccc}
\cdots & H_{k+1}(\mathbf{K},\mathbf{L}) & \xrightarrow{\partial} & H_k(\mathbf{L}) & \xrightarrow{i_*} & H_k(\mathbf{K}) & \xrightarrow{j_*} & H_k(\mathbf{K},\mathbf{L}) & \xrightarrow{\partial} & H_{k-1}(\mathbf{L}) & \cdots \\
\cdots & \downarrow \lambda_* & & \downarrow \lambda_* & & \downarrow \lambda_* & & \downarrow \lambda_* & & \downarrow \lambda_* & \\
\cdots & H_{k+1}(X,A) & \xrightarrow{\partial} & H_k(A) & \xrightarrow{i_*} & H_k(X) & \xrightarrow{j_*} & H_k(X,A) & \xrightarrow{\partial} & H_{k-1}(A) & \cdots
\end{array}$$

The results follows directly from the Five Lemma and the fact that the previously defined chain maps λ pass to morphisms of quotient complexes of relative chains from $C_*(\mathbf{K}, \mathbf{L})$ to $S_*(X, A)$.■

The Homotopy and Excision Properties

In our discussion of simplicial homology the following two facts played important roles:

(1) If P is a polyhedron that is star shaped with respect to some vertex v, then the inclusion from $\{v\}$ to P defines an isomorphism in simplicial homology.

(2) If the polyhedron P is obtained from the polyhedron Q by adjoining a single simplex S (whose boundary must lie in Q), then the inclusion from $(S, \partial S)$ to (P, Q) defines an isomorphism in simplicial homology.

The Homotopy and Excision Properties are just abstract versions of these basic facts.

In order to state the Homotopy Property for pairs of topological spaces, we shall note that two maps of topological space pairs $f, g : (X, A) \to (Y, B)$ are homotopic as maps of pairs if there is a homotopy $H : (X \times [0, 1], A \times [0, 1]) \to (Y, B)$ such that the restriction of H to $(X \times \{0\}, A \times \{0\})$ and $(X \times \{1\}, A \times \{1\})$ are given by f and g respectively

THEOREM 4. (Homotopy invariance of singular homology) *Suppose that $f, g : (X, A) \to (Y, B)$ are homotopic as maps of pairs. Then the associated homomorphisms $f_*, g_* : H_*(X, A) \to H_*(Y, B)$ are equal.*

We have already laid the groundwork for proving this result in Section III.5, and the proof will be given in the Section IV.4. For the time being, we shall simply give three important consequences.

COROLLARY 5. *If $f : X \to Y$ is a homotopy equivalence, then the associated homology maps $f_* : H_*(X) \to H_*(Y)$ are isomorphisms.*

Proof. Let $g : Y \to X$ be a homotopy inverse to f. Since $g \circ f$ is homotopic to the identity on X and $g \circ g$ is homotopic to the identity on Y, it follows that the composites of the homology maps $g_* \circ f_*$ and $f_* \circ g_*$ are equal to the identity maps on $H_*(X)$ and $H_*(Y)$ respectively, and therefore f_* and g_* are isomorphisms.■

COROLLARY 6. *If X is a contractible space and there is a contracting homotopy from the identity to the constant map whose value is given by $y \in X$, then the inclusion of $\{y\}$ in X defines an isomorphism of singular homology groups.*

Proof. Let $i : \{y\} \to X$ be the inclusion map, and let $r : X \to \{y\}$ be the constant map, so that $r \circ i$ is the identity. The contracting homotopy is in fact a homotopy from the identity to the reverse composite $i \circ r$, and therefore $\{y\}$ is a deformation retract of X. By the preceding corollary, it follows that i_* defines an isomorphism of singular homology groups.■

COROLLARY 7. *If $f : (X, A) \to (Y, B)$ is a continuous map of pairs such that the associated maps $X \to Y$ and $A \to B$ are homotopy equivalences, then the homology maps f_* from $H_*(X, A)$ to $H_*(Y, B)$ all isomorphisms.*

Proof. In this case we have a commutative ladder as in Theorem 2, in which the horizontal lines represent the exact homology sequences of (X, A) and (Y, B), while the vertical arrows represent the homology maps defined by the mapping f. Since the mappings from X to Y and from A to B are homotopy equivalences, it follows that all the vertical maps except possibly those involving $H_*(X, A) \to H_*(Y, B)$ are isomorphisms; one can now use the Five Lemma to prove that these remaining vertical maps are also isomorphisms.■

The final property, called *excision*, is the most complicated to state and to prove, and its connection to the second property is relatively remote.

THEOREM 8. (Excision Property) *Suppose that (X, A) is a topological space and that U is an open subset of X such that $U \subset \overline{U} \subset \mathrm{Interior}(A)$. Then the inclusion map from $(X - U, A - U)$ to (X, A) determines an isomorphism in homology.*

A connection between this result and the second property of simplicial homology can be described informally as follows: If we take $B = X - U$, then the inclusion map in the theorem may be rewritten as $(B, B \cap A) \to (B \cup A, A)$. In the second listed property of simplicial homology, the inclusion map can be rewritten in the form $(S, Q \cap S) \to (Q \cup S, Q)$. There is at least a superficial resemblance between each of these and the standard module isomorphism

$$M/M \cap N \quad \cong \quad M + N/N$$

and in fact the similarities turn out to be more than just a coincidence.

We shall continue by proving a stronger analog of property (2) for simplicial homology that was stated above.

THEOREM 9. *Suppose that X is a compact Hausdorff space and $A \subset X$ is a closed subspace such that X is obtained from A by adjoining finitely many k-cells for some $k > 0$. Let*

$$\varphi : A \amalg \left(\{1, \cdots, N\} \times D^k\right) \longrightarrow X$$

be the continuous onto quotient map corresponding to the cell attachments. Then the composite map of pairs

$$\begin{array}{c}\left(\bigcup_j \{j\} \times D^k, \bigcup_j \{j\} \times S^{k-1}\right) \\ \big\downarrow \text{inclusion} \\ \left(A \amalg \left(\{1, \cdots, N\} \times D^k\right), A \amalg \left(\{1, \cdots, N\} \times S^{k-1}\right)\right) \xrightarrow{\varphi} (X, A)\end{array}$$

defines an isomorphism of singular homology groups.

In fact, we shall prove that both of the factors in the composite map also define isomorphisms of homology groups.

COROLLARY 10. *In the setting above the relative homology groups $H_*(X, A)$ are isomorphic to a direct sum of N copies of $H_*(D^k, S^{k-1})$.*■

Proof of Theorem 9. The argument involves detailed work with the constructions of Proposition II.3.4, so we begin by recalling these and expanding upon them.

As before, let $E_1, \cdots, E_N$ be the k-cells, and take

$$\varphi : A \amalg \left(\{1, \cdots, N\} \times D^k\right) \longrightarrow X$$

to be the continuous onto map corresponding to the k-cell attachments. For each $r \in (0, 1]$ let $rD^k \subset D^k$ be the closed disk of radius r centered at the origin, let $F(r) \subset X$ be the image of $\{1, \cdots, N\} \times rD^k$, and let $V(r) = X - F(r)$. It follows that $F(r)$ is a compact (hence closed) subset and $V(r)$ is an open set containing A, and by Proposition II.3.4 we know that A is a strong deformation retract of both $V(r)$ and its closure in X. Note that this closure of $V(r)$ is given by the union of the latter with the image of $\{1, \cdots, N\} \times rS^{k-1}$, where rS^{k-1} is the sphere of radius r which is the point set theoretic frontier of rD^k.

Since A is a strong deformation retract of $\overline{V(\frac{1}{2})}$, it follows from Corollary 7 that the inclusion mapping of pairs ψ defines an isomorphism ψ_* from $H_*(X, A)$ to $H_*\left(X, \overline{V(\frac{1}{2})}\right)$. Since $0 < s < r \leq 1$ implies

$$\overline{V(s)} \subset V(r)$$

it follows from Theorem 8 that the excision mappings

$$e_* : H_*\left(X - V\left(\tfrac{3}{4}\right), \overline{V\left(\tfrac{1}{2}\right)} - V\left(\tfrac{3}{4}\right)\right) \longrightarrow H_*\left(X, \overline{V\left(\tfrac{1}{2}\right)}\right)$$

are isomorphisms. If $0 < s < r \leq 1$ and we let $\textbf{Shell}\,[s, r] \subset D^k$ be the set of points $\mathbf{x}$ such that $|\mathbf{x}| \in [s, r]$, then by construction the mapping φ defines a homeomorphism of pairs

$$\varphi_3 : \{1, \cdots, N\} \times \left(\tfrac{3}{4}D^k, \textbf{Shell}\left[\tfrac{1}{2}, \tfrac{3}{4}\right]\right) \longrightarrow \left(X - V\left(\tfrac{3}{4}\right), \overline{V\left(\tfrac{1}{2}\right)} - V\left(\tfrac{3}{4}\right)\right)$$

and therefore it follows that the homology of the pair on the right is isomorphic to a direct sum of N copies of the homology of the pair $\left(\frac{3}{4}D^k, \mathbf{Shell}\left[\frac{1}{2},\frac{3}{4}\right]\right)$.

We now have the following commutative diagram in which the maps φ_i are defined by φ and all the vertical arrows are associated to inclusion mappings:

$$\begin{array}{ccc}
\left(\{1,\ \cdots\ ,N\}\times D^k, \{1,\ \cdots\ ,N\}\times S^{k-1}\right) & \xrightarrow{\varphi_1} & (X,A) \\
\downarrow \psi' & & \downarrow \psi \\
\left(\{1,\ \cdots\ ,N\}\times D^k, \{1,\ \cdots\ ,N\}\times \mathbf{Shell}\left[\frac{1}{2},1\right]\right) & \xrightarrow{\varphi_2} & \left(X, \overline{V\left(\frac{1}{2}\right)}\right) \\
\uparrow e' & & \uparrow e \\
\left(\{1,\ \cdots\ ,N\}\times \frac{3}{4}D^k, \{1,\ \cdots\ ,N\}\times \mathbf{Shell}\left[\frac{1}{2},\frac{3}{4}\right]\right) & \xrightarrow{\varphi_3} & \left(X - V\left(\frac{3}{4}\right), \overline{V\left(\frac{1}{2}\right)} - V\left(\frac{3}{4}\right)\right)
\end{array}$$

We have already noted that φ_3 is a homeomorphism of pairs and hence induces isomorphisms in singular homology, and we already noted that e is an excision map so it also induces isomorphisms in homology. Furthermore, the map e' is also an excision map and hence induces isomorphisms in homology, and thus it follows that φ_2 defines isomorphisms in homology.

At the beginning of the proof we noted that ψ defines an isomorphism in homology. Since S^{k-1} is a strong deformation retract of $\mathbf{Shell}\left[\frac{1}{2},1\right]$ (push everything out to the boundary radially), it follows that ψ' also defines isomorphisms in homology, and hence it also follows that φ_1 defines isomorphisms in homology, which is precisely the conclusion of the theorem.■

Equivalence of singular and simplicial homology

We are now ready to prove that singular and simplicial homology are naturally equivalent (modulo completing the proofs of Theorems 4 and 8 in the Section IV.4 of the notes).

THEOREM 11. *Let $(X,\mathbf{K})$ be a simplicial complex, let $(A,\mathbf{L})$ determine a subcomplex, and let $\lambda_* : H_*(\mathbf{K},\mathbf{L}) \to H_*(X,A)$ be the natural transformation from simplicial to singular homology that was described in Theorem* 3. *Then λ_* is an isomorphism.*

Proof. The idea is to apply Theorem III.4.7 on natural transformations of homology theories on simplicial complex pairs. In order to do this, we must check that singular homology for simplicial complexes satisfies the five properties $(a)-(e)$ listed shortly before the statement of III.4.7. Property (c) is verified in Proposition IV.1.4, and Properties (a), (b), (d) and (e) are respectively established in Theorem 2, Corollary 7, Theorem 9 and Proposition 1 of this section. Since all these properties hold, Theorem III.4.7 implies that the map λ_* must be an isomorphism for all simplicial complex pairs.■

Homeomorphism types of spheres and Euclidean spaces

At the beginning of these notes we stated the question whether $\mathbb{R}^m$ and $\mathbb{R}^n$ can be homeomorphic if $m \neq n$. We finally have enough machinery to prove the answer is **NO**. The first step is a very simple computation involving simplicial homology.

PROPOSITION 12. *If $n \geq 0$ then $H_q(\Delta_n, \partial\Delta_n) \cong \mathbb{Z}$ if $q = n$ and is trivial otherwise. Furthermore, if $n > 0$ then $H_q(\partial\Delta_{n+1}) \cong \mathbb{Z}$ if $q = 0$ or $q = n$, and it is trivial otherwise.*

We should also note in passing that $H_q(\partial\Delta_1) \cong \mathbb{Z}\oplus\mathbb{Z}$ if $q = 0$ and is trivial otherwise.

COROLLARY 13. *If $n \geq 0$ then $H_q(D^n, S^{n-1}) \cong \mathbb{Z}$ if $q = n$ and is trivial otherwise. Furthermore, if $n > 0$ then $H_q(S^n) \cong \mathbb{Z}$ if $q = 0$ or $q = n$, and it is trivial otherwise.*

Corollary 13 follows from Proposition 12, the existence of the radial projection homeomorphism from $(\Delta_n, \partial\Delta_n)$ to (D^n, S^{n-1}), which is given by Theorem II.3.1, the equivalence of simplicial and singular homology, and the topological invariance of singular homology.∎

Proof of Corollary 12. The easiest way to see the first statement is to compute the ordered simplicial homology of the given pair. In fact, the simplicial chain complex for the standard decomposition of $(\Delta_n, \partial\Delta_n)$ is zero except in degree n, and it is isomorphic to $\mathbb{Z}$ in that case. Thus there are no differentials, and the homology groups are the same as the chain groups in this case. To prove the second statement, consider first the long exact homology sequence, a portion of which is displayed below:

$$\cdots \to H_j(\Delta_{n+1}) \to H_j(\Delta_{n+1}, \partial\Delta_{n+1}) \to H_{j-1}(\partial\Delta_{n+1}) \to H_{j-1}(\Delta_{n+1}) \ \cdots$$

If $j > 1$ then the homology groups of Δ_{n+1} in this part of the sequence are zero and hence we see that $H_j(\Delta_{n+1}, \partial\Delta_{n+1})$ is isomorphic to $H_{j-1}(\partial\Delta_{n+1})$ if $j > 1$. This proves the result for $H_q(\partial\Delta_{n+1})$ when $q > 0$; since $H_q = 0$ for $q < 0$, it only remains to prove the result for $q = 0$. In this case, consider the following piece of the long exact sequence:

$$\cdots \to H_1(\Delta_{n+1}, \partial\Delta_{n+1}) \to H_0(\partial\Delta_{n+1}) \to H_0(\Delta_{n+1})$$

The first group in this piece of the sequence is trivial, and the last group is infinite cyclic, with a generator given by the class of a vertex. This class clearly lies in the image of $H_0(\partial\Delta_{n+1})$ since all vertices are contained in the boundary of the simplex, so the map $H_0(\partial\Delta_{n+1}) \to H_0(\Delta_{n+1}) \cong \mathbb{Z}$ is onto. By exactness and the vanishing of $H_1(\Delta_{n+1}, \partial\Delta_{n+1})$, this map is also 1–1 and hence it must be an isomorphism; this proves the assertion regarding the 0-dimensional homology.∎

COROLLARY 14. *For every $n > 0$, the sphere S^n is* **NOT** *contractible.*

Proof. If a space is contractible, its homology groups are isomorphic to those of a point, but the homology groups of S^n do not have this property.∎

In fact, the homology computation yields the desired result on the homeomorphism types of spheres and Euclidean spaces.

THEOREM 15. *If m and n are positive numbers such that $m \neq n$, then S^m is not homeomorphic to S^n and $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.*

Proof. We start with the statement regarding spheres. Theorem 12 we know that the homology groups of S^m and S^n are not isomorphic if $m \neq n$. Since homeomorphic spaces have isomorphic homology groups, it follows immediately that S^m and S^n cannot be homeomorphic.

In order to derive the corresponding result for $\mathbb{R}^m$ and $\mathbb{R}^n$, we need the following fact from point set topology: *If X and Y are locally compact Hausdorff spaces which are* NOT *compact, and $f : X \to Y$ is a homeomorphism, then X extends to a homeomorphism of one point compactifications $f^\bullet : X^\bullet \to Y^\bullet$, where $f^\bullet$ sends the point at infinity in $X^\bullet$ to the point of infinity in $Y^\bullet$.* Therefore, if $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic then their one point compactifications are also homeomorphic. Since the latter are homeomorphic to S^m and S^n, it follows that if $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic then S^m and S^n are homeomorphic. Since the latter is false if $m \neq n$, it follows that $\mathbb{R}^m$ and $\mathbb{R}^n$ cannot be homeomorphic if $m \neq n$.∎

In fact, we can say considerably more.

PROPOSITION 16. (Invariance of Dimension) *Suppose that X and Y are topological manifolds of dimensions m and n respectively (in other words, they are Hausdorff spaces such that each point has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively). If X and Y are homeomorphic, then $m = n$.*

Proof. Let Z be a topological k-manifold, and let $z \in Z$. Then there is an open neighborhood U of z which is homeomorphic to an open disk W in $\mathbb{R}^k$ such that z corresponds to the center of the open disk; we might as well assume that the center of W is $\mathbf{0}$. Clearly then we have

$$H_*(U, U - \{z\}) \quad \cong \quad H_*(W, W - \{\mathbf{0}\}) \ .$$

Consider now the open covering of Z given by U and $Z - \{z\}$. Then $U \cup (Z - \{z\}) = Z$ and therefore one can use the proof of the Excision Property to show that the inclusion map from $H_*(U, U - \{z\})$ to $H_*(Z, Z - \{z\})$ induces an isomorphism in singular homology (see Proposition IV.4.4A below). Therefore we know that $H_i(Z, Z - \{z\})$ is isomorphic to $\mathbb{Z}$ if $i = k$ and is trivial otherwise.

Now if X and Y are homeomorphic topological manifolds as in the hypothesis of the proposition and the homeomorphism takes $x \in X$ to $y \in Y$, then the homeomorphism induces homology isomorphisms $H_i(X, X - \{x\}) \cong H_i(Y, Y - \{y\})$. Since the first homology group is nonzero if and only if $i = n$ and the second is nonzero if and only if $i = m$, it follows that n and m must be equal.■

The Brouwer Fixed Point Theorem

At this point it is almost traditional to state and prove the Brouwer Fixed Point Theorem. First there is a standard lemma.

LEMMA 17. *For all $n > 0$ the inclusion of S^n in D^{n+1} is not a retract.*

Proof. Since functors take retracts to retracts, if the inclusion were a retract then the induced map in homology would also be a retract, and this in turn would imply that each homology group $H_i(S^n)$ would be a subgroup of the corresponding homology group $H_i(D^{n+1})$. Since this is false for $i = n$, the conclusion follows.■

THEOREM 18. (Brouwer Fixed Point Theorem) *For all $n \geq 0$ every continuous map $f : D^n \to D^n$ has a* fixed point*; in other words, there is a point $\mathbf{x}$ in D^n such that $f(\mathbf{x}) = \mathbf{x}$.*

Proof. If $n = 1$ this is a fairly simple exercise in point set topology, and if $n = 2$ the proof is completed in 205B as follows: First, one proves that S^1 is not a retract of D^2, and then one proves that if there were a map without a fixed point then S^1 would be a retract of D^2. We have established an analog of the first step, and in fact the argument for the second step works for all $n > 0$. One point worth noting is the need to check the continuity of the geometrically described retraction explicitly; this is often left undone in treatments of algebraic topology, but for the sake of completeness we give the details in `brouwer.pdf`.

IV.3 : Computations

(Hatcher, § 2.2)

Before proving the Homotopy and Excision properties for Singular Homology groups, we shall take some time to give some typical uses of homology groups, culminating in a proof of Euler's Formula $F - E + V = 2$ for certain 2-dimensional polyhedra.

Betti numbers and torsion coefficients

We shall start with a result that could have been stated in Unit III.

PROPOSITION 1. *If $(P, \mathbf{K})$ is a simplicial complex of dimension n, then $H_q(P, \mathbf{K}) = 0$ if $q < 0$ or $q > n$, and in the remaining cases $H_q(P, \mathbf{K})$ is a finitely generated abelian group and hence a direct sum of finitely many cyclic groups.*

Since the singular and simplicial homology groups of a simplicial complex are isomorphic, we also have the following conclusion:

COROLLARY 2. *If $(P, \mathbf{K})$ is a simplicial complex of dimension n, then the singular homology groups of P satisfy $H_q(P) = 0$ if $q < 0$ or $q > n$, and in the remaining cases $H_q(P)$ is a finitely generated abelian group and hence a direct sum of finitely many cyclic groups.*■

In the course of proving Proposition 1 we shall need the following basic fact: *If G is a free abelian group on n generators, where n is some nonnegative integer, and H is a subgroup of G, then H is a free abelian group on m generators for some (unique) nonnegative integer $m \leq n$.* — A proof of this result may be found in the previously cited text by Hungerford (see Theorem 1.6 on pages 73 – 74).■

Proof of Proposition 1. This is a purely algebraic result, and we shall prove the conclusion holds for the homology groups of chain complexes C_* such that $C_q = 0$ for $q < 0$ or $q > n$ and C_q is finitely generated in all dimensions. The proposition will follow by applying the algebraic result to the complex of ordered chains $C_*(P, \mathbf{K})$.

Let (C, d) be a chain complex as above, and denote the subgroups of cycles and boundaries in C_q by $\mathcal{Z}_q(C)$ and $\mathcal{B}_{q+1}(C)$ respectively. Then the q^{th} homology $H_q(C)$ is the quotient group $\mathcal{Z}_q(C)/\mathcal{B}_{q+1}(C)$ By the remark in the paragraph before the beginning of this proof, we know that $\mathcal{Z}_q(C)$ is also a finitely generated free abelian group, and therefore its quotient $H_q(C)$ is also finitely generated. In fact if C_q is freely generated by c_q elements then $H_q(C)$ is generated by at most c_q elements.■

By the preceding argument and an algebraic result mentioned near the beginning of these notes, we know that

$$H_q(C) \;\cong\; \mathbb{Z}^{\beta(q)} \oplus \left(\mathbb{Z}_{\tau(1)} \oplus \cdots \oplus \mathbb{Z}_{\tau(s)} \right)$$

where each $\beta(q)$ is a nonnegative integer and the τ_j's are positive integers such that $\tau(j+1)$ divides $\tau(j)$ for all j, and in fact there are unique sequences of integers $\beta(q)$ and $\tau(j)$ with these properties. The number β_q is frequently called the q^{th} *Betti number* of the chain complex (or of a topological space, if the chain complex gives the homology of that space), and the numbers $\tau(j)$ are often called the q^{th} *torsion coefficients*. One can extend the sequence of torsion coefficients to an infinite sequence by setting $\tau(j) = 1$ if $j > s$.

Cellular homology

If P is a polyhedron of positive dimension, the preceding discussion implies that the singular homology groups of P are finitely generated abelian groups. even though the corresponding groups of singular chains are free abelian groups on sets of generators whose cardinalities are equal to $2^{\aleph_0}$.

In fact, the conclusion holds more generally if X has the structure of a finite cell complex by the following result:

THEOREM 3. *Let $(X, \mathcal{E})$ be a finite cell complex of dimension n. Then there is a chain complex $\big(C_*(X,\mathcal{E}), d\big)$ such that the chain groups are finitely generated free abelian in every dimension with $C_q(X,\mathcal{E}) = 0$ if $q < 0$ or $q > n$, and the q-dimensional homology of this chain complex is isomorphic to the singular homology group $H_q(X)$.*

The chain complex will be defined explicitly in terms of singular homology and the cell structure for $(X, \mathcal{E})$, and it will be called the *cellular chain complex*. For each k such that $-1 \leq k \leq n$, let X_k denote the k-skeleton of X, where $X_{-1} = \emptyset$. Specifically, we set $C_q(X,\mathcal{E}) = H_q(X_q, X_{q-1})$ and define the differential d_q to be the following composite:

$$H_q(X_q, X_{q-1}) \quad \xrightarrow{\partial[q]} \quad H_{q-1}(X_{q-1}) \quad \xrightarrow{j[q-1]_*} \quad H_{q-1}(X_{q-1}, X_{q-2})$$

These maps define a chain complex since

$$d_{q-1} \circ d_q \quad = \quad j[q-2]_* \circ \partial[q-1] \circ j[q-1]_* \circ \partial[q]$$

and $\partial[q-1] \circ j[q-1]_* = 0$ because the factors are consecutive morphisms in the long exact homology sequence for (X_{q-1}, X_{q-2}). By the results of the preceding section, *the q-dimensional cellular chain group is isomorphic to a free abelian group on the set of q-cells in $\mathcal{E}$.*

Proof of Theorem 3. The result is immediate if $\dim X = 0$ or -1, in which cases X is a nonempty finite set or the empty set. In this case the cellular chain groups are either concentrated in degree zero (the 0-dimensional case) or are all equal to zero (the (-1)-dimensional case).

We shall prove the result for the explicit cellular chain complex described above by induction on $\dim X$, and for this purpose we assume that the result is true when $\dim X \leq n-1$. The inductive hypothesis then implies that the theorem is true for the $(n-1)$-skeleton X_{n-1}. Now the only difference between the cellular chain complex for X and the corresponding complex for X_{n-1} is that the n-dimensional chain group for the latter is zero while the n-dimensional chain group for the latter is nonzero, and likewise the differentials in both complexes are equal except for the ones going from n-chains to $(n-1)$-chains (in the second case the differential must be zero). It follows that the homology groups of these cell complexes are isomorphic except perhaps in dimensions n and $n-1$.

Similarly, since $H_q(X_n, X_{n-1}) = 0$ if $q \neq n$ or $n-1$, it follows that $H_q(X) \cong H_q(X_{n-1})$ except perhaps in these dimensions. Therefore, we have shown the inductive step except when $q = n$ or $n-1$. It will be necessary to examine these cases more closely.

We shall describe the n-dimensional homology of $C_*(X,\mathcal{E})$ first. By definition the map d_n is a composite $j[q-1]_* \circ \partial[q]_*$, and the factors fit into the following long exact sequences:

$$0 \;=\; H_n(X_{n-1}) \;\longrightarrow\; H_n(X) \;\longrightarrow\; H_n(X, X_{n-1}) \;\longrightarrow\; H_{n-1}(X_{n-1}) \;\cdots$$

$$0 \;=\; H_{n-1}(X_{n-2}) \;\longrightarrow\; H_{n-1}(X_{n-1}) \;\longrightarrow\; H_{n-1}(X_{n-1}X_{n-2})$$

It follows that $H_n(X)$ is isomorphic to the kernel of $\partial[q]_*$ and the map $j[q-1]_*$ is injective. Similarly, it also follows that $H_{n-1}(X)$ is isomorphic to the kernel of $\partial[q-1]_*$ and the map $j[q-2]_*$ is injective. Since $d_q = j[q-1]_* \circ \partial[q]$, it follows that $H_n(X)$ is also isomorphic to the kernel of d_n, and since $C_{n+1}(X,\mathcal{E}) = 0$ it follows that the kernel of d_n is also isomorphic to the n-dimensional homology of $C_*(X, \mathcal{E})$. Thus we now know the theorem is true for all dimensions except possibly $(n-1)$.

In order to describe the $(n-1)$-dimensional homology of $C_*(X, \mathcal{E})$ we shall consider the following diagram, in which both the row and the column are exact:

$$\begin{array}{ccccccc} & & H_{n-1}(X_{n-2}) = 0 & & & & \\ & & \downarrow & & & & \\ \cdots\ H_n(X, X_{n-1}) & \xrightarrow{\partial[n]} & H_{n-1}(X_{n-1}) & \longrightarrow & H_{n-1}(X) & \longrightarrow & H_{n-1}(X, X_{n-1}) = 0 \\ & & \downarrow{\scriptstyle j[n-1]_*} & & & & \\ & & H_{n-1}(X_{n-1}, X_{n-2}) & & & & \end{array}$$

By the exactness of the row we know that $H_{n-1}(X)$ is isomorphic to the quotient group

$$H_{n-1}(X_{n-1}) \,/\, \text{Image } \partial[n]$$

and since $j[n-1]_*$ is injective we know from the previous discussion that $j[n-1]_*$ sends $H_{n-1}(X_{n-1})$ onto the kernel of d_{n-1} (note this map is the same for both X and X_{n-1}). Furthermore, by construction we also know that $j[n-1]_*$ maps the image of $\partial[n]$ onto the image of d_n. If we make these substitutions into the displayed expression above, we see that $H_{n-1}(X)$ is isomorphic to the kernel of d_{n-1} modulo the image of d_n, which proves that the conclusion of the theorem also holds in dimension $n-1$.■

If we let $\mathcal{C}(q) = \{ E^q_\alpha \}$ denote the (finite) set of q-cells for $\mathcal{E}$ and view the cellular chain groups $C_q(X, \mathcal{E})$ as free abelian groups on the sets $\mathcal{C}(q)$ by the preceding construction and result, it follows that for each E^q_α we have

$$d_q\,(\,E^q_\alpha\,) \quad = \quad \sum_{\mathcal{C}(q-1)} \,[\alpha : \beta]\, E^{q-1}_\beta$$

for suitable integers $[\alpha : \beta]$; classically, these coefficients were called *incidence numbers*. Unlike the situation for simplicial chain complexes, there are no general formulas for finding these numbers. If we already know the homology of X from some other result, then it is often possible to recover them by working backwards (*i.e.*, if we know the homology then often there are not many possibilities for the incidence numbers which will yield the correct homology groups).

One condition under which the incidence numbers are recursively computable is if the cell complex is a **regular cell complex**; in other words, each closed n-cell is in fact homeomorphic to to D^n via the attaching map and is a subcomplex in the evident sense of the word (the boundary is a union of cells in the big complex). These will be true for the cell complexes considered in the next subheading.

Here is a very brief summary of the recursive process: Suppose we have worked out the differentials for the chain complex through dimension $n-1$, and we want to find the differentials in dimension n. Let E be an n-cell; by definition, E determines a cell complex which has the homology of a disk. Let ∂E be the subcomplex given by the boundary, so that we have the incidence numbers on ∂E already. It is only necessary to figure out the map from $\mathbb{Z} = C_n(E)$ to $C_{n-1}(E)$. Now the homology of ∂E is just the homology of S^{n-1}, and since $C_n(\partial E) = 0$ it follows that there are no nontrivial boundaries in $C_{n-1}(\partial E)$, so that $H_{n-1}(\partial E) \cong Z$ may be viewed as a subgroup A of $C_{n-1}(\partial E) = C_{n-1}(E)$. Now the image of this copy of $\mathbb{Z}$ in $C_{n-1}(E)$ represents zero in homology since $H_{n-1}(E) = 0$, and therefore there must be some element in $C_n(E)$ which maps to a generator of A. Since $C_n(E)$ is infinite cyclic, it follows that some multiple of the generator $[E]$ for $C_n(E)$ must map to the generator of A. Let $a \in A$ be the generator such that $d(k[E]) = a$; then it follows that $a = k\, d([E])$. But since $d([E])$ is also a cycle, it follows that $d([E]) = m\, a$ for some integer m. Combining these, we see that $a = km\, a$, and since A is torsion free this implies that $km = 1$, so that $k = m = \pm 1$. Thus we must have $d([E]) = \pm a$. the generator of $C_n(E)$. In fact, the exact choice for the sign is unimportant because one obtains the same homology in all cases; we can always choose the generator for $C_n(E)$ so that the incidence number is $+1$.

Convex linear cells

In elementary geometry, the terms *polygon* and *polyhedron* are often used to denote frontiers of bounded open sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ that are defined by finitely many linear equations and inequalities. For example, one has the standard isosceles right triangle in the plane which bounds the compact convex set defined by the inequalities

$$x \;\geq\; 0\,, \qquad y \;\geq\; 0\,, \qquad x + y \;\leq\; 1$$

while standard squares and cubes in the plane and 3-space are defined by

$$0 \;\leq\; x,\; y \;\leq 1\,, \qquad 0 \;\leq\; x,\; y,\; z \;\leq\; 1$$

and the octagon in the plane with vertices

$$(2, \pm 1),\; (-2, \pm 1),\; (1, \pm 2),\; (-1, \pm 2)$$

is defined by the eight inequalities

$$-2 \;\leq\; x,\; y \;\leq\; 2\,, \qquad -3 \;\leq\; x + y \;\leq\; 3\,, \qquad -3 \;\leq\; x - y \;\leq\; 3\,.$$

Convex sets in $\mathbb{R}^n$ defined by finitely many linear equations and inequalities are basic objects of study in the usual theory of linear programming. In particular, it turns out that the sorts of sets we consider are given by all convex combinations of a finite subset of *extreme points* which correspond to the usual geometric notion of vertices. The reference below is the text for Mathematics 120, which covers linear programming and provides some background on the sets considered here, (particularly in Sections 15.4 – 15.8 on pages 264 – 285).

E. K. P. Chong and S. Zak. An Introduction to Optimization. Wiley, New York, 2001. ISBN: 0-471-39126-3.

We defined convex linear cells in Section I.2; recall that a bounded subset $E \subset \mathbb{R}^n$ is a *convex linear cell* (or also as a *rectilinear cell*) if it is defined by finitely many linear equations and inequalities. It follows immediately that such a set is compact and convex.

The main properties of such cells that we shall need are formulated and proved in Section 7 of [MunkresEDT]. Here is a summary of what we need: If we define a k-plane in a real vector space V to be a set of the form $\mathbf{x} + W$, where W is a k-dimensional vector subspace of V, then the *dimension* of a convex linear cell E is equal to the least k such that E lies in a k-plane. If V is an n-dimensional vector space, this dimension is a nonnegative integer which is less than or equal to n. Suppose now that E is k-dimensional in this sense and $\mathbf{P} = \mathbf{x} + W$ is a k-plane containing E; it follows fairly directly that $\mathbf{P}$ is the unique such k-plane. Less obvious is the fact that the interior of E with respect to $\mathbf{P}$ is nonempty.

[For the sake of completeness, here is a sketch of the proof: The cell E must contain a set of $k+1$ points that are affinely independent, for otherwise it would lie in a $(k-1)$-plane. Since a convex linear cell is a closed convex set, it must contain the k-simplex whose vertices are these points, and this set has a nonempty interior in the k-plane $\mathbf{P}$.]

It is convenient to describe a minimal and irredundant set of equations and inequalities which define a convex linear cell E. The unique minimal k-plane containing E can be defined as the set of solutions to a system of $n-k$ independent linear equations, and to describe E it is enough to add a MINIMAL set of inequalities which define E.

Definition. If E is a k-dimensional convex linear cell and we are given an efficient set of defining linear equations and inequalities as in the preceding paragraph, then a $(k-1)$-*dimensional face* of E is obtained by taking the subset for which one of the listed inequalities is replaced by an equation.

For example, in the square the four faces are given by adding one of the four conditions

$$x \;=\; 0, \qquad x \;=\; 1, \qquad y \;=\; 0, \qquad y \;=\; 1$$

to the equations and inequalities defining the square, and for the 2-simplex whose vertices are $(0,0)$, $(1,0)$ and $(0,1)$ one has the three faces defined by strengthening one of the defining inequalities to one of the three equations $x=0$, $y=0$ or $x+y=1$.

It follows immediately that each $(k-1)$-face of E is a convex linear cell, and Lemmas 7.3 and 7.5 on pages 72 – 74 of [MunkresEDT] show that each face described in this manner is $(k-1)$-dimensional. — One can iterate the process of taking faces and define q-faces of E where $-1 \leq q \leq k$; more details appear on page 75 of the book by Munkres (by definition, the empty set is a (-1)-face).

The geometric boundary of E, written $\mathbf{Bdy}(E)$, may be described in two equivalent ways: It is the union of all the lower dimensional faces ofr E, and it is also the point set theoretic frontier of E in $\mathbf{P}$. We shall need the following theorem, which is discussed on pages 71 – 74 of the Munkres book:

PROPOSITION 4. *If $E \subset \mathbb{R}^n$ is a convex linear cell, then the pair $\big(E, \mathbf{Bdy}(E)\big)$ is homeomorphic to (D^k, S^{k-1}).*

We have already shown this result when E is a simplex by constructing a *radial projection homeomorphism*, and as noted on page 71 of Munkres' book a similar construction proves the corresponding result for an arbitrary convex linear k-cell.■

If we combine this proposition with the remaining material on convex linear cells, we obtain the following basic consequence.

PROPOSITION 5. *If E is a convex linear k-cell and $\mathbf{Bdy}(E)$ is its boundary, then these spaces have cell decompositions such that (i) the cells of $\mathbf{Bdy}(E)$ are the faces of dimension less than k, (ii) the cells of E are the cells of $\mathbf{Bdy}(E)$ together with E itself.*■

If we combine the preceding result with Theorem 3, we obtain the following conclusion relating the geometry and algebraic topology of E and its boundary.

COROLLARY 6. *If E and $\mathbf{Bdy}(E)$ are as above, then there exist chain complexes A_* and B_* such the groups A_q are free abelian groups on the sets of nonempty faces of dimension less than k, the groups B_q are free abelian groups on the sets of nonempty faces of dimension $\leq k$, and the homology groups of A_* and B_* are isomorphic to $H_*(S^{k-1})$ and $H_*(D^k)$ respectively.*■

We would like to apply this corollary to derive the formula of Euler stated at the beginning of these notes. This requires an algebraic digression.

Rational homology

Given an arbitrary ring R, one can define *singular homology groups with coefficients in* R using modified singular chain groups $S_*(X;R)$ in which the q^{rmth} group $S_q(X;R)$ is a **free R-module** on the set of singular q-simplices. Boundary homomorphisms can now be constructed as before, and therefore we may define *homology with coefficients in* R in the usual fashion. These groups will be denoted by $H_q(X;R)$. In these notes we shall only be interested in cases where R is either the integers or a field.

In order to proceed, we shall need some algebraic background; the constructions described below work in far greater generality than the situation we consider, but we specialize here to simplify the discussion.

Definition. Let G be an abelian group. The *rationalization* or G, or the *localization of G over the rationals* is formed by a construction very similar to the construction of the rationals from the integers. One starts with ordered pairs (g, r) where $g \in G$ and r is a nonzero integer, and one identifies (g, r) with (h, s) if there is a nonzero integer t such that $t(sg - rh) = 0$ (this is slightly stronger than the condition in the construction of $\mathbb{Q}$ from $\mathbb{Z}$ in which t is always 1). This condition defines an equivalence relation on the set of all ordered pairs, and we let $G_{(0)}$ denote the set of equivalence classes. Formally, the class of (g, r) is supposed to represent an object of the form $r^{-1} \cdot g$, and motivated by this we define addition and multiplication by a rational number as follows:

$$[g,r] \ + \ [h,s] \ = \ [sg + rh, rs] \ , \qquad pq^{-1}[g,r] \ = \ [pg, qr]$$

At this point it is necessary to verify that our definitions of sums and scalar products do not depend upon the choices of representatives for equivalence classes; this is elementary and entirely similar to the corresponding proof for the formal definition of rational numbers in terms of integers. The following result is also elementary:

THEOREM 7. *The object $G_{(0)}$ constructed above is a rational vector space, and the construction also has the following properties:*

(i) *If $g_1, \cdots, g_m$ generate G, then their images under j_G span the rational vector space $G_{(0)}$.*

(ii) *For each abelian group G there is a group homomorphism $j_G : G \to G_{(0)}$ sending $g \in G$ to the equivalence class $[g, 1]$. This map is an isomorphism if G is a rational vector space.*

(iii) *If $f : G \to H$ is a homomorphism then there is an associated linear transformation of rational vector spaces $f_{(0)} : G_{(0)} \to H_{(0)}$ such that the constructions sending an object or morphism Γ to $\Gamma_{(0)}$ define an* ADDITIVE *covariant functor and the maps j_G define a natural transformation from the identity to the associated functor on the category of abelian groups.*

(iv) *The construction sends the infinite cyclic group $\mathbb{Z}$ to $\mathbb{Q}$ and it sends every finite cyclic group to* $\mathbf{0}$. *Furthermore, for all abelian groups G and H we have $[G \oplus H]_{(0)} \cong G_{(0)} \oplus H_{(0)}$.*

In particular, if G is a finitely generated abelian group which is the direct sum of β infinite cyclic groups and several finite cyclic groups, then $G_{(0)}$ is a rational vector space whose dimension is equal to β.

Comments on the proof. Most of the verifications are extremely straightforward and left to the reader, so we shall simply note a few key features. First of all, scalar multiplication by a rational number n/m (where $m \neq 0$) is given by

$$(n/m) \cdot [g, r] \quad = \quad [ng, mr]$$

and similarly the mapping $g_{(0)}$ is defined by the formula

$$f_{(0)}[g, r] \quad = \quad [\, f(g),\, r\,]\ .$$

We shall need the second formula for our next result.■

The following property of the rationalization construction is somewhat less trivial, and it has far-reaching consequences.

THEOREM 8. *The functor $\Gamma \to \Gamma_{(0)}$ sends exact sequences to exact sequences.*

Proof. Every exact sequence is essentially built from short exact sequences; for example, if $A \to B \to C$ is an exact sequence involving $f : A \to B$ and $g : B \to C$, then the sequence is given by fitting together the following sequences:

$$0 \to \mathrm{Ker}(f) \to A \to \mathrm{Image}(f) = \mathrm{Kernel}(g) \to 0$$

$$0 \to \mathrm{Image}(f) = \mathrm{Kernel}(g) \to B \to \mathrm{Image}(g) \to 0$$

$$0 \to \mathrm{Image}(g) \to C \to \mathrm{Cokernel}(g) \to 0$$

Therefore it will be enough to prove the result for short exact sequences. In other words, if $0 \to A \to B \to C \to 0$ is exact, we need to prove the same holds for $0 \to A_{(0)} \to B_{(0)} \to C_{(0)} \to 0$.

We shall only prove that the sequence is exact at the middle object; the proofs at the other two objects are similar and left to the reader. Suppose that $f : A \to B$ is 1–1 and $g : B \to C$ is onto such that the image of f is the kernel of g. Then $g \circ f = 0$ and additivity imply that $g_{(0)} \circ n_{(0)} = 0$, and therefore it follows immediately that the image of $n_{(0)}$ is contained in the kernel of $g_{(0)}$. Suppose now that $[b, t]$ lies in the kernel of $g_{(0)}$. By definitions it follows that there is a nonzero integer s such that $s \cdot g(b) = 0$. By exactness of the original sequence, there is some $a \in A$ such that $f(a) = sb$, and we claim that $n_{(0)}$ maps $[a, st]$ to $[b, t]$. To see this, note that $n_{(0)}[a, st] = [sb, st]$ and the right hand side is equal to to $[b, t]$ because $stb - tsb = 0$.■

The preceding results have the following implication for chain complexes.

COROLARY 9. *Let (C, d) be a chain complex of abelian groups. Then rationalization defines a chain complex $(\, C_{(0)}, d_{(0)}\,)$ of rational vector spaces, and the homology of this chain complex is isomorphic to the rationalized homology groups $H_*(C)_{(0)}$.*■

Euler characteristics and Euler's Formula

We now resume our study of the algebraic topology of convex linear cells. If E is a convex linear cell of dimension k and n_q denotes the number of q-faces for $0 \leq q \leq k$, then direct examination of examples shows that one always obtains the equation

$$n_0 \ - \ n_1 \ + \ n_2 \ \cdots \ + \ (-1)^k n_k \quad = \quad 1$$

in which the final term n_k is always equal to 1 by construction. The machinery of this section provides a means for explaining why this is more than just a coincidence.

Notation. Let (C, d) be a chain complex over the rationals such that only finitely many chain groups C_q are nonzero and the nonzero groups are all finite-dimensional vector spaces over the rationals.

(*i*) Set c_q equal to the dimension of C_q.

(*ii*) Set b_q equal to the rank of d_q.

(*iii*) Set z_q equal to the dimension of the kernel of d_q.

(*iv*) Set h_q equal to the dimension of $H_q(C)$.

It follows immediately that these numbers are defined for all q and are equal to zero for all but finitely many a.

The equation involving the numbers of faces for a convex linear cell depends upon the following algebraic result.

PROPOSITION 10. *In the setting above we have*

$$\sum_q (-1)^q c_q \quad = \quad \sum_q (-1)^q h_q \ .$$

Proof. The main idea of the argument is given on pages 146 – 147 of Hatcher. In analogy with the discussion there, we have $c_q - z_q = b_q$ and $z_q - b_{q+1} = h_q$, so that

$$\sum_q (-1)^q h_q \quad = \quad \sum_q (-1)^q (z_q - b_{q+1}) \quad = \quad \sum_q (-1)^q z_q \ - \ \sum_q (-1)^q b_{q+1} \quad =$$

$$\sum_r (-1)^r z_r \ + \ \sum_r (-1)^r b_r \quad = \quad \sum_q (-1)^q c_q$$

proving that the two sums in the proposition are equal.■

COROLLARY 11. *Suppose that $(X, \mathcal{E})$ is a finite cell complex with c_q cells in dimension $q \geq 0$, and suppose that $H_q(X)$ is isomorphic to a direct sum of β_q infinite cyclic groups plus a finite group. Then we have*

$$\sum_{q \geq 0} (-1)^q c_q \quad = \quad \sum_{q \geq 0} (-1)^q \beta_q \ .$$

The statement regarding convex linear cells follows immediately from Corollary 11 and Proposition 5. — In general, the topologically invariant number on the right hand side is called the **Euler characteristic** of X and is written $\chi(X)$.

Proof. Let A_* be the chain complex over the rational numbers with $A_q = C_q(X, \mathcal{E})_{(0)}$ and the differential given by rationalizing d_q. It then follows that $\dim A_q = c_q$ and $\dim H_q(A) = \beta_q$. The corollary then follows by applying Proposition 10.■

The "classical" formula of Euler is the 2-dimensional case of the following result:

THEOREM 12. (Generalized Euler's Formula) *Let $E \subset \mathbb{R}^{n+1}$ be an $(n+1)$-dimensional convex linear cell, and suppose that E and* $\mathbf{Bdy}(E)$ *have f_r faces of dimension r for $0 \leq r \leq n$ (note that $r = n+1$ is excluded). Then the alternating sum*

$$\sum_{r=0}^{n} (-1)^r n_r$$

is equal to 2 *if* n *is even and* 0 *if* n *is odd.*

We should note that the alternating sum is also equal to the Euler characteristic of $\mathbf{Bdy}(E)$.

Proof of Theorem 12. Since the homology of E is isomorphic to the homology of a point, we know that $\beta_0 = 1$ and $\beta_q = 0$ otherwise. By the preceding discussion we know that

$$n_0 \ - \ n_1 \ + \ n_2 \ \cdots \ + \ (-1)^{n+1} n_{n+1} \ \ = \ \ 1$$

where $n_{n+1} = 1$. Therefore the alternating sum

$$\sum_{r=0}^{n} (-1)^r n_r$$

is equal to $1 - (-1)^{n+1} \ = \ 1 + (-1)^n$, which is 2 if n is even and 0 if n is odd.■

If $n = 3$ this formula is equivalent to the standard identity $F - E + V = 2$.

We shall conclude this section with another simple example:

PROPOSITION 13. *Suppose that* $(X, \mathcal{E})$ *is a connected* 1*-dimensional cell complex (i.e., a graph) with* E *edges and* V *vertices. Then* $H_1(X)$ *is isomorphic to a free abelian group on* $1-E+V$ *generators.*

The methods of 205B show that $\pi_1(X, x)$ is a free group on the same number of generators; in the final section of this unit we shall see how these results are related. subgroup of the free abelian chain group $C_1(X, \mathcal{E})$

Proof. Since X is arcwise connected (why?) and thus its zero-dimensional singular homology is infinite cyclic, it follows that $\beta_0 = 1$. Therefore Corollary 11 implies that $1 - \beta_1 = V - E$ and therefore we may retrieve β_1 easily from the cell structure data by the formula $\beta_1 = 1 + E - V$.■

IV.4 : Proofs of homotopy invariance and Excision

(Hatcher, §§ 2.1 – 2.3)

In this section we shall complete the proof that singular homology satisfies all the Eilenberg-Steenrod properties by showing that singular homology satisfies the Homotopy and Excision Properties. The proof of the former will rely heavily on material from Section III.5 of these notes.

Homotopy invariance

We begin with a simple example:

PROPOSITION 0. *For each* $t \in [0,1]$ *let* $i_t : X \to X \times [0,1]$ *denote the slice inclusion* $i_t(x) = (x,t)$, *Then* i_0 *and* i_1 *are homotopic.*

Proof. The identity map on $X \times [0,1]$ defines a homotopy from i_0 to i_1.■

This observation will be useful in our proof of the homotopy property for singular homology groups.

Proof of Theorem IV.2.4. (Homotopy Invariance). We shall first show that it suffices to prove the theorem for the mappings i_0 and i_1 described in Proposition 0. For suppose we have continuous mappings $f, g : X \to Y$ and a homotopy $H : X \times [0,1] \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. Then we also have

$$f_* \quad = \quad (H \circ i_0)_* \quad = \quad H_* \circ (i_0)_* \quad = \quad H_* \circ (i_1)_* \quad = \quad (H \circ i_1)_* \quad = \quad g_*$$

showing that f and g define the same maps in homology.

To prove the result for the mappings in Proposition 0 we shall in fact prove that *the chain maps* $(i_0)_\#$ *and* $(i_1)_\#$ *from* $S_*(X)$ *to* $S_*(X \times [0,1])$ *are chain homotopic*. — The results of Section III.5 will then imply that the homology maps $(i_0)_*$ and $(i_1)_*$ are equal.

In Section III.5 we noted the existence of simplicial chains

$$P_{q+1} \quad \in \quad C_{q+1}(\Delta_q \times [0,1])$$

such that $P_0 = 0$, $P_1 = \mathbf{y}_0\mathbf{x}_0$ and more generally

$$dP_{q+1} \quad = \quad (i_1)_\# \mathbf{1}_q \ - \ (i_0)_\# \mathbf{1}_q \ - \ \sum_j (-1)^j (\partial_j \times 1)_\# P_q$$

where $\mathbf{1}_q = \mathbf{e_0} \ \cdots \ \mathbf{e}_q \in C_q(\Delta_q)$, the map $\partial_j = \partial_j^{[q]} : \Delta_{q-1} \to \Delta_q$ is affine linear onto the face opposite $\mathbf{e}_j$, and $(-)_\#$ generically denotes an associated chain map. Recall that the existence of the chains P_{q+1} was proved inductively, the key point being that since $\Delta_q \times \mathbf{I}$ is acyclic, such a chain exists if the boundary of

$$(i_1)_\# \mathbf{1}_q \ - \ (i_0)_\# \mathbf{1}_q \ - \ \sum_j (-1)^j (\partial_j \times 1)_\# P_q$$

is equal to zero.

To construct the chain homotopy $K : S_q(X) \to S_{q+1}(X \times [0,1]$, let $T : \Delta_q \to X$ be a free generator of $S_q(X)$ and set $K(T) = (T \times \mathrm{id}_{[0,1]})_\# P_{q+1}$. We then have

$$d\,K(T) \quad = \quad d \circ (T \times \mathrm{id}_{[0,1]})_\# P_{q+1} \quad = \quad (T \times \mathrm{id}_{[0,1]})_\# \circ d(P_{q+1}) \quad =$$

$$(T \times 1)_\# \circ (i_1)_\# \mathbf{1}_q \ - \ (T \times 1)_\# \circ (i_0)_\# \mathbf{1}_q \ - \ \sum_j (-1)^j d \circ (T \circ \partial_j \times 1)_\# P_q \quad =$$

$$(i_1)_\# \circ T_\#(\mathbf{1}_q) \ - \ (i_0)_\# \circ T_\#(\mathbf{1}_q) \ - \ \sum_j (-1)^j (T \circ \partial_j \times 1)_\# d(P_q) \quad =$$

$$(i_1)_\#(T) \ - \ (i_0)_\#(T) \ - \ K \circ d(T) \ .$$

Therefore K defines a chain homotopy between $(i_1)_\#$ and $(i_0)_\#$ as required.∎

Barycentric subdivision and small singular chains

Using the acyclicity of $C_*(\Delta_q)$ we may inductively construct chains $\beta_q \in C_q\big(B(\Delta_q)\big)$ (simplicial chains on the barycentric subdivision) such that $\beta_0 = \mathbf{1}_0$ and

$$d(\beta_q) \quad = \quad \sum_j (-1)^j \, (\partial_j)_\# \beta_{q-1}$$

for $q \geq 0$. If X is a topological space, then we may define a graded homomorphism $\beta : S_*(X) \to S_*(X)$ such that for each singular simplex $T : \Delta_q \to X$ we have $\beta(T) = T_\#(\beta_q)$.

LEMMA 1. *The graded homomorphism β is a map of chain complexes. Furthermore, if A is a subspace of X then β maps $S_*(A)$ into itself.*

Proof. We have $d \circ \beta(T) = d \circ T_\#(\beta_q) = T_\# \circ d(\beta_q)$, and the last term is equal to

$$T_\# \left(\sum_j (-1)^j \, (\partial_j)_\# \beta_{q-1} \right) \quad = \quad \sum_j (-1)^j \, (T \circ \partial_j)_\# \beta_{q-1}$$

which in turn is equal to $\beta\big(\, d(T) \,\big)$.■

The significance of the barycentric subdivision chain map is that it takes a chain in a given homology class and replaces it by a chain which is in the same homology class but is composed of smaller pieces; in fact, if one applies barycentric subdivision sufficiently many times, it is possible to find a chain representing the same homology class such that its chain are arbitrarily small. Justifications of these assertions will require several steps.

The first objective is to prove that the barycentric subdivision map is chain homotopic to the identity. As in previous constructions, this begins with the description of some universal examples.

PROPOSITION 2. *There are singular chains $L_{q+1} \in S_{q+1}(\Delta_n)$ such that $L_1 = 0$ and $d(L_{q+1}) = \beta_q - \mathbf{1}_q - \sum_j \, (-1)^j (\partial_j)_\#(L_q)$.*

By convention we take $L_0 = 0$.

Sketch of proof. Once again, the idea is to construct the chains recursively. Since Δ_q is acyclic, we can find a chain with the desired properties provided the difference

$$\beta_q \; - \; \mathbf{1}_q \; - \; \sum_j \, (-1)^j (\partial_j)_\#(L_q)$$

is a cycle. One can prove this chain lies in the kernel of d_q by using the recursive formulas for $d_q(\beta_q)$, $d_q(\mathbf{1}_q)$, and $d_q(L_q)$.■

PROPOSITION 3. *If X is a topological space and $A \subset X$ is a subspace, then the identity and the barycentric subdivision maps on $S_*(X, A)$ are chain homotopic.*

Proof. It will suffice to construct a chain homotopy on $S_*(X)$ that sends the subcomplex $S_*(A)$ to itself, for one can then obtain the relative statement by passage to quotients.

Define homomorphisms $W : S_q(X) \to S_{q+1}(X)$ on the standard free generators $T : \Delta_q \to X$ by the formula

$$W(T) \quad = \quad T_\# L_{q+1} \ .$$

By construction, if $T \in S_q(A)$ then $W(T) \in S_{q+1}(A)$. The proof that W is a chain homotopy uses the recursive formula for L_{q+1} and is entirely analogous to the proof that the map K in the proof of Theorem IV.2.4 is a chain homotopy.■

Small singular chains

We have noted that barycentric subdivision takes a cycle and replaces it by a homologous cycle composed of smaller pieces and that if one iterates this procedure then one obtains a chain whose pieces are arbitrarily small. Not surprisingly, we need to formulate this more precisely.

Definition. Let X be a topological space, and let $\mathcal{F}$ be a family of subsets whose interiors form an open covering of X. A singular chain $\sum_i n_i T_i \in S_q(X)$ is said to be *$\mathcal{F}$-small* if for each i the image $T_i(\Delta_q)$ lies in a member of $\mathcal{F}$. Denote the subgroup of $\mathcal{F}$-small singular chains by $S_*^{\mathcal{F}}(X)$. It follows immediately that the latter is a chain subcomplex of $S_*^{\mathcal{F}}(X)$; furthermore, if $A \subset X$ and we define $S_*^{\mathcal{F}}(A)$ to be the intersection of $S_*^{\mathcal{F}}(X)$ and $S_*^{\mathcal{F}}(A)$, then we may define relative $\mathcal{F}$-small chain groups of the form

$$S_*^{\mathcal{F}}(X,A) \quad = \quad S_*^{\mathcal{F}}(X)/S_*^{\mathcal{F}}(A) \ .$$

Note further that the barycentric subdivision maps send $\mathcal{F}$-small chains into $\mathcal{F}$-small chains.

THEOREM 4. *For all (X,A) and $\mathcal{F}$, the inclusion mappings $S_*^{\mathcal{F}}(X,A) \to S_*(X,A)$ define isomorphisms in homology.*

Proof. It is a straightforward algebraic exercise to prove that if L is a chain homotopy from the barycentric subdivision map β to the identity, then for each $r \geq 1$ the map $(1 + \ \cdots \ + \beta^{r-1}) \circ L$ defines a chain homotopy from β^r to the identity.

Let $\mathcal{U}$ be the open covering of X obtained by taking the interiors of the sets in $\mathcal{F}$.

Suppose first that we have $u \in H_*(X,A)$ and u is represented by the cycle $z \in S_*(X,A)$. Write $z = \sum_i n_i T_i$ and construct open coverings $\mathcal{W}_i$ of Δ_q by $\mathcal{W}_i = T_i^{-1}(\Delta_q)$. Then by the Lebesgue Covering Lemma there is a positive integer r such that for each i, every simplex in the r^{th} barycentric subdivision of Δ_q lies in a member of $\mathcal{W}_i$. It follows immediately that $\beta^r(z)$ is $\mathcal{F}$-small. Since β^r is a chain map, it follows that $\beta^r(z)$ is also a cycle in both $S_*(X,A)$ and the subcomplex $S_*^{\mathcal{F}}(X,A)$, and since β is chain homotopic to the identity it follows that

$$u \quad = \quad \beta_*(u) \quad = \quad \cdots \quad = \quad (\beta_*)^r(u) \quad = \quad (\beta^r)_*(u)$$

and hence u lies in the image of the homology of the small singular chain group.

To complete the proof we must show that if two cycles in $S_*^{\mathcal{F}}(X,A)$ are homologous in $S_*(X,A)$ then they are also homologous in $S_*^{\mathcal{F}}(X,A)$. Let z_1 and z_2 be the cycles, and let $dw = z_2 - z_1$ in $S_*(X,A)$. As in the preceding paragraph there is some t such that $\beta^t(w) \in S_*^{\mathcal{F}}(X,A)$. Since β^t is a chain map and is chain homotopic to the identity, it follows that we have

$$[z_2] \quad = \quad (\beta^t)_*[z_2] \quad = \quad [\beta^t(z_2)] \quad = \quad [\beta^t(z_1)] \quad = \quad (\beta^t)_*[z_1] \quad = \quad [z_1]$$

in the $\mathcal{F}$-small homology $H_*^{\mathcal{F}}(X,A)$. Therefore we have shown that the map from $H_*^{\mathcal{F}}(X,A)$ to $H_*(X,A)$ is also injective, and hence it must be an isomorphism.■

Application to Excision

We recall the hypotheses of the Excision Property: A pair of topological spaces (X,A) is given, and we have an open subset $U \subset X$ such that $\overline{U} \subset \text{Int}(A)$. Excision then states that the inclusion map of pairs from $(X-U, A-U)$ to (X,A) defines isomorphisms of singular homology groups.

Predictably, we shall use the previous results on small chains. Let $\mathcal{F}$ be the family of subsets given by A and $X-U$. Then by the hypotheses we know that the interiors of the sets in $\mathcal{F}$ form an open covering of X, and by definition the subcomplex $S_*^{\mathcal{F}}(X)$ is equal to $S_*(A) + S_*(X-U)$. Therefore the chain level inclusion map from $S_*(X-U, A-U)$ to $S_*(X,A)$ may be factored as follows:

$$S_*(X-U, A-U) \quad = \quad S_*(X-U)/S_*(A-U) \quad = \quad S_*(X-U)/\left(\, S_*(A) \cap S_*(X-U) \, \right) \quad \longrightarrow$$

$$(S_*(A) + S_*(X-U)) / S_*(A) \quad = \quad S_*^{\mathcal{F}}(X,A) \quad \subset \quad S_*(X,A)$$

Standard results in group theory imply that the last morphism on the top line is an isomorphism, and the preceding theorem shows that the last morphism on the second line is an isomorphism. Therefore if we pass to homology we obtain an isomorphism from $H_*(X-U, A-U)$ to $H_*(X,A)$, which is precisely the statement of the Excision Property.■

The same methods also yield the following result:

PROPOSITION 4A. *If U and V are open subsets of a topological space, then the maps in singular homology induced by the inclusions $(U, U \cap V) \subset (U \cup V, V)$ are isomorphisms.*■

Mayer-Vietoris sequences

One of the most useful results for computing fundamental groups is the Seifert-van Kampen Theorem. There is a similar principle that can be applied to find the homology groups of a space X presented as the union of two open subsets U and V; in fact, the result in homology does not require any connectedness hypotheses on the intersection.

THEOREM 5. (Mayer-Vietoris Sequence for open sets in singular homology.) *Let X be a topological space, and let U and V be open subsets such that $X = U \cup V$. Denote the inclusions of U and V in X by i_U and i_v respectively, and denote the inclusions of $U \cap V$ in U and V by g_U and g_V respectively. Then there is a long exact sequence*

$$\cdots \to H_{q+1}(X) \to H_q(U \cap V) \to H_q(U) \oplus H_q(V) \to H_q(X) \to \cdots$$

in which the map from $H_*(U) \oplus H_*(V)$ to $H_*(X)$ is given on the summands by $(i_U)_*$ and $(i_V)_*$ respectively, and the map from $H_*(U \cap V)$ to $H_*(U) \oplus H_*(V)$ is given on the factors by $-(g_U)_*$ and $(g_V)_*$ respectively (note the signs!!).

Proof. Let $\mathcal{U}$ be the open covering of X whose sets are U and V, and let $S_*^{\mathcal{U}}(X)$ be the chain complex of all $\mathcal{U}$-small chains in $S_*(X)$. Then we have

$$S_*^{\mathcal{U}}(X) \quad = \quad S_*(U) \ + \ S_*(V) \quad \subset S_*(X)$$

(note that the sum is not direct) and hence we also have the following short exact sequence of chain complexes, in which the injection is given by the chain map whose coordinates are $-(g_U)_\#$ and $(g_V)_\#$ and the surjection is given on the respective summands by $(i_U)_\#$ and $(i_V)_\#$:

$$0 \longrightarrow S_*(U \cap V) \longrightarrow S_*(U) \oplus S_*(V) \longrightarrow S_*^{\mathcal{U}}(X) \longrightarrow 0$$

The Mayer-Vietoris sequence is the long exact homology sequence associated to this short exact sequence of chain complexes combined with the isomorphism $H_*^{\mathcal{U}}(X) \cong H_*(X)$.■

In simplicial homology one also has a Mayer-Vietoris sequence, but for much different types of subspaces. Specifically, if $\mathbf{K}_1$ and $\mathbf{K}_2$ are subcomplexes of some $\mathbf{K}$, where $(P, \mathbf{K})$ is a simplicial complex, then the corresponding Mayer-Vietoris sequence has the following form:

$$\cdots \to H_{q+1}(\mathbf{K}) \to H_q(\mathbf{K}_1 \cap \mathbf{K}_2) \to H_q(\mathbf{K}_1) \oplus H_q(\mathbf{K}_2) \to H_q(\mathbf{K}) \to \cdots$$

It is possible to construct a unified framework that will include both of these exact sequences, but we shall not do so here because it involves numerous further results about simplicial complexes. However, it is important to note that in general one does NOT have a Mayer-Vietoris sequence in singular homology for presentations of a spaces X as a union of two closed subsets, and *this even fails for compact subsets of the* 2*-sphere.*

Example. Start with the graph Γ_0 of $\sin(1/x)$ for, say, $0 < |x| \leq 1/(2\pi)$, and consider the following set:

$$\Gamma \quad = \quad \Gamma_0 \cup \{\pm 1/(2\pi)\} \times [-2, 0] \cup \{0\} \times [-1, 1] \cup [-1/(2\pi), 1/(2\pi)] \times \{-2\}$$

This is a compact connected subset of the plane, and it has two arc components; namely, the segment $\{0\} \times [-1, 1]$ and its complement. The latter is homeomorphic to an open interval, and hence both arc components are contractible. Therefore we know that $H_q(\Gamma) = 0$ if $q \neq 0$ and $H_0(\Gamma) \cong \mathbb{Z}^2$. Now let B be the set of points (x, y) in $\mathbb{R}^2$ satisfying

$$0 \leq |x| \leq 1/(2\pi) \ \text{ and either } \ -2 \leq y \leq \sin(1/x) \ (x \neq 0) \ \text{ or } \ |y| \leq 1 \ \text{ if } \ x = 0 \ .$$

It follows immediately that $B = \text{Interior}(B) \cup \Gamma$, where the two subsets on the right hand side are disjoint. Viewing $\mathbb{R}^2 \subset S^2$ in the usual way, let $A = S^2 - \text{Interior}(B)$. It is straightforward to show that the subset $\{-\frac{3}{2}\} \times [-1/(2\pi), 1/(2\pi)]$ is a strong deformation retract of B; specifically, the retraction r sends (x, y) to $(x, -\frac{3}{2})$ and the homotopy is given by $t \cdot r(x, y) + (1-t) \cdot (x, y)$. Therefore we know that the singular homology groups of Γ and B are zero in all positive dimensions.

If there was an exact Mayer-Vietoris sequence

$$\cdots \to H_q(\Gamma) \to H_q(A) \oplus H_q(B) \to H_q(S^2) \to H_{q-1}(\Gamma) \cdots$$

then the results of the preceding paragraph would imply that $H_q(A) \cong H_q(S^2)$ for all $q \geq 2$, and in particular that the map $H_2(A) \to H_2(S^2)$ is nontrivial. Now A is a proper subset of S^2, and it is elementary to prove the following result:

LEMMA 6. *If $n > 0$ and A is a proper subset of S^n, then the inclusion map induces the trivial homomorphism from $H_n(A)$ to $H_n(S^n) \cong \mathbb{Z}$.*

Proof of Lemma 6. If $\mathbf{p}$ is a point of S^n that does not lie in A, then the homology map defined by inclusion factors as a composite

$$H_n(A) \to H_n(S^n - \{\mathbf{p}\}) \to H_n(S^n)$$

and this map is trivial because the complement of $\mathbf{p}$ is homeomorphic to $\mathbb{R}^n$ and the n-dimensional homology of the latter is trivial.■

This result and the discussion in the paragraph preceding the lemma yield a contradiction, and the source of this contradiction is our assumption that there is an exact Mayer-Vietoris sequence.

WHAT GOES WRONG IN THE EXAMPLE? In order to obtain an exact Mayer-Vietoris sequence for closed subsets, one generally needs an extra condition on the regularity of the inclusion maps. One standard type of condition on the closed subsets is that one can find arbitrarily small open neighborhoods such that the subsets are deformation retracts of these neighborhoods. This definitely fails for $\Gamma \subset \mathbb{R}^2$, for if such a neighborhood existed then there would be an open subset of $\mathbb{R}^2$ that would be connected but not arcwise connected.

IV.5 : Homology and the fundamental group

(Hatcher, § 2.A)

There is a simple but important relationship between the fundamental group $\pi_1(X, x)$ of a pointed arcwise connected space and the 1-dimensional homology $H_1(X) \cong H_1(X, \{x\})$.

Definition. Let $[S^1] \in H_1(S^1)$ be the homology class represented by the singular 1-simplex

$$T(1-s, s) \quad = \quad \big(\cos 2\pi s, \, \sin 2\pi s\big)$$

so that T corresponds to the standard counterclockwise parametrization of the unit circle under the identification of $[0,1]$ with the 1-simplex whose vertices are $(1,0)$ and $(0,1)$. The *Hurewicz* (hoo-RAY-vich) map $h : \pi_1(X, x) \to H_1(X)$ is given by taking a representative f of $\alpha \in \pi_1(X, x)$ and setting $h(\alpha) = f_*([S^1])$. By homotopy invariance, this class does not depend upon the choice of a representative.

The main theorem is easy to state,

THEOREM 1. *The mapping h defines a group homomorphism. More important, if X is arcwise connected, then h is onto and its kernel is the commutator subgroup of $\pi_1(X, x)$.*

The assertion in the first sentence of the theorem is verified on page 167 of Hatcher; the proof of the assertion in the second sentence will take the remainder of this section.

Suppose that (X, x) is a pointed space such that X is arcwise connected. The *Eilenberg subcomplex* $\overline{S_*}(X) \subset S_*(X)$ is the chain subcomplex generated by all singular simplices $T : \Delta_q \to X$ which send each vertex of Δ_q to the chosen basepoint x.

PROPOSITION 2. *Under the conditions given above, the inclusion of the Eilenberg subcomplex defines an isomorphism in singular homology.*

Sketch of proof. For each $y \in X$ there is a continuous curve joining y to x, and hence for each singular 0-simplex given by a point y there is a singular 1-simplex $P(y)$ such that $P(y) \circ \partial_1$ is the constant function with value x and $P(y) \circ \partial_0$ is the constant function with value y; clearly it is possible to choose $P(x)$ to be the constant function, and we shall do so. Starting from this, we claim by induction on q that for each singular q-simplex $T : \Delta_q \to X$ there is a continuous map

$$P(T) : \Delta_q \times [0,1] \ \longrightarrow \ X$$

with the following properties:

(i) The restriction of $P(T)$ to $\Delta_q \times \{0\}$ is given by T, and the restriction of $P(T)$ to $\Delta_q \times \{1\}$ is given by a singular simplex in the Eilenberg subcomplex.

(ii) If T lies in the Eilenberg subcomplex, then $P(T)$ is equal to $T \times \mathrm{id}_{[0,1]}$.

(iii) For each face map $\partial_i : \Delta_{q-1} \to \Delta_q$ we have $P(T \circ \partial_i) = P(T) \circ (\partial_i \times \mathrm{id}_{[0,1]})$.

To complete the inductive step, one uses (iii) and the first property in (i) to define $P(T)$ on $\Delta_q \times \{0\} \ \cup \ \partial\Delta_q \times [0,1]$, and then one extends this to all of $\Delta_q \times [0,1]$ using the Homotopy Extension Property.

Let i denote the inclusion of the Eilenberg subcomplex, and define a map ρ from $S_*(X)$ to the Eilenberg subcomplex by taking $\rho(T)$ to be the restriction of $P(T)$ to $\Delta_q \times \{1\}$. The property

(iii) ensures that ρ is a chain map, and we also know that $\rho \circ i$ is the identity on the Eilenberg subcomplex. The proof of the proposition will be complete if we can show that $i \circ \rho$ is chain homotopic to the identity. The proof of this is very similar to the proof of homotopy invariance. Let $\mathbf{P}_{q+1} \in S_{q+1}(\delta_q \times [0,1])$ be the standard chain used in that proof, and define

$$E(T) \quad = \quad \big(P(T)\big)_{\#}\mathbf{P}_{q+1} \ .$$

Then the properties of $\mathbf{P}_{q+1}$ and its boundary imply this defines a chain homotopy from the identity to $i \circ \rho$.■

Conclusion of the proof of Theorem 1. We shall use the following commutative diagram:

$$\begin{array}{ccccc}
F_2(X,x) & \xrightarrow{\ \mathbf{abel}\ } & \overline{S_2}\,(X) & \xrightarrow{\ =\ } & \overline{S_2}\,(X) \\
\Big\downarrow \delta & & \Big\downarrow d_2 & & \Big\downarrow d_2 \\
F_1(X,x) & \xrightarrow{\ \mathbf{abel}\ } & \overline{S_1}\,(X) & \xrightarrow{\ =\ } & \overline{S_1}\,(X) \\
\Big\downarrow \mathbf{can} & & \Big\downarrow \mathbf{can}' & & \Big\downarrow \mathbf{class} \\
\pi_1(X,x) & \xrightarrow{\ \mathbf{abel}\ } & \pi_1^{\mathbf{ab}}(X,x) & \xrightarrow{\ h'\ } & H_1(X)
\end{array}$$

Many items in this diagram need to be explained. On the bottom line, $\pi_1^{\mathbf{ab}}$ denotes the abelianization of the fundamental group formed by factoring out the (normal) commutator subgroup, and the Hurewicz map has a unique factorization as $h' \circ \mathbf{abel}$, where **abel** refers to the canonical surjection from π_1 to its quotient modulo the commutator subgroup. The groups $F_j(X,x)$ are the free groups on the free generators for the Eilenberg subcomplexes $\overline{S_*}\,(X)$, and **abel** generically denotes the passage from free groups to the corresponding free abelian groups. The maps d_2 and class are merely the relevant maps for the Eilenberg subcomplex, the map **can** ′” is the abelianization of the map **can** taking a free generator $T : \Delta_1 \to X$, which is merely a closed curve in X based at x, to its homotopy class in the fundamental group. Finally, δ is a nonabelian boundary map defined on free generators by

$$\delta(T) \quad = \quad [T \circ \partial_2] \cdot [T \circ \partial_0] \cdot [T \circ \partial_1]^{-1} \ .$$

Observe that the composite $\mathbf{can} \circ \delta$ is trivial and hence its abelianization $\mathbf{can}' \circ d_2$ is also trivial.

Proof that the Hurewicz map is onto. Suppose we are given a cycle $z = \sum_i \ n_i T_i$ in the Eilenberg subcomplex. and we let $\gamma(T_i) \in F_1(X,x)$ denote the free generator corresponding to T_i. Then it follows immediately from the commutative diagram that the homology class u represented by z satisfies

$$u \ = h(\alpha) \ , \qquad \text{where} \quad \alpha \quad = \quad \prod_i \big[\mathbf{can}\big(\gamma(T_i)\big)\big]^{n_i} \ .$$

Proof that the reduced Hurewicz map (i.e., its factorization through the abelianization of the fundamental group) is injective. Suppose that $h(\alpha) = 0$ and that the free generator $y \in F_1(X,x)$ represents α. Then it follows that $\mathbf{abel}(y) = d_2(w)$ for some 2-chain w, and if $w' \in F_2(X,x)$ projects to w then $y = \delta(w) \cdot v$, where v lies in the commutator subgroup of $F_1(X,x)$. Since $\mathbf{can} \circ \delta$ is trivial, it follows that the image of y in $\pi_1^{\mathbf{abel}}$ is trivial. Finally, since the image of y in π_1 is α, it also follows that the image of α in $\pi_1^{\mathbf{abel}}$ is trivial, or equivalently that α lies in the commutator subgroup.■

V. Geometric applications

Now that we have constructed homology groups, it is natural to ask what sorts of information these "algebraic pictures" of spaces can yield. This unit describes some of the most basic things that can be done with the subject. The importance of homology groups in analyzing homotopy classes of maps from one space to another are illustrated by two fundamental results whose proofs appear in most comprehensive (as opposed to introductory) texts on algebraic topology, and they can be found in Hatcher.

SPECIAL CASE OF HOPF'S THEOREM. *Let P be a finite n-dimensional polyhedron such that $H_{n-1}(P)$ has no elements of finite order. Then there is a $1-1$ correspondence between the set of homotopy classes $[P, S^n]$ and the algebraic homomorphisms from $H_n(P)$ to $H_n(S^n) \cong \mathbb{Z}$.*

There is also a version of Hopf's Theorem for n-dimensional polyhedra for which $H_{n-1}(P)$ has elements of finite order, but we do not have the background needed to state it here. Since the result obviously also holds if P is merely homeomorphic to a polyhedron, it follows that two continuous maps from S^n to itself are homotopic if and only if they induce the same homomorphism from $H_n(S^n) \cong \mathbb{Z}$ to itself; such a homomorphism is determined by its value on a generator and thus determines a number called the *degree*. We shall look at this concept further in Section V.1.

SIMPLY CONNECTED CASE OF J. H. C. WHITEHEAD'S THEOREM. *Suppose that P and Q are finite* simply connected *polyhedra and $f : P \to Q$ is a continuous map such that for each $i \geq 0$ the induced map of homology $f_* : H_i(P) \to H_i(Q)$ is an isomorphism. Then f is a homotopy equivalence.*

The converse is an immediate consequence of the functoriality and homotopy invariance of homology groups. There are versions of Whitehead's Theorem for connected finite polyhedra that are not simply connected, but once again we do not have the background needed to formulate such a result here. However, it is important to note that the non-simply connected case requires stronger hypotheses than the condition that f defines isomorphisms of ordinary homology groups (specifically, one needs to know that f induces an isomorphism of fundamental groups and isomorphisms on the homology groups of the universal covering spaces for P and Q).

V.1 : Degree theory

(Hatcher, § 2.2)

Definition. If $n > 0$ and $f : S^n \to S^n$ is a continuous mapping, then the *degree* of f is the unique integer d such that the map $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by d (recall that $H_n(S^n) \cong \mathbf{Z}$ and every homomorphism of the latter to itself is multiplication by some integer).

Several properties of the degree are immediate:

(1) If f is the identity, then the degree of f is 1.

(2) If f is a constant map, then the degree of f is 0.

(3) If f and g are homotopic, then their degrees are equal.

(4) If f and g are continuous maps from S^n to itself, then the degree of $f \circ g$ is equal to the degree of f times the degree of g.

(5) If h is a homeomorphism of S^n to itself, then the degree of h and h^{-1} is ± 1, and the degree of $h \circ f \circ h^{-1}$ is equal to the degree of f.

(6) If $n = 1$ and $f(z) = z^m$ (complex arithmetic), then the degree of f is equal to m.

The last property is the only one which is nontrivial. It follows because (a) the map f_* from $\pi_1(S^1, 1) \cong \mathbf{Z}$ is multiplication by m, (b) the Hurewicz map from $\pi_1(S^1, 1)$ to $H_1(S^1)$ is an isomorphism, (c) the Hurewicz map defines a natural transformation of functors from the fundamental group to 1-dimensional singular homology.■

For all $n \geq 2$, there is a standard recursive process for constructing continuous maps from S^n to itself with arbitrary degree.

PROPOSITION 1. *Let $f : S^{n-1} \to S^{n-1}$ be a continuous mapping of degree d, and let $\Sigma(f) : S^n \to S^n$ be defined on $(x, t) \in S^n \subset \mathbb{R}^n \times \mathbb{R}$ by*

$$\Sigma(f)\big(x, t\big) \quad = \quad \left(\sqrt{1 - t^2} f(x),\, t \right) .$$

Then the degree of $\Sigma(f)$ is also equal to d.

COROLLARY 2. *If $n \geq 1$ and d is an arbitrary integer, then there exists a continuous mapping $g : S^n \to S^n$ whose degree is equal to d.*

The case $n = 1$ of the corollary is just (6), above, and the proposition supplies the inductive step to show that if the corollary is true for $(n - 1)$ then it is also true for n.■

Proof of Proposition 1. We should check first that the map $\Sigma(f)$ is continuous. This is immediate from the formula for all points except the north and south poles, and at the latter one can check directly that if $\varepsilon > 0$ then we can take $\delta = \varepsilon$.

Define D^n_+ and D^n_- to be the subsets of S^n on which the last coordinates are nonnegative and nonpositive respectively. It follows immediately that S^n is formed from S^{n-1} by attaching two n-cells corresponding to $D^n_\pm$. This and the vanishing of the homology of disks in positive dimensions imply that all the arrows in the diagram below are isomorphisms:

$$H_{*-1}(S^{n-1}) \leftarrow H_*(D^n_+, S^{n-1}) \rightarrow H_*(S^n, D^n_-) \leftarrow H_*(S^n)$$

Furthermore, the mappings f and $\Sigma(f)$ determine homomorphisms from each of these homology groups to themselves such that the following diagram commutes:

$$\begin{array}{ccccccc}
H_{n-1}(S^{n-1}) & \xleftarrow{\cong} & H_n(D^n_+, S^{n-1}) & \xrightarrow{\cong} & H_n(S^n, D^n_-) & \xleftarrow{\cong} & H_*(S^n) \\
\downarrow f_* & & \downarrow \Sigma(f)_* & & \downarrow \Sigma(f)_* & & \downarrow \Sigma(f)_* \\
H_{n-1}(S^{n-1}) & \xleftarrow{\cong} & H_n(D^n_+, S^{n-1}) & \xrightarrow{\cong} & H_n(S^n, D^n_-) & \xleftarrow{\cong} & H_*(S^n)
\end{array}$$

It follows immediately that the degrees of f and $\Sigma(f)$ must be equal.■

Here is another basic property:

PROPOSITION 3. *If $f : S^n \to S^n$ is continuous and the degree of f is nonzero, then f is onto.*

Proof. If the image of f does not include some point $\mathbf{p}$, then f_* has a factorization of the form

$$H_n(S^n) \ \to \ H_n(S^n - \{\mathbf{p}\}) \ \to \ H_n(S^n)$$

and this homomorphism is trivial because the middle group is zero.■

Linear algebra and degree theory

We shall start with orthogonal transformations.

PROPOSITION 4. *Suppose that T is an orthogonal linear transformation of $\mathbb{R}^n$, where $n \geq 2$, and let $f_T : S^{n-1} \to S^{n-1}$ be the corresponding homeomorphism of S^{n-1}. Then the degree of f_T is equal to the determinant of T.*

Sketch of proof. We shall use a basic fact about orthogonal matrices; namely, if A is an orthogonal matrix then there is another orthogonal matrix B such that $B \cdot A \cdot B^{-1}$ is equal to a block sum of 2×2 rotation matrices plus a block sum of 1×1 matrices such that at most one of the latter has an entry of -1 (and the rest must have entries of 1).

Every 2×2 rotation matrix can be joined to the identity by a path consisting entirely of 2×2 rotation matrices. Therefore it follows that f_T is homotopic to f_S, where S is a diagonal matrix with at most one entry equal to -1 and all others equal to 1. Clearly the degrees of f_S and f_T are equal, and likewise the determinants of S and T must be equal (by continuity of the determinant and the fact that its value for an orthogonal matrix is always ± 1). Thus the proof reduces to showing that the degree of f_S is equal to -1 if there is a negative diagonal entry and is equal to 1 if there are no negative diagonal entries. — In fact, the second statement is obvious since T and f_T are identity mappings in this case.

Therefore everything reduces to showing that the degree of f_S is equal to -1. We can use Proposition 2 to show that the result is true for all n if it is true for $n = 2$, and the truth of the result when $n = 2$ follows immediately from Property (6) of degrees that was stated at the beginning of this document.■

We shall now consider an arbitrary invertible linear transformation T from $\mathbb{R}^n$ to itself. Such a map is a homeomorphism and thus extends to a map $T^\bullet$ of one point compactifications from S^n to itself.

THEOREM 5. *In the setting above, the degree of $T^\bullet$ is equal to the sign of the determinant of T.*

The proof of this result requires some additional input.

LEMMA 6. *Suppose that we are given a continuous curve T_t defined for $t \in [0,1]$ and taking values in the set of all invertible linear transformations on $\mathbb{R}^n$ (equivalently, invertible $n \times n$ matrices). Then $T_0^\bullet$ is homotopic to $T_1^\bullet$.*

Proof of Lemma 6. We would like to define a homotopy by the formula $H_t = T_t^\bullet$, and we can do so if and only if the latter is continuous at every point of $\{\infty\} \times [0,1]$. The latter in turn reduces to showing the following: *For each $t \in [0,1]$ and $M > 0$ there are numbers $\delta > 0$ and $P > 0$ such that $|s - t| < \delta$ and $|v| \geq P$ imply $|T_s(v)| \geq M$.*

Let $\|T\|$ be the usual norm of a linear transformation given by the maximum value of $|T|$ on the unit sphere. It follows immediately that the norm is a continuous function in (the matrix entries associated to) T. It follows that

$$|T_s(v)| \quad \geq \quad \|T_s^{-1}\| \cdot |v|$$

and since the inverse operation is also continuous it follows that $\|T_s^{-1}\|$ is a continuous function of s. In particular, if $\|T_t^{-1}\| = B > 0$ then we can find $\delta > 0$ such that $|s - t| < \delta$ implies $\|T_s^{-1}\| > B/2$, and hence if $|v| > 2M/B$ and $|t - s| < \delta$ then $T_s(v)| \geq M$, as required.■

Proof of Theorem 5. Both the degree of $T^\bullet$ and the sign of the determinant are homomorphisms from invertible matrices to $\{\pm 1\}$, and therefore it will suffice to prove the theorem for a set of linear transformations which generate all the invertible linear transformations. Not surprisingly, we shall take this set to be the linear transformations given by the elementary matrices.

Let $E_{i,j}$ denote the $n \times n$ matrix which has a 1 in the (i, j) entry and zeros elsewhere. Then the function sending $t \in [0, 1]$ to $I + tE_{i,j}$ defines a curve from the elementary matrix $I + E_{i,j}$ to the identity. Therefore the associated linear transformation determines a map which is homotopic to the identity, and consequently the degree and determinant sign agree for elementary linear transformations given by adding a multiple of one row to another.

Similarly, if $D(k, r)$ is a diagonal matrix which has ones except in the k^{th} position and a positive real number r in the latter position, then there is a continuous straight line curve joining the matrix in question to the identity, and this matrix takes values in the group of invertible diagonal matrices. It follows that the degree and determinant sign agree for elementary linear transformations given by multiplying one row by a positive constant.

We are now left with elementary matrices given by either multiplying one row by -1 or by interchanging two rows. These two types of matrices are similar, so both the degrees and determinant signs are equal in each case. Therefore it will suffice to check that the degree and determinant sign agree when one considers an elementary matrix given by multiplying a single row by -1.

By Proposition 2 and the invariance of our numerical invariants under similarity, it will suffice to consider the case where $n = 2$ and we are multiplying the second row by -1. Let $W \subset \mathbb{R}^2$ be the open disk of radius 2 about the origin, so that there is a canonical homeomorphism from $W - \{\mathbf{0}\}$ to $S^1 \times (0, 2)$. Now the map $T^\bullet$ sends $S^2 - \{\mathbf{0}\}$ to itself and likewise for W and S^1. Excision and homotopy invariance now yield the following chain of isomorphic homology groups:

$$H_1(S^1) \leftarrow H_1(W - \{\mathbf{0}\}) \rightarrow H_2(W, W - \{\mathbf{0}\}) \leftarrow H_2(S^2, S^2 - \{\mathbf{0}\}) \longrightarrow H_2(S^2)$$

As in Proposition 3, one has associated maps of homology groups to form a corresponding commutative diagram, and from this diagram one sees that the degree of $T^\bullet$ is equal to the degree of the map determined by $T^\bullet$ on S^1. Since the map on S^1 is merely the mapping sending z to z^{-1}, it follows that the degree is equal to -1, and of course this is the same as the sign of the determinant.■

The Fundamental Theorem of Algebra

One can use degree theory to prove the Fundamental Theorem of Algebra. All proofs of the latter involve some analysis and plane topology, and one advantage of the degree-theoretic proof is that the role of topology is particularly easy to recognize. This proof can also be generalized to obtain a generalization of the Fundamental Theorem of Algebra to polynomials with quaternionic coefficients (this was done by Eilenberg and Niven in the nineteen forties).

We start with an argument that is similar to the proof in the last part of Theorem 5.

PROPOSITION 7. *The map ψ^m of the complex plane sending z to z^m (where m is a positive integer) extends continuously to a map of one point compactifications sending the point at infinity to itself, and the degree of the compactified map is equal to m.*

Proof. The existence of a continuous extension follows because if $M > 0$ then $|z| > M^{1/m}$ implies $|z^m| > M$.

It follows that ψ^m sends $\mathbb{C} - \{\mathbf{0}\}$ to itself. Of course, the map also sends S^1 to itself and this map has degree m, so a diagram chase plus the naturality of the Hurewicz homomorphism imply that ψ^m_* is multiplication by m on $H_1(\mathbb{C} - \{\mathbf{0}\}) \cong \mathbf{Z}$. Diagram chases now show that ψ_* is multiplication by m on

$$H_2(\mathbb{C}, \mathbb{C} - \{\mathbf{0}\}) \quad \cong \quad H_2(S^2, S^2 - \{\mathbf{0}\}) \quad \cong \quad H_2(S^2)$$

and thus the degree of the compactified map is equal to m.■

The following result is standard.

PROPOSITION 8. *If p is a nonconstant monic polynomial, then p extends continuously to a map of one point compactifications sending the point at infinity to itself.*

Sketch of proof. We need to show that if $M > 0$ then there is some $\rho > 0$ such that $|z| > \rho$ implies $|p(z)| > M$. One easy way of doing this is to begin by writing p as follows:

$$p(z) \quad = \quad z^m \cdot \Big(1 \; + \; \frac{a_{n-1}}{z} \; + \; \cdots \; + \; \frac{a_0}{z^n}\Big)$$

If we write the expression inside the parentheses as $1 + b(z)$, then it is clear that if $|z|$ is sufficiently large (say $|z| > N$) then $|b(z)| < \frac{1}{2}$. It follows immediately that if $M > 0$ and $|z| > 2M^{1/m} + N$ then $|p(z)| > M$.■

The Fundamental Theorem of Algebra will now be a consequence of Proposition 3 and the following generalization of Proposition 8:

PROPOSITION 9. *If p is a nonconstant monic polynomial of degree $m \geq 1$, then the degree of the compactified map $p^\bullet$ is equal to m.*

Proof. It will suffice to show that $p^\bullet$ is homotopic to $(\psi^m)^\bullet$.

Define a homotopy from ψ^m to p on the set where $|z| \geq N + 1$ by $h_t(z) = z^m(1 + t\,b(z)\,)$. By the Tietze Extension Theorem, one can extend this to a homotopy over all of $\mathbb{C}$. As in the previous argument, if $M > 0$ and $|z| > 2M^{1/m} + N + 1$ then $|h_t(z)| > M$ for all t. One can then argue as in the first paragraph of the proof of Lemma 6 to show that $p^\bullet$ is homotopic to $(\psi^m)^\bullet$.■

V.2 : Classical theorems of Jordan and Brouwer

(Hatcher, § 2.B; Munkres, 61–64)

Most of this has become standard in algebraic topology texts, and we shall quote Hatcher as appropriate. The following result corresponds to the first half of Proposition 2B.1 on page 169 of that reference.

PROPOSITION 1. *If $A \subset S^n$ is homeomorphic to D^k for some $k < n$, then the $H_i(A)$ is infinite cyclic if $i = 0$ and trivial otherwise.*

Since Hatcher's statement involves reduced homology and this concept has not yet been discussed in these notes, we shall do so now. There are (at least) two ways of looking at the reduced homology of a space X. If P is a space with one point and $c : X \to P$ is the constant map, then the reduced homology $\widetilde{H}_*(X)$ may be viewed as the kernel of the homomorphism c_* in homology. If X is nonempty and $b : P \to X$ maps the point in P to an arbitrary point in X, then $c \circ b$ is the identity on P, and it follows that there is a direct sum decomposition

$$H_*(X) \;\cong\; \widetilde{H}_*(X) \;\oplus\; H_*(P)\ .$$

This has the following consequences:

(1) If $i \neq 0$, then $H_i(X) \cong \widetilde{H}_*(X)$.

(2) If $i - 0$, then $H_i(X) \cong \widetilde{H}_*(X) \oplus \mathbb{Z}$. — In particular, X is arcwise connected if and only if $\widetilde{H}_0(X)$ is trivial.

It follows immediately that if X is a nonempty space and $b \in X$, then the reduced homology of X is isomorphic to the homology of the pair $(X, \{b\})$ (verify this!). Using this description, one can prove the following result which is needed in Hatcher's (standard) proof of Proposition 1:

PROPOSITION 2. (Reduced Mayer-Vietoris Sequence in singular homology) *Let X be a topological space, and let U and V be open subsets such that $X = U \cup V$ and $U \cap V$ is nonempty. Denote the inclusions of U and V in X by i_U and i_v respectively, and denote the inclusions of $U \cap V$ in U and V by g_U and g_V respectively. Then there is a long exact sequence as in Theorem* IV.4.5 *in which ordinary homology groups are replaced by reduced homology groups.*

Sketch of proof. Let $b \in U \cap V$. Then there is a short exact sequence of chain complexes

$$0 \longrightarrow S_*(U \cap V, \{b\}) \longrightarrow S_*(U, \{b\}) \oplus S_*(V, \{b\}) \longrightarrow S_*^{\mathcal{U}}(X, \{b\}) \longrightarrow 0$$

analogous to the one which appears in the proof of Theorem IV.4.5, and the long exact homology sequence of this short exact sequence of chain complexes will be the reduced Mayer-Vietoris sequence.■

Note on the proof of Proposition 1. In order to use the relative Mayer-Vietoris sequence it is necessary to know from the start that A is a proper subset of S^n; however, A cannot be equal to S^n because the homology groups of A and S^n are not isomorphic.■

We shall state the Jordan-Brouwer Separation Theorem in a slightly more detailed version than the one in Hatcher:

THEOREM 3. (Jordan-Brouwer Separation Theorem.) *Let $n \geq 2$, and suppose that $A \subset S^n$ is homeomorphic to S^{n-1}. Then $S^n - A$ contains two components, and A is the frontier of each component.*

Note on the proof. The existence of two components is shown in the second half of Hatcher's previously cited Proposition 2B.1 (*q.v.*).

It remains to prove that points of A are limit points of each components. Suppose that $S^n - A$ is the union of the two open, connected, disjoint subsets U and V.

Assume that not every point of A is a limit point of both U and V. Without loss of generality, it is enough to consider the case where $x \in A$ is not a limit point of V. Since $x \notin V$, it follows that there is some open set W_0 in S^n such that $x \in W_0$ and $W_0 \cap V = \emptyset$.

Consider the open set $W_0 \cap A$ in A; since the latter is homeomorphic to S^{n-1}, it follows that there is a subneighborhood of the form $A - E$, where $E \subset A$ is homeomorphic to a closed $(n-1)$-disk and $A - E$ is homeomorphic to an open $(n-1)$-disk centered at x. If $W = W_0 \cap S^n - E$, then W is still open in S^n and we still have $x \in W$ and $W \cap V = \emptyset$.

By construction we have $S^n - E = U \cup A - E \cup V$ where the pieces are pairwise disjoint. Furthermore, we have $A - E \subset W$ and hence $U \cup W$ is an open set of $S^n - E$ which is disjoint from V and contains U and $A - E$. Therefore it follows that $S^n - E$ is a union of the nonempty disjoint open sets $U \cup W$ and V and hence is disconnected. On the other hand, since E is homeomorphic to a closed disk we know that $S^n - E$ is connected, so we have a contradiction. The source of this contradiction was our assumption that x was not a limit point of V, and hence this must be false. Therefore x must be a limit point of V, and as noted above it follows that every point of A is a limit point of both U and V.■

With the preceding results at our disposal, we can prove the following basic result exactly as in Hatcher:

THEOREM 4. (Invariance of Domain, Brouwer) *Let U and V be open subsets of $\mathbb{R}^n$ for some $n \geq 2$, and let $h : U \to V$ be continuous and $1-1$. Then h is an open mapping, and in particular $h[U]$ is an open subset of $\mathbb{R}^n$.*■

The name of the result refers to the fact that if V is homeomorphic to an open subset of $\mathbb{R}^n$, then V must also be an open subset of $\mathbb{R}^n$.

APPLICATION TO MANIFOLDS WITH BOUNDARY. If $n \geq 0$, ten a *topological n-manifold with boundary* is a Hausdorff space M such that every point has an open neighborhood which is homeomorphic to an open subset of either $\mathbb{R}^n$ or $\mathbb{R}^n_+$, where the latter is all points in $\mathbb{R}^n$ whose last coordinate is nonnegative. The previously defined notion of topological n-manifold may be viewed as a special case of this concept where the second alternative (open in $\mathbb{R}^n_+$) is not allowed. The *(manifold) interior* of M is defined to be the set of all points which have an open neighborhood homeomorphic to an open subset in $\mathbb{R}^n$, and the *(manifold) boundary* ∂M is the set of all points x for which there is an open neighborhood U homeomorphic to an open subset $W \subset \mathbb{R}^n_+$ such that x corresponds to a point in the intersection $W \cap \mathbb{R}^{n-1}$, where we view $\mathbb{R}^{n-1}$ as a subset of $\mathbb{R}^n_+$ by the standard identification and inclusion $\mathbb{R}^{n-1} \cong \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n_+$. It follows immediately that the manifold interior of M is an open subset of M and a topological n-manifold as defined previously, and the manifold boundary is a closed subset of M which is a topological $(n-1)$-manifold as defined previously; if $\partial M = \emptyset$, then M is a topological manifold in the previously defined sense (without boundary).

Perhaps the simplest nontrivial example of an n-manifold with boundary is $\mathbb{R}^n$. It follows immediately that the interior of $\mathbb{R}^n$ contains all points $(x_1, \cdots, x_n)$ such that $x_n > 0$ and the boundary consists of all points such that $x_n = 0$. Our terminology and some simple drawings suggest that boundary points are not interior points in the sense of the preceding definition, and using Invariance of Domain we can prove this:

PROPOSITION 5. *Let M be a topological manifold with boundary. Then no point in ∂M is an interior point of M.*

Proof. Suppose that $x \in \partial M$. Then x has an open neighborhood U which is homeomorphic to an open subset W of $\mathbb{R}^n_+$ such that $W \cap \mathbb{R}^{n-1}$ is nonempty and x corresponds to a point $(y, 0)$ in this intersection. If x is also an interior point of M, then there is an open subset W_0 of W which is homeomorphic to an open subset of $\mathbb{R}^n$. By Invariance of Domain the subset W_0 is actually open in $\mathbb{R}^n$, and as such it contains all points of the form (y, t) such that $-h < t < 0$ for some $h > 0$. However, by construction W does not contain any such points, and therefore W_0 does not either, so we have a contradiction. The source of this contradiction was our assumption that $W_0 \subset W$ was homeomorphic to an open subset in $\mathbb{R}^n$, and therefore such an open subneighborhood cannot exist. This is precisely the assertion in the proposition.■

Manifolds with boundary play an extremely important role in geometric topology. Many of the most basic examples arise as follows: Let U be open in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}$ be a function with continuous partials such that if $f(\mathbf{x}) = \mathbf{0}$ then $\nabla f(\mathbf{x})$ is nonzero; typical example of such functions are given by nonconstant linear functions and many quadratic polynomial functions, most notably $f(\mathbf{x}) = 1 - |\mathbf{x}|^2$. Basic results from multivariable calculus then imply that $f^{-1}(\mathbb{R}_+)$ is a manifold with boundary. Of course, if we take $f(x) = 1 - |x|^2$, then the inverse image is just the unit disk defined by $|x|^2 \leq 1$, and the boundary of this manifold is just the unit sphere defined by $|x|^2 = 1$. Another example is the doughnut-shaped region in $\mathbb{R}^3$ defined by the cylindrical coordinate inequality $(r-1)^2 + z^2 \leq \frac{1}{4}$.

Further results

Since the 2-dimensional case of the Jordan-Brouwer Separation Theorem is just the Jordan Curve Theorem which is proved in Chapter 10 of Munkres by other methods, we shall indicate how several of the results from that chapter can be retrieved using the methods developed here.

The starting point is Theorem 63.5 on page 392 of Munkres. This states that if A and B are closed subsets of S^2 such that $S^2 - A$ and $S^2 - B$ are connected and $A \cap B$ consists of two points, then $S^2 - (A \cup B)$ has exactly two components. If A and B are **arcwise connected by simple arcs** in the sense that every pair of points can be joined by a simple curve (one which is 1–1), then one can prove Theorem 63.5 by considering the reduced Mayer-Vietoris Sequence for the pair of open sets $S^2 - A$ and $S^2 - B$ as follows:

Under the given hypotheses we have the partial exact sequence

$$H_1(S^2 - A) \oplus H_1(S^2 - B) \longrightarrow H_1\big(S^2 - (A \cap B)\big) \cong \mathbb{Z} \longrightarrow \widetilde{H_0}\big(S^2 - (A \cup B)\big) \to 0$$

and this shows immediately that the open subset $S^2 - (A \cup B)$ has at most two components. On the other hand, we have simple closed curves in A and B with images C and D and endpoints given by the two points in $A \cap B$. This yields chains of inclusions

$$S^2 - A \subset S^2 - C \subset S^2 - (A \cap B)\,, \qquad S^2 - B \subset S^2 - D \subset S^2 - (A \cap B)\,.$$

Since the 1-dimensional homology groups of $S^2 - C$ and $S^2 - D$ both vanish, it follows that the inclusions above induce the trivial map in 1-dimensional homology. Therefore the left hand map in the exact sequence above must be trivial, and it follows that $H_1\big(S^2 - (A \cap B)\big) \cong \mathbb{Z}$ maps bijectively to $\widetilde{H_0}\big(S^2 - (A \cup B)\big)$ and therefore $S^2 - (A \cup B)$ must have exactly two components.■

This is weaker than the conclusion in Munkres, but it is strong enough to yield all of the applications to embedding graphs in the plane which appear in Section 64 of that book. The reason for this is that a connected graph is arcwise connected by simple arcs (verify this!). This is also enough to prove Exercise 1 on page 393 of Munkres as well as a weak versions of Exercise 2 on the same page. In the latter case, the result can be established if we assume that the subset $D \subset S^2$ is homeomorphic to a connected graph.

Another application along the same lines is the following *Duality Theorem*:

THEOREM 6. *Let $A \subset S^2$ be homeomorphic to a graph. Then for all integers k there are isomorphisms $H_{2-k}(S^2 - A) \cong H_k(S^2, A)$.*■

These isomorphisms also have good naturality properties with subcomplex inclusions $A \subset B$ (in other words, the pair (B, A) is homeomorphic to a pair consisting of a graph and a subcomplex of the latter with respect to some simplicial decomposition) given by the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(H_k(S^2,B),bbZ) & \xrightarrow{\mathrm{Hom}(i_*,\mathbb{Z})} & \mathrm{Hom}(H_k(S^2,A),bbZ) \\ \downarrow \cong & & \downarrow \cong \\ H_{2-k}(S^2-B) & \xleftarrow{j_*} & H_{2-k}(S^2-A) \end{array}$$

Unfortunately, proving this naturality property requires a substantial amount of additional input (compare the discussion in the third full paragraph on page 393 of Munkres), so we shall not attempt to do so. One can use this naturality to derive special cases of other results in Chapter 10 of Munkres; typical examples are Lemmas 61.2 and 62.2.

V.3 : Simplicial approximation

(Hatcher, § 2.C)

The treatment in Hatcher is fairly standard, so we shall only discuss a few issues here.

PROPOSITION 1. *Let $g : \mathbf{K} \to \mathbf{L}$ be a simplicial map, let $|g|$ be the associated continuous map of underlying topological spaces, and let λ_* denote the standard natural transformation obtained from the chain complex inclusion $C_*(\mathbf{K}) \to S_*(P)$, where P is the polyhedron with simplicial decomposition $\mathbf{K}$. Then $\lambda_* \circ g_* = |g|_* \circ \lambda_*$.*

This follows immediately from the construction of λ, for if $\mathbf{v}_0 \ \cdots \ \mathbf{v}_q$ is one of the free generators for $C_q(\mathbf{K})$, then its image under the associated simplicial chain map associated to g is $g(\mathbf{v}_0) \ \cdots g(\mathbf{v}_q)$i, and under the chain map $\lambda(\mathbf{L})_\#$ this goes to $|g|_\# \circ \lambda(\mathbf{K})_\# \, (\mathbf{v}_0 \ \cdots \ \mathbf{v}_q)$.■

COROLLARY 2. *Suppose that $(P, \mathbf{K})$ and $(Q, \mathbf{L})$ are simplicial complexes, and let $f : P \to Q$ be continuous. Suppose that $r > 0$ and $g : B^r(\mathbf{K}) \to \mathbf{L}$ are such that g is a simplicial approximation to f, and let $\beta_r : C_*(\mathbf{K}) \to C_*(B^r(\mathbf{K})\,)$ be the iterated barycentric subdivision map. Then $f_* \circ \lambda_* = \lambda_* \circ g_* \circ (\beta_r)_*$.*

Sketch of proof. We have an analog of β_r defined from $S_*(P)$ to itself, and by the results leading to the proof of the Excision Property this map is chain homotopic to the identity. From this it follows that $|g|_* \circ \lambda_* = \lambda_* \circ g_* \circ (\beta_r)_*$. Since g is a simplicial approximation to f we know that $f_* = |g|_*$, and if we make this substitution into the equation in the preceding sentence we obtain the assertion in the corollary.■

Of course, the point of the corollary is that one can compute the map in homology associated to f using the simplicial approximation g.

Given a continuous function f as above, one natural question about simplicial approximations is to find the value(s) of r for which there is a simplicial approximation $g : B^r(\mathbf{K}) \to \mathbf{L}$. The result below shows that in many cases we must take r to be very large.

PROPOSITION 3. *Suppose that $(P, \mathbf{K})$ and $(Q, \mathbf{L})$ are simplicial complexes, and let $f : P \to Q$ be continuous. Let $r_0(f) > 0$ be the smallest value of r such that f is homotopic to a simplicial map $g : B^r(\mathbf{K}) \to \mathbf{L}$. Then the following hold:*

(i) The number $r_0(f)$ depends only upon the homotopy class of f.

(*ii*) *If the set of homotopy classes* $[P,Q]$ *is infinite, then for each positive integer* M *there are infinitely many homotopy classes* $[f_n]$ *such that* $r_0(f_n) > M$.

Proof. The first part follows immediately from the definition, so we turn out attention to the second. Recall that a simplicial map is completely determined by its values on the vertices of the domain.

Suppose now that $\mathbf{L}$ has b vertices and $B^r(\mathbf{K})$ has a_r. There are b^{a_r} different ways of mapping the vertices of $B^r(\mathbf{K})$ to those of $\mathbf{L}$; although some of these might not arise from a simplicial map, we can still use this to obtain a finite upper bound on the number of simplicial maps from $B^r(\mathbf{K})$ to $\mathbf{L}$, and we also have a finite upper bound on the number of simplicial maps from $B^r(\mathbf{K})$ to $\mathbf{L}$ for all $r \leq M$ if M is any fixed positive integer. It follows that there are only finitely many homotopy classes for which $r_0 \leq M$.∎

In particular, by the results of Section V.1 we can apply this proposition to $[P,Q]$ where P and Q are both homeomorphic to S^n for some $n \geq 1$.

V.4 : The Lefschetz Fixed Point Theorem

(Hatcher, § 2.C)

Once again the treatment in Hatcher is fairly standard, so we shall only concentrate on a few issues.

From the viewpoint of these notes, the Lefschetz number is obtained using the traces of various maps on rational chain groups or cohomology groups. The proof that the alternating sum of traces is the same for simplicial chains and simplicial homology is a special case of the following result:

PROPOSITION 1. *Suppose that* C_* *is a chain complex of rational vector spaces such that each* C_q *is finite-dimensional and only finitely many are nontrivial, and let* $T : C_* \to C_*$ *be a chain map. Then*

$$\sum_q (-1)^q \,\mathrm{trace}\, T_q \quad = \quad \sum_q (-1)^q \,\mathrm{trace}\, (T_*)_q \ .$$

The proof of this combines the method of Proposition IV.4.10 with the following result:

LEMMA 2. *Let* V *be a finite-dimensional vector space over a field, let* W *be a vector subspace, and suppose that* $T : V \to V$ *is a linear transformation such that* $T[W] \subset W$*. Let* T_W *be the associated linear transformation from* W *to itself, and let* $T_{V/W}$ *denote the linear transformation from* V/W *to itself which sends* $\mathbf{v} + W$ *to* $T(\mathbf{v}) + W$ *for all* $\mathbf{v} \in V$ *(this is well-defined). Then* $\mathrm{trace}\,(T) = \mathrm{trace}\,(T_W) + \mathrm{trace}\,(T_{V/W})$.

Proof of Lemma 2. Pick a basis $\mathbf{w_1}, \cdots, \mathbf{w}_k$ for W and extend it to a basis for V by adding vectors $\mathbf{u}_{k+1}, \cdots, \mathbf{u}_n$. It follows that the vectors $\mathbf{u}_{k+1} + W, \cdots, \mathbf{u}_n + W$ form a basis for V/W. If we now let $\mathbf{v}$ denote either $\mathbf{v}$ or $\mathbf{w}$ and as usual write

$$T(\mathbf{v}_j) \quad = \quad \sum_i a_{i,j}\, \mathbf{v}_i$$

then the traces of T, T_W and $T_{V/W}$ are given by the sums of the scalars $a_{i,i}$ from 1 to n in the case of T, from 1 to k in the case of T_W, and from $k+1$ to n in the case of $T_{V/W}$.∎

As noted above, Proposition 1 follows by applying the same method used in Proposition IV.3.10 with the dimensions c_q, z_q, b_q and h_q replaced by the traces of the corresponding linear transformations.■

Vector fields on S^2

We may think of a *tangent vector field* on the sphere S^2 as a continuous map $\mathbf{X} : S^2 \to \mathbb{R}^3$ such that $\mathbf{X}(\mathbf{u})$ is perpendicular to $\mathbf{u}$ for all $\mathbf{u} \in S^2$ (in other words, the value of $\mathbf{X}$ at a point $\mathbf{u}$ in S^2 is the tangent vector to a curve passing through $\mathbf{u}$). One can use the Lefschetz Fixed Point Theorem to prove the following fundamental result on such vector fields.

THEOREM 3. *If $\mathbf{X}$ is a tangent vector field on S^2, then there is some $\mathbf{u} \in S^2$ such that $\mathbf{X}(\mathbf{u}) = \mathbf{0}$.*

Proof. Suppose that the vector field is everywhere nonzero. If we set

$$\mathbf{Y}(\mathbf{u}) \quad = \quad |\mathbf{X}(\mathbf{u})|^{-1} \cdot \mathbf{X}(\mathbf{u})$$

then $\mathbf{Y}$ is a continuous vector field such that $|\mathbf{Y}|$ is always equal to 1, so that $\mathbf{Y}$ defines a continuous map from S^2 to itself. By the perpendicularity condition we know that $\mathbf{Y}(\mathbf{u}) \neq \mathbf{u}$ for all $\mathbf{u}$, and therefore by the Lefschetz Fixed Point Theorem we know that the Lefschetz number of $\mathbf{Y}$ must be zero.

We now claim that $\mathbf{Y}$ defines a continuous map from S^2 to itself which is homotopic to the identity. Specifically, take the homotopy

$$H(\mathbf{u}, t) \quad = \quad \cos\left(\frac{t\pi}{2}\right) \cdot \mathbf{Y}(\mathbf{u}) \; + \; \sin\left(\frac{t\pi}{2}\right) \cdot \mathbf{u}$$

which which moves $\mathbf{u}$ to $\mathbf{Y}(\mathbf{u})$ along a 90° great circle arc. Since $\mathbf{Y}$ is homotopic to the identity, it follows that its Lefschetz number equals the Lefschetz number of the identity, which is $\chi(S^2) = 2$. This contradicts the conclusion of the preceding paragraph; the source of this contradiction was our assumption that $\mathbf{X}(\mathbf{u}) \neq \mathbf{0}$ for all $\mathbf{u}$, and therefore it follows that there is some $\mathbf{u}_0 \in S^2$ such that $\mathbf{X}(\mathbf{u}_0) = \mathbf{0}$.■

In fact, the same argument goes through virtually unchanged for all even-dimensional spheres. On the other hand, every odd-dimensional sphere does admit a tangent vector field which is everywhere nonzero. One quick way to construct an example is to take the vector field on $S^{2n+1} \subset \mathbb{R}^{2n+2}$ given by the formula

$$\mathbf{X}(x_1, x_2, x_3, x_4, \; \cdots \; , x_{2n+1}, x_{2n+2}) \quad = \quad (-x_2, x_1, -x_4, x_3, \; \cdots \; , -x_{2n+2}, x_{2n+1}) \; ;$$

if we view $\mathbb{R}^{2n+2}$ as $\mathbb{C}^{n+1}$, then the vector field sends a vector $\mathbf{z} = (z_1, \; \cdots \; , z_{n+1})$ to $i\,\mathbf{z}$.

Geometric interpretation of the Lefschetz number. Suppose that P is a polyhedron which is homeomorphic to a compact smooth manifold M (without boundary), and let $f : M \to M$ be a smooth self-map. Basic results on approximating mappings on smooth manifolds imply that f is homotopic to a smooth map $g : M \to M$ such that g has only finitely many fixed points and for each fixed point $x \in M$ the associated linear map of the tangent space $T(x)$ at x

$$L_f(x) \quad = \quad \mathbf{T}(g)_x : T(x) \;\longrightarrow\; T(x)$$

has the property that $L_f(x) - \mathrm{id}_{T(x)}$ is an isomorphism (in such cases the fixed point set is said to be isolated and nondegenerate). For each fixed point x one can define a *local fixed point index* $\Lambda(g)_x$ to be the sign of the determinant of $L_f(x) - \mathrm{id}_{T(x)}$. Under these conditions the Lefschetz number of g turns out to be given by

$$\Lambda(g) \quad = \quad \sum_{g(x)=x} \Lambda(g)_x \ .$$

Proving this is beyond the scope of these notes and requires the notion of *local fixed point index*. In the paper cited below, a set of axioms for fixed point indices of smooth maps is given, and Chapter 7 of the text by Dold explains how such indices are related to the Lefschetz number as described here:

A. Dold. *Lectures on Algebraic Topology*. (Second Edition). *Springer–Verlag, New York etc.*, 1980.

M. Furi, M. P. Pera, and M. Spadini. *On the uniqueness of the fixed point index on differentiable manifolds*. Fixed point theory and its applications **2004**, 251–259.

V.5 : Dimension theory

(Munkres, § 50)

At the beginning of these notes, we mentioned the following question:

Is there some purely topological way to describe the intuitive notion of $n-$dimensionality, at least for spaces that are relatively well-behaved?

Of course, in linear algebra there is the standard notion of dimension, and this concept has far-reaching consequences for understanding dimensions in geometry. A topological approach to describing the dimensions of at least some spaces is implicit in our proof for Invariance of Dimension (see Proposition IV.2.16), which can be used to define a notion of dimension for topological spaces which locally look like an open subset of $\mathbb{R}^n$ for some fixed $n \geq 0$. There is an extensive literature on topological approaches to defining the dimensions of spaces. Our purpose here is to discuss one particularly important example known as the *Lebesgue covering dimension*; for reasonably well-behaved classes of spaces this is equivalent to other frequently used concepts of dimension. Here are some printed and online references for topological dimension theory:

W. Hurewicz and H. Wallman. *Dimension Theory* (Revised Edition, Princeton Mathematical Series, Vol. 4). *Princeton University Press, Princeton*, 1996.

K. Nagami. *Dimension Theory* (with an appendix by Y. Kodama, Pure and Applied Mathematics Series, Vol. 37). *Academic Press, New York*, 1970.

J. Nagata. *Modern Dimension Theory* (Second Edition, revised and extended; Sigma Series in Pure Mathematics, Vol. 2). *Heldermann-Verlag, Berlin*, 1983.

`http://en.wikipedia.org/wiki/Lebesgue_covering_dimension`

`http://en.wikipedia.org/wiki/Dimension` http://en.wikipedia.org/wiki/Inductive_dimension

FRACTAL DIMENSIONS. There are several notions of *fractal dimension* for subsets of $\mathbb{R}^n$ which depend on the way in which an object is embedded in $\mathbb{R}^n$ and not just the subset's underlying topological structure; for example, various standard examples of nonrectifiable curves in the plane have fractal dimensions which are numbers strictly between 1 and 2. Such objects are interesting for a variety of reasons, but they are beyond the scope of this course so we shall only give two online references here:

http://en.wikipedia.org/wiki/Fractal_dimension

http://www.warwick.ac.uk/~masdbl/dimension-total.pdf

The basic setting

We shall base our discussion upon the material in Section 50 of Munkres. For the sake of clarity we shall state the main definition and mention some standard conventions.

Definition. Let X be a topological space, let n be a nonnegative integer, and let $\mathcal{U}$ be an indexed open covering of X. Then we shall say that *the open covering $\mathcal{U}$ has order at most n* provided every intersection of the form

$$U_{\alpha(0)} \ \cap \ \cdots \ \cap \ U_{\alpha(n)}$$

is empty, and we shall say that *the space X has Lebesgue covering dimension $\leq n$* provided every open covering $\mathcal{U}$ of X has a refinement $\mathcal{V}$ of order $\leq n$. Frequently we shall write $\dim X \leq n$ if the Lebesgue covering dimension is at most n.

We shall say that $\dim X = n$ (the Lebesgue covering dimension is equal to n) if $\dim X \leq n$ is true but $\dim X \leq n-1$ is not. By convention, the Lebesgue covering dimension of the empty set is taken to be -1, and we shall write $\dim X = \infty$ if $\dim X \leq n$ is false for all n.

Munkres states and proves many fundamental results about the Lebesgue covering dimension, and we shall not try to copy or rework most of his results here. Instead, our emphasis in this section will be on the following key issues:

(1) Describing precise connections between the topological theory of dimension as in Munkres and the algebraic notions of k-dimensional homology groups for various choices of k.

(2) Using the methods of these notes to give an alternate proof of Theorem 50.6 in Munkres; namely, if $A \subset \mathbb{R}^n$ is compact, then the topological dimension of A satisfies $\dim A \leq n$.

(3) Using algebraic topology to prove that the topological dimension of an n-dimensional polyhedron is in fact **equal** to n (the results in Munkres show that this dimension is at most n).

We shall begin by addressing the dimension question in (2); one reason for doing this is that the approach taken here will play a crucial role in our treatment of the subject.

MUNKRES, THEOREM 50.6. *If A is a compact subset of $\mathbb{R}^n$, then $\dim A \leq n$.*

Alternate proof. We know that there is some very large hypercube K of the form $[-M, M]^n$ which contains A, and we also know that A is closed in this hypercube. By Theorem 50.1 on pages 306–307 of Munkres, it is enough to show that the hypercube has dimension at most n. Since every hypercube has a simplicial decomposition with simplices of dimension $\leq n$, it will suffice to prove the following result:

LEMMA 1. *If $P \subset \mathbb{R}^m$ is a polyhedron with an n-dimensional simplicial decomposition, then the topological dimension of P is at most n.*

If we know this, then we know that the hypercube, and hence A, must have topological dimension $\leq n$.■

Proof of Lemma 1. Let $\mathcal{U}$ be an open covering of the hypercube K, and let $\varepsilon > 0$ be a Lebesgue number for $\mathcal{U}$. Using barycentric subdivisions, we can find an n-dimensional simplicial decomposition of K whose simplices all have diameter less than $\varepsilon/2$. Therefore if $\mathbf{v}$ is a vertex of this simplicial decomposition, then the open set $\mathbf{Openstar}(\mathbf{v})$ is contained in some element of $\mathcal{U}$. Now these sets form an open covering of K (see Section 2.C of Hatcher), and therefore these open stars form a finite open refinement of $\mathcal{U}$. Since an intersection of open stars $\cap_i$ $\mathbf{Openstar}(\mathbf{v}_i)$ is nonempty if and only if the vertices $\mathbf{v}_i$ lie on a simplex from the underlying simplicial decomposition, the n-dimensionality of the decomposition implies that every intersection of $(n+2)$ distinct open stars must be empty. This is exactly the criterion for the covering by open stars to have order at most $(n+1)$. Therefore we have shown that $\mathcal{U}$ has a finite open refinement with at most this order, which means that the topological dimension of K is at most n.■

The discussions of the first and third issues are closely related, and they use the material on partitions of unity on pags 225–226 of Munkres (see Theorem 36.1 in particular).

Definitions. Let X be a $\mathbf{T_4}$ space, and let $\mathcal{U}$ be a finite open covering of X. Set $\mathbf{Vec}(\mathcal{U})$ equal to the (finite-dimensional) real vector space with basis given by the sets in $\mathcal{U}$, and define the **nerve** of $\mathcal{U}$, written $\mathfrak{N}(\mathcal{U})$, to be the simplicial complex whose simplices are given by all vertex sets of the form $U_{\alpha(0)}, \ \cdots \ , U_{\alpha(q)}$ such that

$$U_{\alpha(0)} \ \cap \ \cdots \ \cap \ U_{\alpha(q)} \quad \neq \quad \emptyset \ .$$

By construction, the vertices of this simplicial complex are all symbols of the form $[U_\alpha]$, where U_α is nonempty and belongs to $\mathcal{U}$.

If $\{ \varphi_\alpha \}$ is a partition of unity which is subordinate to (= dominated by) $\mathcal{U}$, then there is a *canonical map* k_φ from X to $\mathfrak{N}(\mathcal{U})$ given by the partition of unity:

$$k_\varphi(x) \quad = \quad \sum \ \varphi_\alpha(x) \cdot [U_\alpha]$$

Different partitions of unity yield different maps, but we have the following:

CLAIM: *For each finite open covering $\mathcal{U}$, all canonical maps from X to $\mathfrak{N}(\mathcal{U})$ are homotopic to each other.*

Proof of the claim. For each choice of x and canonical maps φ_0, φ_1, we know that the points $\varphi_i(x)$ lie on the simplex whose vertices are all $[U_\alpha]$ such that $x \in U_\alpha$. Thus the straight line segment joining $\varphi_0(x)$ to $\varphi_1(x)$ also lies on this simplex, and hence also lies in the nerve of $\mathcal{U}$. In other words, the image of the straight line homotopy from φ_0 to φ_1 is always contained in $\mathfrak{N}(\mathcal{U})$, and therefore the two canonical maps into $\mathfrak{N}(\mathcal{U})$ are homotopic.■

In the special case where $(P, \mathbf{K})$ is a simplicial complex and $\mathcal{U}$ is the open covering given by open stars of vertices (see Hatcher for the definitions), the canonical map(s) from P to the nerve of $\mathcal{U}$ can be described very simply as follows:

PROPOSITION 2. *Let P, $\mathbf{K}$ and $\mathcal{U}$ be as above, and for each vertex $\mathbf{v}$ of $\mathbf{K}$ define the* extended barycentric coordinate function $\mathbf{v}^* : P \to [0,1]$ *as follows: If $\mathbf{x} \in A$ for some simplex A which contains $\mathbf{v}$ as a vertex, let $\mathbf{v}^*(\mathbf{x})$ denote the barycentric coordinate of $\mathbf{x}$ with respect to $\mathbf{v}$, and if $\mathbf{x}$ lies on* *a simplex A which does not contain $\mathbf{v}$ as a vertex, set $\mathbf{v}^*(\mathbf{x}) = 0$ (it follows immediately that this map is well-defined and continuous). Define a map $\kappa : P \to \mathfrak{N}(\mathcal{U})$ by $\kappa(\mathbf{x}) = \sum \mathbf{v}^*(\mathbf{x}) \cdot \mathbf{v}$. Then κ defines a homeomorphism from P to $\mathfrak{N}(\mathcal{U})$, and every canonical map with respect to the open covering $\mathcal{U}$ is homotopic to κ.*

Sketch of proof. First of all, the barycentric coordinate functions are well-defined, for if $\mathbf{x}$ lies on a simplex A with vertex $\mathbf{v}$ and also on a simplex B for which $\mathbf{v}$ is not a vertex, then it follows that the barycentric coordinate of $\mathbf{x}$ with respect to $\mathbf{v}$ must be zero. The assertion that κ defines a homeomorphism from P to the nerve of $\mathcal{U}$ follows because κ maps the simplices of $\mathbf{K}$ bijectively to the simplices of $\mathfrak{N}(\mathcal{U})$; more precisely, there is a 1–1 correspondence of simplices and each simplex of $\mathbf{K}$ is sent to a simplex of the nerve by a bijective affine map.

Finally, the proof that κ is homeomorphic to a canonical map associated to a partition of unity follows from the same considerations which appear in the proof that two canonical maps are homotopic (for every $\mathbf{x} \in P$, there is a simplex in the nerve containing both $\kappa(\mathbf{x})$ and the value of a canonical map at $\mathbf{x}$).■

Čech homology groups

The idea behind singular homology groups is that one approximates a space by maps **from** simplicial complexes (in particular, simplices) into a space X. Dually, the idea behind Čech homology groups is that one approximates a space by maps **into** simplicial complexes. Constructions of this type play an important role in the theory and applications of machinery from algebraic topology, but we shall only focus on what we need. As is often the case, the first step is to construct some necessary algebraic machinery.

Inverse systems and inverse limits

The definition of Čech homology requires the notion of *inverse limit*; special cases of this concept appear in Hatcher, but since we need the general case we must begin from scratch.

Definition. A *codirected set* is a pair $(A, \prec)$ consisting of a set A and a binary operation $\prec$ such that the following hold:

(a) (Reflexive Property) For all $x \in A$ we have $x \prec x$.

(b) (Transitive Property) If $x, y, z \in A$ are such that $x \prec y$ and $y \prec z$, then $x \prec z$.

(c) (Lower Bound Property) For all $x, y \in A$ there is some $w \in A$ such that $w \prec x$ and $w \prec y$.

These are similar to the defining conditions for a partially ordered set, but we do **not** assume the symmetric property (so $x \prec y$ and $y \prec x$ does not necessarily imply $x = y$), and the Lower Bound Property does not necessarily hold for a partially ordered set which is not linearly ordered. On the other hand, if a partially ordered set is a **lattice** (*i.e.*, finite subsets always have least upper bounds and greatest lower bounds), then it is a codirected set.

The basic example of a codirected set in Hatcher is given by the positive integers $\mathbb{N}^+$ with the **reverse** of the usual partial ordering, so that $a \prec b$ if and only if $b \geq a$.

Given a codirected set $(A, \prec)$, there is an associated category $\mathsf{CAT}\,(A, \prec)$ for which $\mathsf{Morph}\,(x, y)$ is nonempty if and only if $x \prec y$, and in this case $\mathsf{Morph}\,(x, y)$ contains exactly one element.

Definition. Let $(A, \prec)$ be a codirected set, and let $\mathbf{C}$ be a category. An *inverse system* in $\mathbf{C}$ indexed by $(A, \prec)$ is a covariant functor F from $\mathsf{CAT}\,(A, \prec)$ to $\mathbf{C}$. If $a \prec b$, then the value of F on the unique morphism $a \to b$ is frequently denoted by notation like $f_{a,b}$; in other words, $f_{a,b} = F(a \prec b)$.

There is a closely related concept of *inverse limit* for inverse systems. One can do this in purely categorical terms, but we are only interested in working with inverse limits over categories of modules. For inverse systems $F = \{F(a)\}$ of modules, the inverse limit

$$\lim_{\leftarrow} \quad = \quad \text{inv} \lim_{A} \; F(a) \quad = \quad \text{proj} \lim_{A} \; F(a)$$

is defined to be the set of all $x = (x_a)$ in $\prod_A \; F(a)$ such that for each $a \prec b$ we have $f_{a,b}(x_a) = x_b$. For each $a \in A$ the map p_a denotes projection onto the a-coordinate.

Inverse limits have the following universal mapping property, which in fact characterizes the construction.

PROPOSITION 3. *Suppose that F is an inverse system as above, and suppose that we are given a module L with maps $q_a : L \to F(a)$ such that $f_{a,b} \circ q_a = q_b$ whenever $a \prec b$. Then there is a unique homomorphism $h : L \to \lim_{\leftarrow} F(a)$ such that $g_a = f_a \circ h$ for all a.*

This is an immediate consequence of the definitions.■

There are straightforward analogs of the inverse limit construction for may categories (sets, compact Hausdorff spaces, groups, ...), and we shall leave the details of setting up such objects to the reader as an exercise.

Frequently it is important to recognize that inverse limits of directed systems can be given by inverse limits over "good" subobjects. We shall say that $B \subset A$ is a *codirected subobject* if B is a subset, the binary relation is the restriction of the binary relation on A, and the Lower Bound Property still holds on B (however, if $w \in A$ is such that $w \prec b, a$ we do not necessarily assume that $w \in B$; we only assume that there is some $w' \in B$ with $w' \prec a, b$). We shall say that such a object is *cofinal* if for each $x \in A$ there is some $y \in B$ such that $y \prec x$.

Example. Let γ be a cardinal number, and let $\mathsf{Cov}_{\gamma}(X)$ be the family of indexed open coverings of X such that the cardinality of the indexing set is at most γ. We shall say that an indexed open covering $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ is an *indexed refinement* of $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ if there is a map of indexing spaces $j : B \to A$ such that $V_\beta \subset U_{j(\beta)}$ for all β; note that if $\mathcal{V}$ is a refinement of $\mathcal{U}$ in the usual sense then by the Axiom of Choice we can always find a function j with the required properties. — Suppose now that X is a compact metric space and $\mathsf{FinCov}\,(X)$ is a set of all finite indexed open coverings whose indexing sets are subsets of the set $\mathbb{N}$ of nonnegative integers. If $\mathbb{A}$ is a subset of $\mathsf{FinCov}\,(X)$ such that for each $k > 0$ there is an open covering $\mathcal{A}_k \in \mathbb{A}$ whose (open) subsets all have diameter less than $1/k$, then a Lebesgue number argument implies that $\mathbb{A}$ is cofinal in $\mathsf{FinCov}\,(X)$.

Given a cofinal subobject B and an inverse system F on A, then there is an associated inverse system $F|B$. The following crucial observation suggests the importance an usefulness of such restricted inverse systems.

PROPOSITION 4. *Suppose that we are given the setting above, and let B be a cofinal subobject. Then there is a canonical isomorphism from $\lim_{\leftarrow} F$ to $\lim_{\leftarrow} F|B$.*

Proof. By definition, the inverse limit L_A over all of A is a submodule of $P_A = \prod_{a \in A} \; F(a)$ and the inverse L_B limit over B is a submodule of $P_B = \prod_{b \in B} \; F(b)$. Let $\varphi_0 : P_A \to P_B$ be given by the projections onto the factors $F(b)$; since the operations in the product are defined coordinatewise, it follows immediately that φ_0 is a module homomorphism.

By construction it follows that φ_0 maps L_A to L_B. If $\varphi : L_A \to L_B$ be the homomorphism defined by φ_0, the objective is to prove that φ is an isomorphism. It is straightforward to verify that φ is onto. Suppose now that we are given $x = (x_a)$ and $y = (y_a)$ such that $\varphi(x) = \varphi(y)$. Then $x_b = y_b$ for all $b \in B$, and we need to show that this implies $x_a = y_a$ for all a. Let $\alpha \in A$ be arbitrary, and choose $\beta \in B$ such that $\beta \prec \alpha$. Then we have $x_\alpha = f_{\beta,\alpha}(x_\beta)$ and $y_\alpha = f_{\beta,\alpha}(y_\beta)$. Since we are assuming that $y_\beta = x_\beta$, it follows that $y_\alpha = x_\alpha$.■

Definition and properties of Čech homology

Suppose that X is a compact Hausdorff space, let $A \subset X$ be a closed subspace, and let $\mathsf{FinCov}\,(X, A)$ denote the codirected set of all pairs $(\mathcal{U}, \mathcal{U}|A)$, where $\mathcal{U}$ is a finite open covering of X and $\mathcal{U}|A$ denotes its restriction to A with all empty intersections deleted; the binary relation

$$\beta \quad = \quad (\mathcal{V}, \mathcal{V}|A) \quad \prec \quad (\mathcal{U}, \mathcal{U}|A) \quad = \quad \alpha$$

is taken to mean that $(\mathcal{V}, \mathcal{V}|A)$ is an indexed refinement of $(\mathcal{U}, \mathcal{U}|A)$. Since we are working with indexed refinements, it follows that the map of indexing sets will define a simplicial mapping of nerve pairs

$$j_{\beta,\alpha} : (N_\beta, N'_\beta) \quad = \quad \Big(\mathfrak{N}(\mathcal{V}), \mathfrak{N}(\mathcal{V}|A)\Big) \quad \longrightarrow \quad \Big(\mathfrak{N}(\mathcal{U}), \mathfrak{N}(\mathcal{U}|A)\Big) \quad = \quad (N_\alpha, N'_\alpha)$$

and therefore we obtain an inverse system of simplicial complex pairs and simplicial mappings. If we take the simplicial or singular chain complexes associated to such a system we obtain inverse systems of chain complexes, and if we pass to homology we obtain inverse systems of homology groups; at the chain complex level the inverse systems are different, but their homology groups are the same.

Definition. If X is a compact Hausdorff space and $A \subset X$ is a closed subspace, then the **Čech homology groups** $\check{H}_q(X, A)$ are the inverse limits of the inverse systems $H_q(N_\alpha, N'_\alpha)$, where α runs through all pairs $(\mathcal{U}, \mathcal{U}|A)$.

Presumably we have introduced these groups because they have implications for dimension theory, and one can also ask if these groups can be computed for finite simplicial complexes. The next two results confirm these expectations.

THEOREM 5. *If X is a compact Hausdorff space whose Lebesgue covering dimension is $\leq n$ and A is a closed subset of X, then $\check{H}_q(X, A) = 0$ for all $q > n$.*

Proof. The condition on the Lebesgue covering dimension implies that every finite open covering $\mathcal{U}$ of X has a (finite) refinement such that each subcollection of $n+2$ open subsets from $\mathcal{U}$ has an empty intersection. This condition means that the nerve of $\mathcal{U}$ has no simplices with $n+2$ vertices and hence no simplices of dimension $\geq n+1$; in other words, the (geometric) dimension of the nerve is at most n. By Proposition 4 and the assumption on the Lebesgue covering dimension, we know that the Čech homology of (X, A) can be computed using open coverings for which each subcollection of $n+2$ open subsets from $\mathcal{U}$ has an empty intersection, and hence the Čech homology is an inverse limit of homology groups of simplicial complexes with dimension $\leq n$. Since the q-dimensional homology of such complexes vanishes if $q > n$, it follows that the same is true for the inverse limit groups when $q > n$, and therefore we must have $\check{H}_q(X, A) = 0$ for all $q > n$.■

The next main result states that the Čech homology for a simplicial complex pair is the same as the homology we have already defined. a more general result:

THEOREM 6. *If X is a compact Hausdorff space and $A \subset X$ is a closed subspace, then there is a canonical mapping φ_∞ from $H_*(X, A)$ to $\check{H}_*(X, A)$ (the* singular-Čech comparison map*), where* *the groups on the left are singular homology groups. If X is a polyhedron with some simplicial* $\mathbf{K}$ *such that A is a subcomplex with respect to this decomposition, then the singular-Čech comparison map is an isomorphism.*

Before proving this result, we shall use the conclusion to derive the main implications for dimension theory.

THEOREM 7. (i) *For all $n \geq 0$, the Lebesgue covering dimension of the disk D^n is equal to n.*

(ii) *If $(P, \mathbf{K})$ is a simplicial complex whose geometric definition is equal to n, then the Lebesgue covering dimension of P is also equal to n.*

(iii) *If $A \subset \mathbb{R}^n$ is a compact subset with a nonempty interior, then the Lebesgue covering dimension of A is equal to n.*

(iv) *If $\mathbf{Q} = [0,1]^\infty$ is the Cartesian product of countably infinitely many copies of the unit interval (the so-called* Hilbert cube*), then the Lebesgue covering dimension of $\mathbf{Q}$ is equal to ∞.*

Proof. We shall take these in order.

Proof of (i). By the discussion at the beginning of this section (or the corresponding discussion in Munkres), we know that the Lebesgue covering dimension of D^n is at most n, so we need to show that it cannot be $\leq (n-1)$. We shall exclude this by deriving a contradiction from it. If the Lebesgue covering dimension was strictly less than n, then it would follow that $\check{H}_n(D^n, A)$ would vanish for all closed subsets $A \subset D^n$. By Theorem 6 we know that $\check{H}_n(D^n, S^{n-1}) \cong H_n(D^n, S^{n-1})$, and since the latter is isomorphic to $\mathbb{Z}$ it follows from Theorem 5 that the Lebesgue covering dimension cannot be $\leq n-1$. Therefore this dimension must be equal to n.■

Proof of (ii). This follows immediately from (i) and Theorem 50.2 of Munkres (see page 307 for details).■

Proof of (iii). By the discussion at the beginning of this section we know that the Lebesgue covering dimension of A is $\leq n$. Since A has a nonempty interior, it follows that A contains a closed subset which is homeomorphic to D^n. This means that the Lebesgue covering dimension of A must be at least as large as the Lebesgue covering dimension of D^n, which is n. Combining these observations, we conclude that the Lebesgue covering dimension of A is equal to n.■

Proof of (iv). Let $H\langle n\rangle \subset \mathbf{Q}$ be the subset of all points whose coordinates satisfy $x_k = 0$ for $k \geq n+1$. Then it follows that $H\langle n\rangle$ is a closed subset of $\mathbf{Q}$ which is homeomorphic to D^n, and therefore we have $n = \dim H\langle n\rangle \leq \dim \mathbf{Q}$ for all n.■

Remark. The preceding result implies that the Lebesgue covering dimension does not behave well with respect to quotients, even if the space and its quotient are polyhedra. In particular, if $f : X \to Y$ is a continuous and onto mapping of compact Hausdorff spaces, then in general we cannot say anything about the relation between the Lebesgue covering dimensions of X and Y even if we know that both numbers are finite. The simplest counterexamples are given by the continuous surjection from $[0,1]$ to $[0,1]^2$ given by the Peano curve (described in Section 44 of Munkres) and the usual first coordinate projection from $[0,1]^2$ to $[0,1]$; in the first case the dimension increases when one passes to the quotient, and in the second case the dimension decreases (which is what one reasonably expects). Of course, if we take f as above to be an identity map, then the dimension does not change.

We shall discuss the behavior of dimensions under taking products after proving Theorem 6.

Proof of Theorem 6. We begin by proving the general statement. If $\mathcal{U}$ is an open covering of X and A is a closed subset of X, then we have seen that a partition of unity subordinate to $\mathcal{U}$ defines a canonical map from X into the nerve $\mathfrak{N}(\mathcal{U})$, and by construction this map sends A into $\mathfrak{N}(\mathcal{U}|A)$. We have also seen that the homotopy class of this map is well defined (at least when $A = \emptyset$, but the same argument implies that the canonical maps of pairs associated to different partitions of unity will be homotopic as maps of pairs). Therefore we have homomorphisms

$$(k_\alpha)_* : H_*(X, A) \longrightarrow H_*\big(\mathfrak{N}(\mathcal{U}), \mathfrak{N}(\mathcal{U}|A)\big)$$

and we need to show that these yield a map into the inverse limit of the groups on the right hand side, which is true if and only if

$$(k_\alpha)_* \quad = \quad (j_{\beta\alpha})_* \circ (k_\beta)_*$$

for all α and β such that $\beta \prec \alpha$. But if the latter holds, then it follows that the composite $j_{\beta\alpha} \circ (k_\beta)$ defines a canonical map into the nerve pair (N_α, N'_α), and therefore this composite is homotopic to k_α; therefore the associated maps in homology are equal, and this implies that we have the desired homomorphism φ_∞ into the inverse limit $\check{H}_*(X, A)$.

We must now show that the singular-Čech comparison map φ_∞ is an isomorphism if X is a polyhedron with simplicial decomposition $\mathbf{K}$ and A corresponds to a subcomplex of $(X, \mathbf{K})$. Let $r > 0$, and let $\mathcal{W}_r$ be the open covering by open stars of vertices in the r^{th} barycentric subdivision $B^r(\mathbf{K})$. Then by construction we have $\mathcal{W}_{r+1} \prec \mathcal{W}_r$ for all r, and a Lebesgue number argument shows that the set of all open coverings $\mathcal{W}_r$ determines a cofinal subset of $\mathsf{FinCov}\,(X)$. If (N_r, N'_r) denotes the nerve pair associated to $\mathcal{W}_r$, then it follows that $\check{H}_*(X, A)$ is isomorphic to the inverse limit of the groups $H_*(N_r, N'_r)$.

If we can show that the canonical maps k_r into N_r all define isomorphisms from $H_*(X, A)$ to $H_*(N_r, N'_r)$, then the map into the inverse limit will be an isomorphism for the following reasons:

(1) If $\varphi_\infty(u) = 0$, then $(k_r)_*(u) = 0$ for all r, and since each of these maps is an isomorphism it follows that $u = 0$.

(2) If v lies in the inverse limit, then v has the form $(v_1, v_2, \cdots)$ where $v_r = (j_{r,r+1})_*(v_{r+1})$ for all r. Since k_r defines an isomorphism, it follows that $v_r = (k_r)_*(u_r)$ for some unique $u_r \in \check{H}(X, A)$, and if we can show that $u_r = u_{r+1}$ for all r then it will follow that $v = \varphi_\infty(u)$. But the previous equations imply that

$$(k_r)_*(u_{r+1}) \quad = \quad (j_{r,r+1})_* \circ (k_{r+1})_*(u_{r+1}) \quad = \quad (j_{r,r+1})_*(v_{r+1}) \;\; v_r \quad = \quad (k_r)_*(u_r)$$

and since $(k_r)_*$ is injective it follows that $u_{r+1} = u_r$.

To conclude the proof, we note that the relative version of Proposition 2 implies that the map of pairs determined by each k_r is homotopic to a homeomorphism of pairs.■

As noted before, this concludes tha proof that the Lebesgue covering dimension of D^n is equal to n. It is also possible to prove the following result:

THEOREM 8. *For every $n \geq 0$ the Lebesgue covering dimension of $\mathbb{R}^n$ is equal to n.*

Sketch of proof. The exercises at the end of Section 50 in Munkres (see pages 315–316) provide machinery for extending results on covering dimensions to "reasonable" noncompact spaces. In particular, Exercise 8 shows that the Lebesgue covering dimension of $\mathbb{R}^n$ is at most n. Since the dimension of the closed subspace D^n is equal to n, it follows that the Lebesgue covering dimension of $\mathbb{R}^n$ is at least n, and therefore it must be exactly n.■

One cap proceed similarly to extend the conclusions for Exercises 9 and 10 on page 316 of Munkres. Specifically, *every (second countable) topological n-manifold has Lebesgue covering dimension equal to n, and if $A \subset \mathbb{R}^n$ is a close subset with nonempty interior, then the Lebesgue covering dimension of A is also equal to n.*■

Note. For topological n-manifolds, second countability is equivalent to the σ-compactness condition which appears on page 316 of Munkres (proof?).

Dimensions of products

The standard homeomorphism $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{m+n}$ strongly suggests the following question:

QUESTION. *If we know that the Lebesgue covering dimensions of the nonempty compact Hausdorff spaces X and Y are m and n respectively, does it follow that the Lebesgue covering dimension of the product $X \times Y$ is equal to $m + n$?*

In the next subheading we shall prove the following result:

PROPOSITION 9. *If X and Y are compact Hausdorff spaces whose Lebesgue covering dimensions are m and n respectively, then the Lebesgue covering dimension of the product $X \times Y$ is less than or equal to $m + n$.*

We shall derive this result as an immediate consequence of Proposition 18 below.

If we assume that our spaces are somewhat reasonable, then we can prove a stronger and more satisfying result:

PROPOSITION 10. *In the setting of Proposition 9, suppose that $X = \cup_i A_i$ and $Y = \cup_j B_j$ where the sets A_i and B_j are all homeomorphic to k-disks for suitable values of k. Then the Lebesgue covering dimension of $X \times Y$ is equal to $m + n$.*

Proof of Proposition 10. By Theorem 50.2 of Munkres and finite induction, it follows that the dimension of $X \times Y$ is equal to the maximum of the dimensions of the closed subsets $A_i \times B_j$. On the other hand, the same result implies that there are some indices p and q such that A_p is homeomorphic to D^m and B_q is homeomorphic to D^n (otherwise the dimensions of X and Y would be strictly less than m and n). Since $D^m \times D^n$ is homeomorphic to D^{m+n} it follows that $X \times Y$ has a closed subset with Lebesgue covering dimension equal to $m + n$. On the other hand, we also know that the dimension of each disk A_i is at most m and the dimension of each disk B_j is at most n, so the dimension of $X \times Y$ is at most $m + n$. If we combine these, we find that the dimension of $X \times Y$ is equal to $m + n$.■

Counterexamples to the general question

Although Propositions 9 and 10 may suggest that the formula $\dim(X \times Y) = \dim X + \dim Y$ holds more generally, it is possible to construct examples where the left hand side is less than the right. The first examples of this sort were discovered by L. S. Pontryagin; here is a reference to the original paper:

> **L. S. Pontryagin.** *Sur une hypothèse fondementale de la théorie de la dimension.* Comptes Rendus Acad. Sci. (Paris) **190** (1930), 1105–1107.

In Pontryagin's example one has $X = Y$ and $\dim X = 2$ but $\dim(X \times X) = 3$. By the following result, these are the lowest dimensions in which one can have $\dim(X \times Y) < \dim X + \dim Y$.

DIMENSION ESTIMATES FOR PRODUCTS. *Let X and Y be nonempty compact metric spaces. Then the following hold:*

(a) *If* $\dim Y = 0$, *then* $\dim X \times Y = \dim X$.

(b) *If* $\dim Y = 1$, *then* $\dim X \times Y = \dim X + 1$.

(c) *If* $\dim Y \geq 2$, *then* $\dim X \times Y \geq \dim X + 1$.

Proofs of these results are beyond the scope of this course, so we shall limit ourselves to mentioning some key points which arise in the proofs.

The proof of the first statement is actually fairly direct, and it only requires a small amount of additional machinery. Proofs of the second and third statements using an alternate approach to defining topological dimensions (the *weak inductive* or *Menger-Urysohn dimension*) are due to Hurewicz (we should note that the Menger-Urysohn definition is the one which appears in Hurewicz and Wallman). Here is a reference to the original paper.

> **W. Hurewicz.** *Sur la dimension des produits cartésiens.* Annals of Mathematics **36** (1935), 194–197.

There is a brief indication of another way to retrieve (b) at the top of page 241 in the book by Nagami (however, this requires a substantial amount of input from algebraic topology). One proof of (c) can be obtained by combining (b) with the following existence theorem: *If Y is a compact metric space such that $n = \dim Y$ is finite and $0 < k < n$, then there is a closed subset $B \subset Y$ such that* $\dim B = k$. — This result and the equivalence of the Lebesgue and Menger-Urysohn dimensions for compact metric spaces are discussed in an appendix to this section.

Spaces for which $\dim(X \times Y) < \dim X + \dim Y$ are generally far removed from the sorts of objects studied in most of topology, but it is important to recognize their existence. On the other hand, even though there is no general product formula for the dimensions of compact metric spaces, the validity of the formula for many well-behaved examples (see Proposition 9) leads one naturally to look for necessary and sufficient conditions under which one has $\dim(X \times Y) = \dim X + \dim Y$. Here is one reference which answers the question:

> **Y. Kodama.** *A necessary and sufficient condition under which* $\dim(X \times Y) = \dim X + \dim Y$. Proc. Japan. Acad. **36** (1960), 400–404.

As in several previously cited cases, the proofs of the main results in this paper rely heavily on input from algebraic topology.

Further results

We shall consider two issues related to the discussion of dimension theory:

1. Giving an example of a compact subset of $\mathbb{R}^2$ for which the singular and Čech homology groups are not isomorphic.

2. Showing that a compact subset of $\mathbb{R}^n$ has Lebesgue covering dimension n if and only if it has a nonempty interior (one can then use the previously cited exercises in Munkres to show that the same conclusion holds for arbitrary closed subsets). The machinery developed for this question will also yield a proof of Proposition 9 on the Lebesgue covering dimensions of cartesian products.

The example for the first problem will be the *Polish circle*, and our discussion will be based upon the following online reference:

http://math.ucr.edu/∼res/math205B/polishcircle.pdf

The key to studying the Čech homology of arbitrary compact subsets in $\mathbb{R}^n$ is a fundamental *continuity property* which does not hold in singular homology.

Continuity in Čech homology

The results in Chapter IX of Eilenberg and Steenrod show that Čech homology is functorial with respect to continuous maps of compact Hausdorff spaces. Given this, we can the basic result very simply.

THEOREM 11. (Continuity Property) *Suppose that X is a subspace of some Hausdorff topological space E, and suppose further that there are compact subsets $X_\alpha \subset E$ such that $X = \cap_\alpha X_\alpha$ for all α and the family X_α is closed under taking finite intersections. Then we have*

$$\check{H}_*(X) \quad \cong \quad \lim_{\leftarrow} \ \check{H}_*(X_\alpha) \ .$$

If $E = \mathbb{R}^n$ for some n, then it is always possible to find such a family of compact subsets X_n such that $X_{n+1} \subset X_n$ for all n and X_n is a finite union of hypercubes of the form

$$\prod_{i=1}^{n} \left(x_i, \, x_i + \frac{1}{2^n} \right)$$

where each x_i is a rational number expressible in the form $p_i/2^n$ for some integer p_i. For example, one can take X_n to be the union of all such cubes which have a nonempty intersection with X.

Reference for the proof of Theorem 11. A proof is given on page 261 of Eilenberg and Steenrod (specifically, see theorem X.3.1).■

Remark. One can also make the singular-Čech comparison map into a natural transformation of covariant functors, but we shall not do this here because it is not needed for our purposes except for a remark following the proof of Theorem 15 (as before, details may be found in Chapters IX and X of Eilenberg and Steenrod).

Singular and Čech homology of the Polish circle

As in the previously cited document

http://math.ucr.edu/∼res/math205B/polishcircle.pdf

the Polish circle P is defined to be the union of the following curves:

(1) The graph of $y = \sin(1/x)$ over the interval $0 \leq x \leq 1$.

(2) The vertical line segment $\{1\} \times [-2, 1]$.

(3) The horizontal line segment $[0, 1] \times \{-2\}$.

(4) The vertical line segment $\{0\} \times [-2, 1]$.

One important fact about the Polish circle is that it is arcwise connected but not locally arcwise connected. The proof of this is analogous to the discussion on page 66 of the online notes

http://math.ucr.edu/∼res/math205A/gentopnotes2008.pdf which shows that the space B, which is given by closure (in $\mathbb{R}^2$) of the graph of $\sin(1/x)$ for $x > 0$, is connected but not arcwise connected. For the sake of completeness, we shall indicate how one modifies the argument to show the properties of P stated above. First of all, since P is the union of four arcwise connected subspaces $A \cup B \cup C \cup D$ such that $A \cap B$, $B \cap C$ and $C \cup D$ are all nonempty, the arcwise connectedness of P follows immediately. To prove that P is not arcwise connected, we need the following result, whose proof is similar to the previously cited argument which shows that B is not arcwise connected:

LEMMA 12. *Let Y be a compact, arcwise connected, locally arcwise connected topological space, let $f : Y \to P$ be continuous, and suppose that $a_0 \in Y$ is such that the first coordinate of $f(a_0)$ is zero and $f(a_0) \neq (0, -2)$. Then there is an arcwise connected open neighborhood V of a_0 in Y such that $f[V]$ is contained in the intersection of Y with the $y-$axis.*■

This observation has far-reaching consequences for the fundamental group and singular homology of P, all of which come from the following:of c in Y

PROPOSITION 13. *Let Y and f be as in the preceding lemma. Then there is some $\varepsilon > 0$ such that $f[Y]$ is disjoint from the open rectangular region $(0, \varepsilon) \times (-2, 2)$.*

In terms of the presentation of P given above, this means that $f[Y]$ is contained in the union of $B \cup C \cup D$ with the graph of $\sin(1/x)$ over the interval $[\varepsilon, 1]$. This subspace M_ε is homeomorphic to a closed interval and as such is contractible. Therefore Proposition 13 has the following application to the algebraic-topological invariants of the Polish circle:

THEOREM 14. *If P is the Polish circle, then $\pi_1(P, p)$ is trivial for all $p \in P$, and the inclusion of $\{p\}$ in P induces an isomorphism of singular homology groups.*

Proof of Theorem 14, assuming Proposition 13. We begin with the result on the fundamental group. Suppose that γ is a closed curve in P based at p. By Proposition 13 we know that the image of γ lies in M_ε for some $\varepsilon > 0$, so that the class of γ in $\pi_1(P, p)$ lies in the image of $\pi_1(M_\varepsilon, p)$. Since M_ε is contractible, it follows that the image of $\pi_1(M_\varepsilon, p)$ in $\pi_1(P, p)$ is trivial, and therefore the latter must also be trivial.

The proof for singular homology is similar. If $z \in S_q(P)$ is a cycle, then there is some M_ε such that $p \in M_\varepsilon$ and z lies in the image of $S_q(M_\varepsilon)$. Of course, this means that the class u represented by z lies in the image of the homomorphism $H_q(M_\varepsilon) \to H_q(P)$, and since M_ε is contractible it follows that this image is trivial if $q > 0$. On the other hand, if $q = 0$, then the arcwise connectedness of all the spaces implies that the various inclusion maps all induce isomorphisms in 0-dimensional singular homology.■

Proof of Proposition 13. Let E denote the inverse image of the intersection of P with $\{0\} \times [-\frac{3}{2}, 1]$. Then for each $c \in E$ there is an arcwise connected open neighborhood V_c of c in Y such that $f[V_c]$ is contained in the intersection of Y with the $y-$axis. Let W_c be an open neighborhood of c whose closure is contained in V_c. By continuity E is closed in Y and hence E is a compact subset, so there is a finite subcollection of the sets W_c, say $\{W_1, \cdots, W_n\}$, which covers E.

Define $G \subset Y$ to be the closed subset

$$Y \; - \; \cup_{i=1}^{n} W_i$$

so that $f[G]$ is compact and disjoint from $P \cap \{0\} \times [-\frac{3}{2}, 1]$. If $A \subset P$ is the piece of the graph of $\sin(1x)$ described above, then it follows that the second coordinates of all points in $f[G] \cap A$ are positive and by compactness must be bounded away from zero; in other words, there is some $\varepsilon > 0$ such that $f[G] \cap A$ is disjoint from $(0, \varepsilon) \times \mathbb{R}$. But this means that

$$f[Y] \quad = \quad f[G] \, \cup \, \left(\cup_{i=1}^{n} f[W_i] \right)$$

must be disjoint from $(0, \varepsilon) \times (-2, 2)$.■

In contrast to the preceding, we have the following result:

THEOREM 15. *The Čech homology groups of the Polish circle P are given by $\check{H}_q(P) = \mathbb{Z}$ if $q = 0, 1$ and zero otherwise.*

The results on Čech homology groups in Eilenberg and Steenrod show that these groups are functorial for continuous mappings and that homotopic mappings induce the same algebraic homomorphisms in Čech homology. If we combine this with Theorem 15 and the results on singular homology, we see that *the Polish circle P is a space which is simply connected and has the singular homology of a point, but P is* not *a contractible space.*

An alternate proof of the preceding statement is given in

`http://math.ucr.edu/~res/math205Bcommentaries.pdf`

which establishes the existence of a continuous map $r|P$ from P to S^1 that is not homotopic to a constant. In fact, it follows that the map $r|P$ induces an isomorphism from $\check{H}_*(P)$ to $\check{H}(S^1) \cong H_*(S^1)$.

Proof. We shall prove this using the continuity property of Čech homology as stated above, and we shall use the presentation of P as an intersection of the decreasing closed subsets B_n in the previously cited `polishcircle.pdf`. Since $P \ = \ \cap_n \, B_n$ it follows that

$$\check{H}_* \quad \cong \quad \lim_{\leftarrow} \; \check{H}_*(B_n)$$

and since each B_n is homeomorphic to a finite simplicial complex (describe this explicitly — it is fairly straightforward), we can replace Čech homology with singular homology on the right hand side. It will suffice to prove that each B_n is homotopic to a circle and the inclusion mappings $B_{n+1} \subset B_n$ are all homotopy equivalences. We shall do this using the subspaces C_n from the `polishcircle` document.

By construction, C_n is a subset of B_n, and we claim that C_n is a deformation retract of B_n. Let X_n be the closed rectangular box

$$\left[\frac{2}{(4n+3)\pi} \right] \; \times \; [-1, 1]$$

(the piece shaded in blue in the third figure), and let Q_n denote the bottom edge of X_n defined by the equation $y = -1$. It follows immediately that Q_n is a strong deformation retract of X_n; since the closure of $B_n - X_n$ intersects X_n in the two endpoints of Y_n, we can extend the retract $X_n \to Y_n$ and homotopy $X_n \times [0, 1] \to X_n$ by taking the identity on $\overline{B_n - X_n}$ to extend the retraction and the trivial homotopy from the identity to itself on $\overline{B_n - X_n}$. This completes the proof that C_n is a strong deformation retract of B_n.

By construction the space C_n is homeomorphic to the standard unit circle, and furthermore it is straightforward to check that the composite

$$C_{n+1} \quad \subset \quad B_{n+1} \quad \subset \quad B_n \quad \longrightarrow \quad C_n$$

(where the last map is the previously described homotopy inverse) must be a homeomorphism which is the identity off the points which lie in the vertical strip

$$\left(\frac{2}{4n+7}, \frac{2}{4n+3}\right) \times \mathbb{R}$$

and on this strip it is the flattening map which sends a point $(x, y) \in C_{n+1}$ to $(x, -1) \in C_n$. Therefore the map in homology from $H_q(C_{n+1})$ to $H_q(C_n)$ is an isomorphism of infinite cyclic groups in dimensions 0 and 1 and of trivial groups otherwise, and it follows that the map from $H_q(B_{n+1})$ to $H_q(B_n)$ is also an isomorphism of of infinite cyclic groups in dimensions 0 and 1 and of trivial groups otherwise. As in the proof of the second half of Theorem 6, it follows that $\check{H}_q(X)$ must be infinite cyclic if $q = 0$ or 1 and trivial otherwise.■

In fact, as noted before the proof of Theorem 15 one can show that a standard map from P to S^1 induces isomorphisms in Čech homology. This requires the naturality property of the comparison map from singular to Čech homology.

Dimensions of nowhere dense subsets

We have seen that if A is a compact subset of $\mathbb{R}^n$ with a nonempty interior, then the Lebesgue covering dimension of A is equal to n; we shall conclude this section with a converse to this result. In order to prove the converse we shall need some refinements of the ideas which arise in the proof of the embedding theorem stated as Theorem 50.5 in Munkres (see pages 311–313).

Definition. Let $(X, \mathbf{d})$ be a metric space, let $f : X \to Y$ be a continuous map of topological spaces, and let $\varepsilon > 0$. We shall say that f is an *ε−map* if for all $u, v \in X$ the equation $f(u) = f(v)$ implies that $\mathbf{d}(u, v) \leq \varepsilon$; an equivalent formulation is that for all $y \in Y$ the diameter of the level set $f^{-1}[\{y\}]$ is less than or equal to ε.

Clearly a continuous map f is 1–1 if and only if it is an ε−map for all $\varepsilon > 0$ (equivalently, it suffices to have this condition for all numbers of the form $1/k$ where k is a positive integer or all numbers of the form 2^{-k} where k is a positive integer).

We shall need the following result, which is entirely point set-theoretic.

LEMMA 17. *Let $(X, \mathbf{d}_X)$ and $(Y, \mathbf{d}_Y)$ be compact metric spaces, let $\varepsilon > \varepsilon' > 0$, and let $f : X \to Y$ be a continuous ε'−map. Then there is a $\delta > 0$ such that if $A \subset Y$ has diameter less than or equal to δ, then $f^{-1}[A]$ has diameter less than ε.*

Proof. Let $\eta = \frac{1}{2}(\varepsilon + \varepsilon')$ and let $K_\eta \subset X \times X$ be the set of all (x_1, x_2) such that $\mathbf{d}_X(x_1, x_2) \geq \eta$. Then K_η is a closed (hence compact) subset of $X \times X$ and $f \times f[K_\eta]$ is a compact subset of $Y \times Y$ which is disjoint from the diagonal Δ_Y because f is an ε'−map. It follows that the restriction of the distance function $\mathbf{d}_Y$ to $f \times f[K_\eta]$ is bounded away from zero by a positive constant h; in other words, if $U_h \subset Y \times Y$ is the set of all $(y_1, y_2) \in Y \times Y$ such that $\mathbf{d}_Y(y_1, y_2) \leq h/2$, then $(y_1, y_2) \notin f \times f[K_\eta]$.

Suppose now that the diameter of A is less than $\delta = h/2$; then we have $A \times A \subset U_h$, and it follows that if $(p, q) \in f^{-1}[A]$, then $\mathbf{d}_Y\big(f(p), f(q)\big) < \delta$, and this means that (p, q) cannot lie in K_η because the image of the latter under $f \times f$ is disjoint from U_h, which contains $A \times A$. In other words, if the diameter of A is less than δ, then the diameter of $f^{-1}[A]$ must be less than or equal to η, which is less than ε.■

The next result gives a method for approximating n-dimensional compact metric spaces by n-dimensional simplicial complexes.

PROPOSITION 18. *Let X be a compact metric space, and let n be a nonnegative integer. Then the Lebesgue covering dimension of X is $\leq n$ if and only if for every $\varepsilon > 0$ there is an $\varepsilon-$map from X into some n-dimensional polyhedron P.*

Proof. Suppose first that the Lebesgue covering dimension of X is $\leq n$. Take the open covering of X by open disks of radius $\varepsilon/2$ about the points of X, and extract a finite subcovering

$$\mathcal{U} \quad = \quad \left\{ N_{\varepsilon/2}(x_1), \ \cdots \ , \ N_{\varepsilon/2}(x_m) \right\} .$$

Let $\{ \varphi_j \}$ be a partition of unity subordinate to this finite covering, and consider the canonical map k from X to $\mathfrak{N}(\mathcal{U})$. If $k(u) = k(v)$, then $\varphi_j(u) = \varphi_j(v)$ for all j; at least one of these values must be positive, and therefore we can find some j such that $u, v \in N_{\varepsilon/2}(x_j)$. Since the latter implies $\mathbf{d}(u,v) \leq \text{diameter} N_{\varepsilon/2}(x_j) \leq \varepsilon$, it follows that k is an $\varepsilon-$map.

As usual, with respect to this metric there is a Lebesgue number $\eta > 0$ for this open covering. Let $0 < \varepsilon' < \varepsilon < \eta$, and let $f : X \to P$ be an $\varepsilon'-$map from X to some polyhedron P of dimension $\leq n$. By the preceding lemma there is some $\delta > 0$ such that if $A \subset Y$ has diameter less than δ then $f^{-1}[A]$ has diameter less than ε.

Take a sufficiently large barycentric subdivision of P such that all simplices have diameter at most $\delta/2$, and let $\mathcal{V}$ be the open covering given by the inverse images (under f) of open stars of the vertices in P. Then the intersection of any $n+2$ open subsets in $\mathcal{V}$ is empty; if we can show that $\mathcal{V}$ is a refinement of $\mathcal{U}$, then we are done. But the open stars of vertices in P all have diameter at most δ, and thus by Lemma 17 their inverse images have diameters which are at most ε. Since ε is less than a Lebesgue number for $\mathcal{U}$, it follows that each of the open subsets in $\mathcal{V}$ must be contained in some open set from $\mathcal{U}$, and thus $\mathcal{V}$ is an open refinement of $\mathcal{U}$ such that every subcollection $n+2$ subsets in $\mathcal{V}$ has an empty intersection.∎

Before proceeding, we shall show that Proposition 18 yields the previously stated result about the dimensions of Cartesian products (namely, $\dim(X \times Y) \leq \dim X + \dim Y$). In this argument we assume that $\dim X$ and $\dim Y$ are both finite; it is straightforward to verify that if X and Y are $\mathbf{T_1}$ spaces and either $\dim X = \infty$ or $\dim Y = \infty$, then $\dim(X \times Y) = \infty$ (look at the contrapositive statement).

Proof of Proposition 9. Suppose that $\dim X \leq m$ and $\dim Y \leq n$, and let $\varepsilon > 0$. By Proposition 18, it will suffice to construct an ε-map from $X \times Y$ to some polyhedron T of dimension at most $m + n$. For the sake of definiteness, in this argument the metrics on products are given by the $\mathbf{d}_2$ metrics associated to metrics on the factors (using the notation of the 205A notes).

The construction is fairly straightforward. By the dimension hypotheses and Proposition 18 we know there are $(\varepsilon/\sqrt{2})$-maps $f : X \to P$ and $g : Y \to Q$, where P and Q are polyhedra of dimension at most m and n respectively. It follows that the product map $f \times g : X \times Y \to P \times Q$ is an ε-map into a polyhedron whose dimension is at most $m + n$.∎

Using Proposition 18, we can prove the result on the dimensions of nowhere dense subsets mentioned above.

THEOREM 19. *Suppose that $A \subset \mathbb{R}^n$ is compact and nowhere dense. Then the Lebesgue covering dimension of A is at most $n - 1$.*

The estimate in the theorem is the best possible estimate because we know that the Lebesgue covering dimension of the nowhere dense subset S^{n-1} is equal to $n - 1$.

Proof. We shall prove that A satisfies the criterion in Proposition 18. One step in the proof involves the following result:

CLAIM. *If* $\mathbf{v}$ *is an interior point of the disk* D^n *where* $n > 0$, *then* S^{n-1} *is a retract of* $D^n - \{\mathbf{v}\}$.

The quickest way to prove this is to take the map $\rho : D^n \times D^n - \text{diagonal} \to S^{n-1}$ constructed in the file `brouwer.pdf` and restrict it to $(D^n - \{\mathbf{v}\}) \times \{\mathbf{v}\}$.

The first steps in the proof are to let $\varepsilon > 0$ and to take a large hypercube Q containing A. We know that Q has a simplicial decomposition, and if we take repeated barycentric subdivisions we can construct a decomposition whose simplices all have diameter less than $\varepsilon/2$. Let σ be an n-simplex in this decomposition. Since σ has a nonempty interior (in the sense of point set topology) and A is nowhere dense in $\mathbb{R}^n$, it follows that there is some interior point $w(\sigma)$ in σ such that $w(\sigma) \notin A$. By the claim above, we know that the boundary $\partial\sigma$ is a retract of $\sigma - w(\sigma)$, and we can piece the associated retractions together to obtain a retraction

$$r : Q \ - \ \left(\bigcup_{\dim \sigma = n} \{w(\sigma)\} \right) \ \longrightarrow \ Q^{[n-1]}$$

where $Q^{[n-1]}$ (the *n-skeleton*) is the union of all simplices in Q with dimension strictly less than n. By construction the set A is contained in the domain of r, and therefore we also obtain a retraction $r : A \to Q^{[n-1]}$. The inverse image of a point z in the codomain is contained in all simplices which contain z, and since these simplices all have diameter less than $\varepsilon/2$, it follows that each set $r^{-1}[\{z\}]$ has diameter less than ε. Therefore we have shown that $r|A$ is an $\varepsilon-$map into the $(n-1)$-dimensional polyhedron $Q^{[n-1]}$. By Proposition 18, it follows that the Lebesgue covering dimension of A is at most $n - 1$.■

Using results from Section 50 of Munkres (including the exercises), it is a straightforward exercise to prove the following generalization of Theorem 19:

COROLLARY 20. *Let* M^n *be a second countable topological n-manifold, and suppose that* $A \subset M$ *is a closed nowhere dense subset of* M^n. *Then the Lebesgue covering dimension of* A *is strictly less than* n.■

V.5. Appendix : The Flag Property

DEFAULT HYPOTHESIS. Unless stated otherwise, all spaces discussed in this Appendix are compact metric spaces with finite Lebesgue covering dimensions.

In our discussion of product formulas for the Lebesgue covering dimension, we noted that $\dim X \times Y > \dim X$ if $\dim Y > 0$, and we gave references for the proof when $\dim Y = 1$. We also asserted that the general case followed quickly from this special case because $\dim Y > 0$ implies the existence of a closed subset $A \subset Y$ with $\dim A = 1$. In fact, we have the following:

PROPOSITION A1. (Flag Property) *Suppose that* X *satisfies the Default Hypothesis and* $\dim X = n > 0$. *Then there is a chain of closed subsets*

$$\{y\} \ = \ A_0 \ \subset \ A_1 \ \subset \ \cdots \ \subset A_n \ = \ X$$

such that $\dim A_k = k$ *for all* k.

Note. The name for this result is motivatived by a standard geometrical concept of a *flag* of subspaces in $\mathbb{R}^n$, which is a sequence of vector subspaces

$$\{\mathbf{0}\} \quad = \quad V_0 \quad \subset \quad V_1 \quad \subset \cdots \subset \quad V_n \quad = \quad \mathbb{R}^n$$

such that $\dim V_k = k$ for all k; of course, there is a similar concept if $\mathbb{R}$ is replaced by an arbitrary field.

The proof of the Flag Property is a fairly direct consequence of equality of the Lebesgue covering dimension and the previously cited Menger-Urysohn or weak inductive dimension for compact metric spaces. Here is a summary of what we need in order to prove the Flag Property:

THEOREM A2. *Let X be a compact metric space such that $\dim X \leq n$, and let $x \in X$. Then there is a countable neighborhood base at x of the form*

$$\mathfrak{B} \quad = \quad \{\, W_1 \quad \supset \quad W_2 \quad \cdots \qquad \}$$

such that for each k the set $\mathsf{Bdy}_x(W_k)$ has dimension at most $n-1$. Conversely, if such neighborhood bases exist for each point of X, then $\dim X \leq n$.

As in Munkres, the boundary (or frontier) $\mathsf{Bdy}_X(E)$ of $E \subset X$ (in X) is the intersection of the limit point sets $\mathbf{L}_X(E) \cap \mathbf{L}_X(X - E)$; since we are working with metric spaces, this is a closed subset of X.

Idea of proof for Theorem A2. The statement in the conclusion is essentially the same as the condition for the Menger-Urysohn dimension of X to be at most n (this is given on page 24 of Hurewicz and Wallman). Therefore the conclusion will follow if we know that the Lebesgue covering dimension and the Menger-Urysohn dimension are equal for compact metric spaces. Virtually every book on dimension theory from the past 50 years contains some abstract version of this equality. More directly, one can use Theorem V.8 on page 67 of Hurewicz and Wallman (in which "dimension" means the Menger-Urysohn dimension) to show that the two definitions are the same for compact metric spaces.

One reason that the standard references for dimension theory phrase things in more abstract terms is that the Lebesgue covering dimension and Menger-Urysohn dimension are not necessarily equal for more general topological spaces (usually it is easy to find examples; see also the Wikipedia article on inductive dimension mentioned earlier).

Proof of Proposition A1. (Compare Hurewicz and Wallman, Proposition III.1.D, pp. 24–25.) If $\dim X = 1$ then X is nonempty and the conclusion follows immediately. Proceeding by induction on the dimension, we shall assume the result is true for compact metric spaces of dimension $\leq n-1$. Suppose that X is an n-dimensional compact metric space. Since $\dim X$ is not less than or equal to $n-1$, Theorem A2 implies the existence of some point $z \in X$ such that for all countable neighborhood bases at z of the form

$$\mathfrak{A} \quad = \quad \{\, V_1 \quad \supset \quad V_2 \quad \cdots \qquad \}$$

we have $\dim(\mathsf{Bdy}_X(V_k)) > n-2$ for infinitely many k (why?). In particular, this holds for the neighborhood base $\mathfrak{B}$ for z described in the statement of Theorem A2 (we know such a neighborhood base exists because $\dim X = n$). It follows that $\dim(\mathsf{Bdy}_X(W_k)) = n-1$ for all such k. Choose a specific m such that $\dim(\mathsf{Bdy}_X(W_m)) = n-1$. By the induction hypothesis, there is a chain of closed subspaces

$$\{y\} \quad = \quad A_0 \quad \subset \quad A_1 \quad \subset \quad \cdots \quad \subset \quad A_{n-1} \quad = \quad \mathsf{Bdy}_X(W_m)$$

and we may extend this to a chain of subspaces as in the conclusion of the proposition by taking $A_n = X$.■

VI. Cohomology

Suppose that $\mathbb{F}$ is a field and (X, A) is a pair of topological spaces. One can then define the q-dimensional cohomology $H^q(X, A; \mathbb{F})$ to be the vector space dual $\mathrm{Hom}_{\mathbb{F}}\big(H_q(X, A; \mathbb{F}), \mathbb{F} \big)$, and this construction extends to a contravariant functor on the category of pairs of spaces and continuous maps. Similarly, by taking adjoint maps of dual spaces we obtain natural coboundary morphisms $\delta : H^q(A; \mathbb{F}) \to H^{q+1}(X, A; \mathbb{F})$ and long exact cohomology sequences for pairs.

One natural question is why one would bother to do this, especially since it follows that $H_q(X; \mathbb{F}) \cong H^q(X; \mathbb{F})$ if X has the homotopy type of a finite cell complex (because the homology is finite dimensional and is isomorphic to its vector space dual). There are two related answers:

(1) Even when mathematical objects and their duals are equivalent, in many cases it is more convenient to work with the dual object rather than the original one, and vice versa. — For example, vector fields and differential 1-forms on a smooth manifold are dual to each other, but they play markedly different roles in the theory of smooth manifolds. In particular, vector fields are better for working with differential equations, while differential forms provide a more convenient way for manipulating expressions like line integrals.

(2) Frequently the dual objects have some extremely useful extra structure which is not easily studied in the original objects. — To continue with our example of vector fields and differential 1-forms, the latter have better functoriality properties, and the exterior derivative construction on differential forms does not have a functorial counterpart for vector fields unless one adds some further structure like a riemannian metric. On the other hand, there can also be some nice structure on the original objects which is not on their duals; for example, the Lie bracket construction on vector fields has no obvious counterpart on 1-forms unless one adds some further structure.

In fact, it turns out that cohomology groups have a useful additional structure; namely, there are natural bilinear **cup product** mappings

$$\cup \;:\; H^p(X, A; \mathbb{F}) \times H^q(X, A; \mathbb{F}) \;\longrightarrow\; H^{p+q}(X, A; \mathbb{F})$$

which do not have comparably simple counterparts in homology. This illustrates the second point about objects and their duals. Later in these notes we shall illustrate how the first point manifests itself in homology and cohomology.

A useful result

At several points in this unit we shall need the following result on **acyclic** (no homology) chain complexes.

THEOREM 0. *Let C_* be a chain complex such that $C_k = 0$ for $k < M$ for some integer M and each C_k is free abelian on some set of generators G_k. Then $H_*(C) = 0$ in all dimensions if and only if there is a contracting chain homotopy $D_q : C_q \to C_{q+1}$ (for all q) such that $Dd + dD = \mathbf{id}_C$.*

Proof. If D exists then clearly the homology is zero by the usual sort of argument. Conversely, suppose that $H_*(C) = 0$, and let m be the first degree in which C is nonzero. We may construct D_m as follows: If $T \in G_m$, then $dT = 0$ and hence $T = du$ for some $u \in C_{m+1}$. Define $D_m(T) = u$ and extend the map using the freeness property. Now suppose by induction that we have defined D_k for $k \leq N - 1$.

Let T be an element of the free generating set G_N. Then we need to find an element $u_T \in C_{n+1}$ so that $du_T = T - DdT$. Since the complex has no homology, such a class exists if and only if the right hand side is a boundary. But now we have a familiar sort of inductive calculation:

$$d(T - DdT) \ = \ dT - dDdT \ = \ dT - (1 - Dd)dT \ = \ dT - dT - DddT \ = \ -DddT \ = \ D0T \ = \ 0$$

Hence we can define $D(T) = u_T$ and extend by freeness. This completes the inductive step and proves the existence of the contracting chain homotopy.■

VI.1 : The basic definitions

(Hatcher, §§ 3.1–3.2)

We begin by defining the singular cohomology of a space with coefficients in an arbitrary $\mathbb{D}$-module, where $\mathbb{D}$ is a commutative ring with unit (a setting broad enough to contain coefficients in fields, the integers, and quotients of the latter). However, we shall quickly specialize to the case of fields in order to minimize the amount of algebraic machinery that is needed.

Definition. Let (X, A) be a pair of topological spaces, and let π be a module over the ring $\mathbb{D}$ as above. The **singular cochain complex** $\big(S^*(X, A; \pi),\ \delta\big)$ of (X, A) **with coefficients in** π is defined with $S^q(X, A) = \mathrm{Hom}\big(S_q(X, A),\ \pi\big)$ and the *coboundary mapping*

$$\delta^{q-1} : S^{q-1}(X, A; \pi) \ \longrightarrow \ S^q(X, A; \pi)$$

given by the adjoint map $\mathrm{Hom}(d_q, \pi)$.

Many basic properties of singular cochain complexes follow immediately from the definitions, including the following:

PROPOSITION 1. *(i) We have* $\delta^q \circ \delta^{q-1} = 0$.

(ii) The singular cochain complex is contravariantly functorial with respect to continuous mappings on pairs of topological spaces.

The first of these follows because $d_q \circ d_{q+1} = 0$ and the functor $\mathrm{Hom}(-,-)$ is additive, while the second is basically just a consequence of the definition and the covariant functoriality of the singular chain complex.■

Before going further, we shall define the q-dimensional **singular cohomology** $H^q(X, A; \pi)$ of (X, A) **with coefficients in** π to be the kernel of δ^q modulo the image of δ^{q-1}. Elements of the kernel are usually called **cocycles**, and elements of the image are usually called **cobooundaries**. As in the case of singular chain complexes, it follows that the map of singular cochains

$$f^{\#} : S^*(Y, B; \pi) \ \longrightarrow \ S^*(X, A; \pi)$$

associated to a continuous map $f : (X, A) \to (Y, B)$ will pass to a homomorphism

$$f^* : H^*(Y, B; \pi) \ \longrightarrow \ H^*(X, A; \pi)$$

and this makes singular cohomology into a contravariant functor on pairs of spaces and continuous maps.

If (X,A) is a pair of topological spaces, then for each q we know that $S_q(X) \cong S_q(A) \oplus S_q(X,A)$ **as free abelian groups** (but this is **NOT** an isomorphism of chain complexes!), and from this it follows that for each q we have a split short exact sequence of modules

$$0 \longrightarrow S^*(X,A;\pi) \xrightarrow{j^\#} S^*(X;\pi) \xrightarrow{i^\#} S^*(A;\pi) \longrightarrow 0$$

where $j : X \to (X,A)$ and $i : A \to X$ are the usual inclusions. As in the case of singular chains, this leads to a natural **long exact cohomology sequence**; to simplify the notation we shall omit the coefficient module π in the display below:

$$\cdots \quad H^{k-1}(A) \quad \xrightarrow{\delta} \quad H^k(X,A) \quad \xrightarrow{j^*} \quad H^k(X) \quad \xrightarrow{i^*} \quad H^k(A) \quad \xrightarrow{\delta} \quad H^{k+1}(X,A) \quad \cdots$$

As in the case of homology, this sequence extends indefinitely to the left and right.

Notational convention. The contravariant algebraic maps induced by inclusions are often called **restriction** maps; one motivation for this terminology is that a map like $i^\#$ restricts attention from objects defined for X to objects defined only for the subspace A (for example, consider the restriction map from continuous real valued functions on X to those defined on A, which is defined by composing a function $f : X \to \mathbb{R}$ with the inclusion mapping i).

We can now proceed as in the study of singular homology to prove homotopy invariance, excision, and Mayer-Vietoris theorems for singular cohomology; informally speaking, one need only apply the functor $\mathrm{Hom}(-,\pi)$ to everything in sight, including chain homotopies. At some points one needs Theorem 0 to conclude that if C_* is an acyclic, free abelian chain complex, then it has a contracting chain homotopy and the latter implies that $\mathrm{Hom}(C_*,\pi)$ has no nonzero cohomology (verify this!).

Cup products

We shall now assume that our coefficients π are a commutative ring with unit, which we shall call $\mathbb{D}$.

Definition. Let X be a space; then the *augmentation mapping* $\varepsilon_X(\mathbb{D}) = \varepsilon_X \in S^0(X;\mathbb{D})$ is the homomorphism from $S_0(X)$ to $\mathbb{D}$ which sends each singular 0-simplex $T : \Delta_0 \to X$ to the unit element of $\mathbb{D}$.

The following is an immediate consequence of the definitions.

PROPOSITION 2. *If $f : X \to Y$ is continuous, then $f^\#(\varepsilon_Y) = \varepsilon_X$. Furthermore, $\delta^0(\varepsilon_X) = 0$.*■

The augmentation plays a key role in the multiplicative structure mentioned earlier. Before proceeding, we need some geometric definitions.

Definition. Let p and q be nonnegative integers, and as usual let Δ_{p+q} denote the standard simplex. Then the *front and back faces* $\mathbf{Front}_p(\Delta_{p+q})$ and $\mathbf{Back}_q(\Delta_{p+q})$ are the p- and q-dimensional faces whose vertices are respectively the first $(p+1)$ and last $(q+1)$ vertices of the original simplex. Note that these intersect in the p^{th} vertex of Δ_{p+q}.

Definition. Given two cochains $f \in S^p(X,A;\mathbb{D})$ and $g \in S^q(X,A;\mathbb{D})$, their **cup product** $f \cup g \in S^{p+q}(X,A;\mathbb{D})$ is given as follows: For each standard free generator of $S_{p+q}(X,A)$ — in other words, each singular simplex of X whose image is not entirely contained in A — we define

$$f \cup g(T) \quad = \quad (f|\mathbf{Front}_p) \cdot (g|\mathbf{Back}_q) \ .$$

We then have the following:

PROPOSITION 3. *The cup product is functorial for continuous maps of pairs. Furthermore, it is bilinear and associative, and if $A = \emptyset$ then ε_X is a two sided multiplicative identity.*■

At this point we do not want to address questions about the possible commutativity properties of the cup product. This is a decidedly nonelementary issue, and in several respects it is a fundamental difficulty which has an enormous impact across most if not all of algebraic topology.

Clearly one would hope the cup product will pass to cohomology, and the following result guarantees this:

PROPOSITION 4. *In the notation of the cup product definition, we have*

$$\delta(f \cup g) \quad = \quad (\delta f) \cup g \ + \ (-1)^p \, f \cup (\delta g) \ .$$

In particular, it follows that $f \cup g$ is a cocycle if both f and g are, and if we are given equivalent representatives f' and g' for the same cohomology classes, then $f \cup g - f' \cup g'$ is a coboundary.

Proof. The identity involving the coboundary of $f \cup g$ is derived in Lemma 3.6 on page 206 of Hatcher. If f and g are both coboundaries, the formula immediately implies that $f \cup g$ is also a coboundary. Suppose now that we also have $f - f' = \delta v$ and $g - g' = \delta w$. It then follows that

$$\delta(v \cup g) \quad = \quad (f - f') \cup g \ , \qquad \delta(f' \cup w) \quad = \quad \pm\, f' \cup (g - g') \ .$$

The first of these implies that $f \cup g$ and $f' \cup g$ determine the same cohomology class, while the second implies that $f' \cup g$ and $f' \cup g'$ also determine the same cohomology class.■

In many contexts it is useful to have a slight refinement of the cup product described above. Specifically, if A and B are both subspaces of X, then there is a cochain level cup product

$$S^p(X, A; \mathbb{D}) \times S^q(X, B; \mathbb{D}) \ \longrightarrow \ S^{p+q}(X, A \cup B; \mathbb{D})$$

which is a very slight modification of the original definition. To see this, note first that a homomorphism f from $S_p(X, A)$ to $\mathbb{D}$ is equivalent to a homomorphism F from $S_p(X)$ which vanishes on the subgroup $S_p(A)$, and similarly a homomorphism g from $S_q(X, B)$ to $\mathbb{D}$ is equivalent to a homomorphism G from $S_q(X)$ which vanishes on the subgroup $S_q(B)$. Consider what happens if we form $F \cup G$; this is a cochain on X which vanishes on $A \cup B$ because one factor vanishes on A and the other vanishes on B. Therefore it has a natural interpretation as a cochain in $S^{p+q}(X, A \cup B; \mathbb{D})$. This refined cup product has analogs of all the properties one might expect to generalize from the case $A = B$.

VI.2 : A weak Universal Coefficient Theorem

(Hatcher, § 3.1)

We have already asserted the q-dimensional cohomology of a space is the dual space of the q-dimensional homology if we take coefficients in a field. However, our basic definition is somewhat different from this, so the next step is to verify the assertion at the beginning of this unit. Hatcher formulates and proves more general results (for example, see Theorem 3.2 on page 195). In this course we do not have enough time to develop the homological algebra necessary to prove such a result, and in any case the results for fields are strong enough to yield some important insights.

The Kronecker Index

As usual let $\mathbb{D}$ be a commutative ring with unit, let C_* be a chain complex of $\mathbb{D}$-modules, and define an associated cochain complex by $C^q = \mathrm{Hom}_{\mathbb{D}}(C_q, \mathbb{D})$, with a coboundary map $d^q = \mathrm{Hom}(d_{q+1}, \mathbb{D})$ analogous to the construction for singular cochains. Then evaluation defines a bilinear map $C^q \times C_q \to \mathbb{D}$ which is called the *Kronecker index pairing* and its value at $f \in C^q$ and $x \in C_q$ is usually written as $\langle f, x \rangle$.

LEMMA 1. *Suppose that $f, f' \in C^q$ are cocycles and $x, x' \in C_q$ are cycles such that $f - f' = \delta a$ and $x - x' = db$. Then $\langle f, x \rangle = \langle f', x' \rangle$.*

Proof. For an arbitrary cochain g and chain y it follows immediately that $\langle \delta g, y \rangle = \langle g, dy \rangle$. Therefore we have

$$\langle f, x - x' \rangle \quad = \quad \langle f, db \rangle \quad = \quad \langle \delta f, b \rangle \quad \langle 0, b \rangle \quad = \quad 0$$

and similarly

$$\langle f - f', x' \rangle \quad = \quad \langle \delta a, x' \rangle \quad = \quad \langle a, dx' \rangle \quad \langle a, 0 \rangle \quad = \quad 0$$

which combine to show that $\langle f, x \rangle = \langle f', x' \rangle$.■

COROLLARY 2. *The chain/cochain level Kronecker index pairing passes to a well-defined bilinear pairing from $H^q(C) \times H_q(C)$ to $\mathbb{D}$.*■

Manipulations with dual vector spaces

We now assume that $\mathbb{F}$ is a field. If V is a vector space over $\mathbb{F}$ and U is a subspace of V, then we have a short exact sequence of vector spaces

$$0 \;\to\; U \;\to\; V \;\to\; V/U \;\to\; 0$$

and applying the dual space functor we obtain the following short exact sequence of dual spaces:

$$0 \;\to\; (V/U)^* \;\to\; V^* \;\to\; U^* \;\to\; 0$$

The image of the map from $(V/U)^*$ to V^* is the **annihilator** of U, which consists of all linear functionals which vanish on U and will be denoted by $U^{\dagger}$.

Suppose now that V_1 and V_2 are vector spaces over $\mathbb{F}$ and $T : V_1 \to V_2$ is a linear transformation. Then we can factor T into a composite

$$V_1 \to J_1 \cong J_2 \subset V_2$$

where J_1 is the quotient of V_1 by the kernel of T, the map from J_1 to J_2 is an isomorphism, and J_2 is the image of T. There is also a corresponding factorization for the induced map of dual spaces

$$V_2^* \to J_2^* \cong J_1^* \subset V_1^*$$

These factorizations will be useful in proving the following abstract version of a key result in linear algebra:

PROPOSITION 3. *In the notation above, let $T^* : V_2 \to V_1^*$ be the associated map of dual spaces. Then we have* $(\text{Kernel } T)^\dagger = \text{Image T}^* \subset V_1^*$ *and* $(\text{Image } T)^\dagger = \text{Kernel } T^* \subset V_2^*$.

Proof. By our previous observations we know that $(\text{Kernel } T)^\dagger$ corresponds to $J_1^* = J_2^*$, and since J_2 is the image of T, we have the asserted relationship. Similarly, we know that $(\text{Image}^T)^\dagger$ corresponds to $(V_2/J_2)^*$, and one can check directly that this corresponds to all linear functionals f on V_2 such that $0 = f \circ T = T^*(f)$.■

We now have enough machinery to derive the relationship between homology and cohomology over a field.

PROPOSITION 4. *Let C_* be a chain complex over a field $\mathbb{F}$, and let C^* be the dual cochain complex. Then for each q there is a natural isomorphism from $H^q(C)$ to $H_q(C)^*$.*

Proof. We shall focus on verifying the assertion about the isomorphism first. By definition we know that

$$H^q \quad \cong \quad (\text{Kernel } \delta^q)/(\text{Image } \delta^{q-1}) \ .$$

Using the relationship $\delta = d^*$ we may rewrite the right hand side in the form

$$(\text{Image } d_{q+1})^\dagger/(\text{Kernel } d_q)^\dagger$$

and conclude by noting that the latter subquotient of C_q^* corresponds to

$$H_q^* \quad \cong \quad \Big((\text{Kernel}_q^d)/(\text{Image } d_{q+1}) \Big)^* \ .$$

Under these correspondences and the defining isomorphism

$$H_q \quad \cong \quad \Big((\text{Kernel}_q^d)/(\text{Image } d_{q+1}) \Big)$$

all the standard pairings which evaluate linear functionals at vectors are preserved. In particular, this means that the isomorphism is given by the pairing described in Corollary 2. Now this pairing is natural by construction, and therefore our isomorphism is also natural.■

Only a little more work is needed to derive the description of singular cohomology that we want.

COROLLARY 5. *If (X, A) is a topological space and $\mathbb{F}$ is a field, then for each q there is a natural isomorphism from $H^q(X, A; \mathbb{F})$ to the dual space $H_q(X, A; \mathbb{F})^*$.*

Proof. At this point all we need to do is describe a natural isomorphism

$$S^*(X, A; \mathbb{F}) \quad \cong \quad \text{Hom}(S_*(X, A), \mathbb{F}) \quad \longrightarrow \quad \text{Hom}_{\mathbb{F}}\,(S_*(X, A) \otimes \mathbb{F}, \mathbb{F})$$

because the latter is the cochain complex to which Proposition 4 applies. However, the isomorphism in question is given directly by the universal properties of the tensor product construction sending the chain groups $S_q(X, A)$ to $S_q(X, A) \otimes \mathbb{F}$; in other words, there is a 1–1 correspondence between abelian group homomorphisms from $S_q(X, A)$ to $\mathbb{F}$ and $\mathbb{F}$-linear maps from $S_q(X, A) \otimes \mathbb{F}$ to $\mathbb{F}$.■

If (X, A) is a pair of topological spaces, then similar considerations show that under this isomorphism the connecting morphism in cohomology

$$\delta^* : H^p(A; \mathbb{F}) \ \longrightarrow \ H^{p+1}(X, A; \mathbb{F})$$

corresponds to the map $\text{Hom}_{\mathbb{F}}(\partial, \mathbb{F})$, where $\partial : H_{p+1}(X, ; \mathbb{F}) \to H_p(A; \mathbb{F})$ is the connecting morphism in homology. This reflects the fact that chain complex boundaries and cochain complex coboundaries are adjoint to each other with respect to the Kronecker index pairing; details of the verification are left to the reader.

VI.3 : Examples of cup products

(Hatcher, §§ 3.2, 3.B)

One obvious point about the preceding discussion is that we have not yet produced examples for which the cup product of two positive-dimensional cohomology classes is nontrivial. Our next order of business is to find classes of spaces with this property.

Cross products

Given two topological spaces X and Y, the **cohomology cross product** of $v \in H^p(X; \mathbb{F})$ and $w \in H^q(Y; \mathbb{F})$ is given by

$$v \times w \quad = \quad \pi_X^*(v) \cup \pi_Y^*(w) \quad \in \quad H^{p+q}(X \times Y; \mathbb{F})$$

and there is a corresponding definition of cross product at the cochain level. These maps are functorial with respect to pairs of continuous mappings $f : X \to X'$ and $g : Y \to Y'$, and on the cochain level one has the following analog of a key identity for cup products:

$$\delta(a \times b) \quad = \quad \delta(a) \times b \; + \; (-1)^p \, a \times \delta(b) \; , \qquad \text{where} \;\; a \in S^p(X; \mathbb{F})$$

In fact, the cross product and cup product are equivalent, for if $X = Y$ then one also has the identity

$$v \cup w \quad = \quad \Delta_X^*(v \times w)$$

where $\Delta_X : X \to X \times X$ is the diagonal mapping. There is also a relative cross product pairing

$$\mu \; : \; H^p(X, A; \mathbb{F}) \times H^q(Y, B; \mathbb{F}) \;\longrightarrow\; H^{p+q}(X \times Y, A \times Y \cup X \times B; \mathbb{F})$$

which comes from a pairing at the cochain level, and we also have the following basic identity:

PROPOSITION 1. *In the preceding setting, suppose that $B \neq \emptyset$, and let δ^* generically denote the connecting morphisms in long exact cohomology sequences for pairs. Then for each $v \in H^p(A; \mathbb{F})$ and $w \in H^q(Y; \mathbb{F})$ we have $\delta^*(v \times w) = \delta^*(v) \times w$.*

This follows directly from the cochain level formula for the coboundary of a cross product.■

We can now proceed as in Example 3.11 on pages 210–211 of Hatcher to prove the following nontriviality result for cross products:

PROPOSITION 2. *Let Y be an arbitrary nonempty topological space, let $I = [0, 1]$ denote the unit interval, and let $\partial I = \{0, 1\}$ be its boundary. Denote the generator of $H^1(I, \partial I; \mathbb{F}) \cong \mathbb{F}$ by ω. Then for each $q \geq 0$ the map*

$$L_\omega : H^q(Y; \mathbb{F}) \;\longrightarrow\; H^{q+1}(I \times Y, \partial I \times Y; \mathbb{F}) \; , \qquad L_\omega(a) \quad = \quad \omega \times a$$

is bijective.

This result follows by combining Proposition 1 with the identity $\omega = \delta(\mathbf{u}_1 - \mathbf{u}_0)$, where $\mathbf{u}_t$ is the image of the (augmentation) unit element in $H^0(\{t\}\times Y;\mathbb{F})$ in $H^0(\partial I\times Y;\mathbb{F})$ under the natural splitting isomorphism

$$H^*(\partial I \times Y;\mathbb{F}) \quad\cong\quad H^*(\{0\}\times Y;\mathbb{F}) \;\oplus\; H^*(\{1\}\times Y;\mathbb{F}) \;.\blacksquare$$

Note that $\mathbf{u}_1 + \mathbf{u}_0$ is the unit element in $H^0(\partial I \times Y;\mathbb{F})$.

We also have the following modified version of the preceding result:

PROPOSITION 3. *Let Y be an arbitrary nonempty topological space. Denote the generator of $H^1(S^1;\mathbb{F}) \cong \mathbb{F}$ by Ω. Then for each $q \geq 0$ the map*

$$L_\omega : H^q(Y;\mathbb{F}) \;\longrightarrow\; H^{q+1}(S^1\times Y;\mathbb{F})\;, \qquad L_\omega(a) \;=\; \omega\times a$$

is injective, and its image is the kernel of the map

$$i^* : H^{q+1}(S^1\times Y;\mathbb{F}) \;\longrightarrow\; H^{q+1}(Y;\mathbb{F})$$

where $i : Y \to S^1\times Y$ is the slice inclusion sending $y\in Y$ to $(1,y)$.

Derivation of Proposition 3. For the sake of conciseness we shall omit the coefficients $\mathbb{F}$ in the discussion which follows.

Consider the standard decomposition of S^1 into upper and lower semicircles $D^1_\pm$ consisting of all $(x,y)\in S^1$ such that $y \geq 0$ or $y \leq 0$. Similarly, set $W_\pm$ equal to the complement of $(\mp 1, 0)$, so that $D^1_\pm$ is a strong deformation retract of $W_\pm$ and S_0 is a strong deformation retract of $W_+\cap W_-$. It follows immediately that all mappings in the sequence

$$H^*(S^1\times Y, \{(0,-1)\}\times Y) \;\leftarrow\; H^*(S^1\times Y, W_-\times Y) \;\rightarrow\; H^*(S^1\times Y, D_-\times Y) \;\rightarrow\; H^*(D^1_+\times Y, S^0\times Y)$$

are isomorphisms. Since (D^1, S^0) is homeomorphic to $(I, \partial I)$, the class ω corresponds to a class $\omega' \in H^1(S^1, \{0,-1\})$ under these mappings, and by Proposition 2 we know that cross product multiplication by ω' is injective.

To conclude the proof, we first need to see that if $\mathbf{p}\in S^1$, then there is a direct sum decomposition

$$H^*(S^1\times Y) \quad\cong\quad H^*(S^1\times Y, \{\mathbf{p}\}\times Y) \;\oplus\; H^*(Y)$$

because the restriction map from $H^*(S^1\times Y)$ to $H^*(\{\mathbf{p}\}\times Y) \cong H^*(Y)$ is split surjective (the map induced by projection onto Y is a one-sided inverse). Proposition 2 implies that the image of the cross product map is the entire first summand, and this immediately yields the conclusion we want.$\blacksquare$

Products of cell complexes

If $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ are cell complexes, then there is a product cell complex structure on $X\times Y$ whose cells are given by the following simple rule: If e_X is a p-cell of X and f_Y is a q-cell of Y, then $e_X \times f_Y$ is a $(p+q)$-cell of $X\times Y$. The attachments of the cells are determined by the homeomorphisms

$$(D^p\times D^q, S^{p-1}\times D^q \cup D^p\times S^{q-1}) \quad\cong\quad (D^{p+q}, S^{p+q-1})$$

sending (x,y) with $|x|, |y| \leq 1$ to

$$\frac{\sqrt{|x|^2 + |y|^2}}{\max(|x|, |y|)} \cdot \Big(x,\, y \Big)$$

if $(x,y) \neq (0,0)$ and sending $(0,0)$ to itself; this is continuous at the exceptional point because the ratio in the display is bounded from above by $\sqrt{2}$. If X_i and Y_j denote the respective i- and j-dimensional skeleta, then the n-skeleton of $X \times Y$ is equal to the union of the subspaces $X_p \times Y_q$, where p and q run through all pairs such that $p + q = n$.

Of course, this construction can be iterated any finite number of times. In particular, if we start out with the standard cell decomposition of S^1 with one 0-cell and one 1-cell, for each positive integer n we obtain a corresponding cell decomposition of the torus T^n.

Cohomology of the torus T^n

Let $\mu \in H^1(S^1) \cong \mathbb{F}$ be a generator; as before, we suppress the coefficients because we assume they lie in $\mathbb{F}$. If we express the torus as $S^1 \times T^{n-1}$ and take π_1 and ρ to be the projections onto the two factors, then the preceding results show that $H^*(T^n)$ splits as a direct sum into the image of ρ^* and $\pi_1^*\mu$ times the image of ρ (with respect to the cup product. In particular, it follows that bases for the cohomology of T^n are given by monomials in the elements $\alpha_i = \pi_i^*\mu$, where π_i denotes projection onto the i^{th} factor and we take monomials $\prod_j \ \alpha_{i_j}$ such that the indices i_j are increasing with respect to j. In particular, we have the following:

PROPOSITION 4. *For all positive integers n we have $H^n(T^n) \cong \mathbb{F}$, and the generator is a cup product of 1-dimensional classes.*■

In order to understand the cup product structure completely, it suffices to know how to multiply 1-dimensional classes because an inductive argument shows that they generate the cup product structure. We shall first do this when $n = 2$ and then explain how one can pass to the general case.

Let $e \in H_1(S^1) \cong \mathbb{F}$ be a generator, and choose $\mu \in H^1(S^1)$ such that $\langle \mu, e \rangle = 1$. Let i_1 and i_2 denote slice inclusions of S^1 whose images are $S^1 \times \{1\}$ and $\{1\} \times S^1$ respectively.

PROPOSITION 5. *The classes $i_{1*}e$ and $i_{2*}e$ form a basis for $H_1(T^2)$, and a dual basis for $H^1(T^2)$ with respect to the Kronecker index is given by $\pi_1^*\mu$ and $\pi_2^*\mu$. The cup product $\pi_1^*\mu \cup \pi_2^*\mu$ generates $H^2(T^2)$, and we have $\pi_2^*\mu \cup \pi_1^*\mu = -\pi_1^*\mu \cup \pi_2^*\mu$* (note the sign!). *Furthermore, the cup squares of $\pi_1^*\mu$ and $\pi_2^*\mu$ are trivial.*

Proof. The final statement follows because the maps π_j^* are multiplicative and $\mu^2 = 0$ (since $H^2(S^1)$ is trivial). To prove the assertion about dual bases it suffices to compute the Kronecker indices $\langle \pi_j^*\mu, i_{k*}e \rangle$. These are given by $\langle \mu, \pi_{j*} \circ i_{k*}e \rangle$; since $\pi_j \circ i_k$ is the identity if $j = k$ and constant if $j \neq k$ it follows that the relevant Kronecker indices are 1 if $j = k$ and 0 if $j \neq k$, which shows that the bases are dual to each other. By our previous discussions, we know that $\pi_1^*\mu \cup \pi_2^*\mu$ generates $H^2(T^2)$.

Let $U \in H_2(T^2)$ be a generator, and let τ be the transposition (or twist) map of T^2 sending (z, w) to (w, z). We claim that $\tau_* U = -U$. One way to see this is to consider the attaching map $p : (I \times I, \text{boundary}) \to (T^2, 1-\text{skel.})$ sending (s,t) to $(\exp 2\pi\, i\, s, \exp 2\pi\, i\, t)$. If τ' denotes the corresponding twist map on $I \times I$, then it follows that $p \circ \tau' = \tau \circ p$; since the map from $H_2(T^2)$ to $H_2(T^2, 2-\text{skel.})$ is an isomorphism, the effect of τ_* on $H_2(T^2)$ is completely determined by the effect of τ'_* on $H_2(I \times I, \text{boundary})$. We can compute this directly using a simplicial decomposition of $I \times I$ formed by splitting the latter into two solid triangles along the diagonal, and if we do so we see that the map induced by τ' on relative 2-dimensional cohomology is just multiplication by -1 as asserted in the statement of the proposition.

We shall now apply all this to computing the reverse order cup product $\pi_2^*\mu \cup \pi_1^*\mu$. Since $\pi_1 \circ \tau = \pi_2$ and $\pi_2 \circ \tau = \pi_1$ we have

$$\pi_2^*\mu \cup \pi_1^*\mu \quad = \quad \tau^*\pi_1^*\mu \cup \tau^*\pi_2^*\mu \quad = \quad \tau^*(\pi_1^*\mu \cup \pi_2^*\mu) \ .$$

Now τ_* is multiplication by -1 on 2-dimensional homology, and hence its dual space map, which is τ^*, must be multiplication by -1 on 2-dimensional cohomology. If we combine this with the preceding equations, we find that $\pi_2^*\mu \cup \pi_1^*\mu = -\pi_1^*\mu \cup \pi_2^*\mu$ as claimed.∎

THE GENERAL CASE. Suppose now that we have an arbitrary torus T^n, and as before let pi_i denote projection onto the i^{th} factor, so that we want to see what happens if we compute $\pi_j^*\mu \cup \pi_k^*\mu$ where $j > k$. Let S_{jk} denote the symmetry of T^n given by interchanging the j and k coordinates. Then as in the case $n = 2$ we have

$$\pi_j^*\mu \cup \pi_k^*\mu \quad = \quad S_{jk}^* \left(\pi_j^*\mu \cup \pi_k^*\mu\right) \ .$$

If $j = k + 1$ then we can use the case $n = 2$ and naturality considerations to conclude that the right hand side is equal to $-\pi_k^*\mu \cup \pi_j^*\mu$. More generally, we know that S_{jk} can be written as a composite of an odd number of transpositions S_{uv} where u and v satisfy $u = v + 1$, and if we use such a factorization we can conclude that $\pi_j^*\mu \cup \pi_k^*\mu = -\pi_k^*\mu \cup \pi_j^*\mu$ in all cases. — To summarize, we have shown the following:

THEOREM 6. *The cohomology algebra $H^*(T^n)$ is isomorphic to the exterior algebra $\wedge^*(\mathbb{F}^n)$ such that $H^q(T^n)$ corresponds to $\wedge^q(\mathbb{F}^n)$ and the cup product corresponds to the wedge product.*∎

GRADE-COMMUTATIVITY OF THE CUP PRODUCT. Theorem 6 implies that the cup product in $H^*(T^n)$ is not commutative but satisfies the following related property which is often called **grade-commutativity**:

> *If a is a p-dimensional cohomology class and b is a q-dimensional cohomology class, then $b \cup a = (-1)^{pq}\, a \cup b$.*

This result is stated as Theorem 3.14 on page 215 of Hatcher. Its proof requires some additional digressions, and one argument is worked out on pages 215–217 of Hatcher. More systematic approaches to the study of products in singular homology and cohomology are given in the following references:

Chapter 3 of Davis and Kirk, *Lecture Notes in Algebraic Topology*.
Chapter VII of Dold, *Lectures on Algebraic Topology (Second Edition)*.
Chapers 5–7 of Munkres, *Elements of Algebraic Topology*.
Chapter 12 of Rotman, *Introduction to Algebraic Topology*.
Chapter 5 of Spanier, *Algebraic Topology*.
Chapter 13 of Switzer, *Algebraic Topology — Homology and Homotopy*.
Chapter 5 of Vick, *Homology Theory — An Introduction to Algebraic Topology*.

More general products with spheres

The preceding discussion can be generalized and iterated to yield information on the cup product structure of arbitrary product spaces of the form $T^n \times Y$. The statement of the result will be simpler if we introduce the following algebraic construction:

Definition. Suppose that A_* and B_* are graded modules over a commutative ring with unit $\mathbb{D}$ (in other words, each is a sequence of modules indexed by the integers). Then their graded tensor product $C_* = A_* \otimes_{\mathbb{D}} B_*$ is given by

$$C_n \quad = \quad \bigoplus_{p+q=n} A_p \otimes_{\mathbb{D}} B_q \ .$$

In most cases of interest at this level, the modules A_p and B_q are zero for $p, q < 0$, and in such cases it follows that C_n will also be zero if $n < 0$.

In the remainder of this discussion, we shall assume that our ground ring and coefficients are a field $\mathbb{F}$.

THEOREM 8. *If $n > 0$ and Y is an arbitrary nonempty space, then the cohomology algebra $H^*(T^n \times Y)$ is $\mathbb{F}$-linearly isomorphic to $H^*(T^n) \otimes H^*(Y)$. This isomorphism is natural with respect to Y, and it has the following multiplicative property: If ρ_1 and ρ_2 are projections onto T^n and Y respectively, then for each ordered triple $(a, b, c) \in H^*(T^n) \times H^*(T^n \times Y) \times H^*(Y)$ the class $a \otimes b \otimes c$ corresponds to the cup product $\rho_1^*(a) \cup b \cup \rho_2^*(c)$.*

We have already essentially proved this in the case $n = 1$. Proving the general result by induction on n is fairly straightforward with the tools we have developed, but it is somewhat lengthy and will be left to the reader as an exercise. If we combine this result with the grade-commutativity property described earlier, we can given a complete description of the cup product structure on $H^*(T^n \times Y)$ in terms of $H^*(Y)$.■

If we wish to describe the identity in Theorem 8 abstractly, we can say that $H^*(T^n \times Y)$ is a left $H^*(T^n)$-module and a right $H^*(Y)$-module with mixed associativity (in other words, a *bimodule*).

Similar results hold if T^n is replaced by an arbitrary product of spheres; we shall restrict attention to the case where the product has only one factor.

THEOREM 9. *If $n > 0$ and Y is an arbitrary nonempty space, then there are natural isomorphisms*

$$H^k(S^n \times Y) \ \cong H^k(Y) \ \oplus \ H^{k-n}(Y)$$

such that the first summand is the image of $H^k(Y)$ under the homomorphism π_Y^* induced by the coordinate projection $\pi_Y : S^n \times Y \to Y$. The second summand is isomorphic to the image of the mapping $H^k(S^n \times Y, \{\text{point}\} \times Y) \to H^k(S^n \times Y)$ and is given by taking the cross product with a generator Ω_n of $H^n(S^n, \{\text{point}\}) \cong \mathbb{F}$.

Proof. We shall begin by proving the existence of the isomorphism. Once again, this is known for $n = 1$, so we proceed by induction on n, assuming the result is known for $n - 1$. As before, let $W_\pm$ denote the complements of the north and south poles, so that $W_\pm$ is homeomorphic to $\mathbb{R}^n$, $W_+ \cup W_- = S^n$, and $W^+ \cap W_- \cong S^{n-1} \times \mathbb{R}$. One can then derive the result for S^n using the Mayer-Vietoris cohomology sequence for the decomposition $S^n = W_+ \cup W_-$ and the validity of the result for S^{n-1} (note that $W_\pm \times Y$ is homotopy equivalent to Y and $(W_+ \cap W_-) \times Y$ is homotopy equivalent to $S^{n-1} \times Y$). It follows from the construction and induction hypotheses that one can take the first summand to be the image of $H^*(Y)$ in the product.

Proving the statement about cup products is similar but requires a little more attention to details, and it resembles the usual inductive proof that $H^n(S^n) \cong H^{n-1}(S^{n-1})$. We need to prove that the cross product defines suspension isomorphisms $H^k(S^n \times Y, \{\text{point}\} \times Y) \to H^k(S^n \times Y)$, and by the preceding discussion we know this is true when $n = 1$, so we shall assume inductively that the conclusion is true for $n - 1 \geq 1$.

Consider the following commutative diagram, in which the maps from the object at the upper left are cross products with suitably related generators of $H^{n-1}(S^{n-1},\{\text{pt.}\}) \cong \mathbb{F}$ and $H^n(S^n,\{\text{pt.}\}) \cong \mathbb{F}$:

$$\begin{array}{ccccc}
H^{k-n}(Y) & \longrightarrow & H^{k-1}(S^{n-1}\times Y,\{\text{pt.}\}\times Y) & = & H^{k-1}(S^{n-1}\times Y,\{\text{pt.}\}\times Y) \\
\downarrow & & & & \downarrow \\
H^k(S^n\times Y,\{\text{pt.}\}\times Y) & \longrightarrow & H^k(S^n\times Y, D^n_-\times Y) & \longrightarrow & H^k(D^n_+\times Y, S^{n-1}\times Y)
\end{array}$$

If Y is a point then this diagram gives the suspension isomorphism in cohomology mentioned above. By excision, homotopy invariance and exactness it follows that all the morphisms in the diagram other than the two maps coming from $H^{k-n}(Y)$ are isomorphisms. Now the induction hypothesis implies that the map from $H^{k-n}(Y)$ to $H^{k-1}(S^{n-1}\times Y,\{\text{pt.}\}\times Y)$ is an isomorphism, and therefore it follows from the diagram that the map from $H^{k-n}(Y)$ to $H^k(S^n\times Y,\{\text{pt.}\}\times Y)$ must also be an isomorphism.■

More generally, we have the following basic fact which is verified in Sections 3.2 and 3.B of Hatcher (for example, see Theorem 3.16 on page 219). Once again, the proof of this result requires a deeper study of products in singular theory than we can give in the present course.

THEOREM 10. *Suppose that X and Y are (finite) cell complexes and that $a \in H^p(X)$ and $b \in H^q(Y)$ are nonzero. Then their cross product $a \times b$ is a nonzero element of $H^{p+q}(X\times Y)$.*■

NOTE. The result in Hatcher is formulated for coefficients in more general rings and assumes that either $H^*(X)$ or $H^*(Y)$ is a free graded module over the coefficient ring. If we assume the coefficient ring is a field, then the freeness condition is automatic.

VI.4 : Two applications

(Hatcher, §§ 3.2, 4.2)

Although we have only obtained relatively weak versions of the basic results on products in singular homology and cohomology theory, they suffice to yield two fairly significant results. One is a restriction on the maps in homology associated to a homotopy self-equivalence from $S^{2m}\times S^{2m}$ to itself, and the other is a proof that for all $m > 1$ there is a continuous mapping from S^{4m-1} to S^{2m} which is not homotopic to a constant. The existence of such maps reflects several of the fundamental difficulties one encounters when trying to study homotopy theory.

Cell decompositions for products of spheres

Let n be a positive integer, and let $\mathbb{D}$ be a commutative ring with unit. If we take the simplest cell decomposition for S^n with a 0-cell and an n-cell, then the product construction yields a cell decomposition of $S^n\times S^n$ with one 0-cell, two n-cells and one $2n$-cell. If $n \geq 2$ then there are no possible nonzero differentials in the cellular chain complex for computing $H_*(S^n\times S^n;\mathbb{D})$ and hence one can read off the homology immediately. If $\sigma \in H_n(S^n;\mathbb{D}) \cong \mathbb{D}$ is a generator and i_1, i_2 are the usual slice inclusions, then the classes $i_{1*}\sigma$ and $i_{2*}\sigma$ form a free basis for $H_n(S^n\times S^n;\mathbb{D})$. The top cell of this complex is attached to the n-skeleton, which is a wedge of two copies of S^n by a continuous map

$$P : S^{2n-1} \longrightarrow S^n \vee S^n$$

that we shall call the *universal Whitehead product.*

Let n be as in the preceding paragraph, and let $PT^n \subset T^n$ denote the $(n-1)$-skeleton with respect to the standard cell decomposition of T^n described earlier. Then the quotient space T^n/PT^n is homeomorphic to S^n; let $\kappa : T^n \to S^n$ denote the associated collapsing map. It follows that κ_* and κ^* induce isomorphisms in n-dimensional homology and cohomology (say with field coefficients in the second case). Furthermore, it follows that $\kappa \times \kappa : T^n \times T^n \to S^n \times S^n$ induces a monomorphism in cohomology; verifying this is a fairly straightforward exercise using the corresponding property of κ^*, the known structure of $H^*(S^n \times S^n)$, and the known structure of $H^*(T^n \times F^n)$.

The preceding discussion reduces the computation of the cohomology cup product for $S^n \times S^n$ to questions about the corresponding structure for $T^{2n} = T^n \times T^n$. Here is a formal statement of the conclusions:

PROPOSITION 1. *Let $\Omega \in H^n(S^n)$ be such that the Kronecker index $\langle \Omega, \sigma \rangle = 1$, and let π_1, π_2 denote the projections of $S^n \times S^n$ onto the factors. Then the cohomology classes $\pi_j^* \Omega$ are dual to the homology classes $i_{j*}\sigma$ with respect to the Kronecker index pairing, and these classes satisfy the following conditions:*

(*i*) *Their cup squares are zero.*

(*ii*) *The class $\pi_1^*\Omega \cup \pi_2^*\Omega$ generates $H^{2n}(S^n \times S^n)$.*

(*iii*) *We have the grade-commutative relationship $\pi_2^*\Omega \cup \pi_1^*\Omega \;=\; (-1)^n \pi_1^*\Omega \cup \pi_2^*\Omega$. — In particular, the cup product is commutative if n is even.*

To prove this result, it suffices to look at the image of the cohomology in $H^*(T^{2n})$. If we let $\theta_j \in H^1(T^{2n})$ be dual to the standard basis of $H_1(T^{2n})$ given by slice inclusions of embedded circles, then it follows that $(\kappa \times \kappa)^*$ maps $\pi_1^*\Omega$ and $\pi_2^*\Omega$ to the cup products of $\theta_1, \cdots, \theta_n$ and $\theta_{n+1}, \cdots, \theta_2 n$ respectively. One can read off all the conclusions in the theorem from these identities and the previously determined structure of $H^*(T^{2n})$.■

These computations lead directly to our first application.

THEOREM 2. *Suppose that $m \geq 1$ and f is a homotopy self-equivalence of $S^{2m} \times S^{2m}$. Let σ_1 and σ_2 denote the free basis for $H_{2m}(S^{2m} \times S^{2m}; \mathbb{Z})$ described earlier. Then either the associated map in homology f_* sends the σ_j to $\varepsilon_j \sigma_j$, where $\varepsilon_j = \pm 1$, or else f_* sends σ_1 to $\varepsilon_1 \sigma_2$ and sends σ_2 to $\varepsilon_1 \sigma_1$ where again $\varepsilon_j = \pm 1$.*

All of the possibilities in the theorem can be realized. For the first alternatives this can be done by considering various product of the form 1, $1 \times \rho$, $\rho \times 1$ and $\rho \times \rho$, where ρ is the reflection involution on a sphere, and the second alternatives can be realized by composing the first alternatives with the transposition map τ on $S^{2m} \times S^{2m}$.

Suppose now that n is an arbitrary positive integer. Since $H_n(S^n \times S^n; \mathbb{Z}) \cong \mathbb{Z}^2$, the only general algebraic restriction one can get on a map f_* induced by a homotopy self-equivalence is that it must correspond to a 2×2 matrix over the integers with determinant equal to ± 1. It is fairly simply to construct examples of homotopy self equivalences of T^2 which realize every such matrix (the associated linear transformations of $\mathbb{R}^2$ pass to homeomorphisms of T^2). If n is odd, then the possible 2×2 matrices are also understood, but this is a deeper result; the conclusion is that one can realize every matrix if $n = 1, 3, 7$, while for the remaining odd values of n it is possible to realize every integral 2×2 matrix with determinant ± 1 whose reduction mod 2 is a permutation matrix. For the exceptional odd values of n, one can show this using standard "multiplications" on S^n (complex, quaternionic, and Cayley number multiplication respectively). For the remaining odd values of n, this fact is due to J. F. Adams and was proved in the nineteen fifties.

Proof of Theorem 2. As noted in the preceding paragraph, if σ_1 and σ_2 are the given standard free basis for $H_{2m}(S^{2m} \times S^{2m}; \mathbb{Z}) \cong \mathbb{Z}^2$, then there are integers a, b, c, d such that $ad - bc = \pm 1$ and $f_*(\sigma_1) = a\sigma_1 + b\sigma_2$, $f_*(\sigma_2) = c\sigma_1 + d\sigma_2$. By the naturality of homology with respect to coefficient homomorphisms, it follows that one has a similar description of f_* with rational coefficients. If we tak the dual basis ξ_1, ξ_2 of $H^{2m}(S^{2m} \times S^{2m}; \mathbb{Q})$, then it follows that $f^*\xi_1 = a\xi_1 + c\xi_2$ and $f^*\xi_2 = b\xi_1 + d\xi_2$. Since f preserves cup products and $\xi_j^2 = 0$, the same is true for $f^*(\xi_j)$. But Proposition 1 implies that

$$f^*(\xi_1)^2 \ = \ 2ac\,\xi_1 \cup \xi_2 \ , \qquad f^*(\xi_2)^2 \ = \ 2bd\,\xi_1 \cup \xi_2$$

and since $\xi_1 \cup \xi_2$ is nonzero it follows that $ac = bd = 0$, so that either $a = 0$ or $c = 0$ and also either $b = 0$ and $d = 0$. The cases $a = b = 0$ and $c = d = 0$ both imply that $ad - bc = 0$, so neither can hold, and therefore the only possibilities are $a = d = 0$ or $c = b = 0$. In the first case the condition $ad - bc$ implies that $b, c \in \{\pm 1\}$, while in the second case we must have $a, d \in \{\pm 1\}$. These are precisely the options listed in the theorem.■

Homotopically nontrivial mappings of spheres

If $m < n$ then simplicial approximation implies that every continuous mapping from S^m to S^n is homotopically trivial, and if $m = n$ we know that there are infinitely many homotopy classes of maps $S^n \to S^n$ which can be distinguished homotopically by their degrees; we have not proved that two maps of the same degree are homotopic, but it would not be exceedingly difficult for us to do so at this point (for example, see the argument in Maunder, *Algebraic Topology*, pages 288–291; the statement of this result in Hatcher is Corollary 4.25 on page 361). The important point is that if $m \leq n$, then homotopy classes of maps from S^m to S^n can be distinguished using homology theory. Given that every map from S^m to S^1 is nullhomotopic if $m > 1$, it was natural to hope that all maps $S^m \to S^n$ would be homotopic to constant maps. However, counterexamples began to surface during the nineteen thirties, and describing the homotopy classes of mappings from S^{n+k} to S^n where $k > 0$ turns out to be an exceedingly difficult problem, although it is known that the answer for any specific choice of n and k is finitely computable. We shall limit ourselves to a single class of important examples:

THEOREM 3. *Suppose that m is a positive integer. Then there is a continuous mapping $f : S^{4m-1} \to S^{2m}$ which is not homotopic to a constant.*

In fact, refinements of our methods show that there are infinitely many distinct homotopy classes of such maps. There is actually a very striking converse to this result due to J.-P. Serre: *For all $m, n > 0$, there are only finitely many homotopy classes of continuous mappings from S^n to S^m* **unless** *$m = n$ or m is even and $n = 2m - 1$.*

Proof. Throughout this discussion **the coefficient field will be the rational numbers $\mathbb{Q}$.**

The examples will be composites of the form $\nabla \circ P$, where $P : S^{4m-1} \to S^{2m} \vee S^{2m}$ is the universal Whitehead product described earlier and $\nabla : S^{2m} \vee S^{2m} \to S^{2m}$ folds the two wedge summands together (so its restriction to each summand is the identity). This class is generally known as the *Whitehead product* of the identity map on S^{2m} with itself and denoted by $[\iota_{2m}, \iota_{2m}]$ (compare Hatcher, Example 4.52, page 381). The argument wll require the following relatively elementary observation:

LEMMA 4. *Suppose that $f : S^{p-1} \to A$ is a continuous map into a compact metric space and X is the space obtained by attaching a p-cell to A along f. If f is homotopic to a constant map, then the inclusion of A in X is a retract.*

Proof of Lemma 4. If f is homotopic to a constant, then f extends to a mapping $g : D^p \to A$. Write $X = A \cup E$, where E is the p-cell. Then the retraction from X to A is defined by taking the identity on A and using g to define the mapping on E. By construction, it follows that these definitions fit together to yield a well-defined continuous retraction from X to A.■

Returning to the proof of Theorem 3, let $K(f)$ be the space obtained by adjoining a $4m$-cell to S^{2m} along the mapping $\nabla \circ P$. We then have the following commutative diagram, in which the two horizontal arrows on the left are attaching maps, the middle horizontal arrows are inclusions, and the horizontal arrows on the right are maps which collapse the codomains of the attaching maps to points.

$$\begin{array}{ccccccc}
S^{4m-1} & \stackrel{P}{\longrightarrow} & S^{2m} \vee S^{2m} & \longrightarrow & S^{2m} \times S^{2m} & \longrightarrow & S^{4m} \\
\Big\downarrow = & & \Big\downarrow \nabla & & \Big\downarrow h & & \Big\downarrow = \\
S^{4m-1} & \stackrel{\nabla P}{\longrightarrow} & S^{2m} & \longrightarrow & K(f) & \longrightarrow & S^{4m}
\end{array}$$

This diagram yields the following commutative diagrams in cohomology for each $q > 0$; the rows of these diagrams are short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^q(S^{4m}) & \longrightarrow & H^q(K(f)) & \longrightarrow & H^q(S^{2m}) & \longrightarrow & 0 \\
 & & \Big\downarrow = & & \Big\downarrow h^* & & \Big\downarrow \nabla^* & & \\
0 & \longrightarrow & H^q(S^{4m}) & \longrightarrow & H^q(S^{2m} \times S^{2m}) & \longrightarrow & H^q(S^{2m} \vee S^{2m}) & \longrightarrow & 0
\end{array}$$

It follows that $H^*(K(f))$ is isomorphic to $\mathbb{Q}$ in dimensions $0, 2m, 4m$ and is trivial otherwise. Let θ denote a generator for $H^{2m}\big(K(f)\big)$. It follows that $h^*(\theta)$ is a nonzero multiple of $\xi_1 + \xi_2$, and we might as well choose θ so that it maps to this class in $H^{2m}(S^{2m} \times S^{2m})$. Furthermore, we have

$$h^*(\theta)^2 \quad = \quad 2\,\xi_1 \cup \xi_2 \quad \neq 0$$

so that θ^2 must also be nonzero in $H^{4m}\big(K(f)\big)$.

We claim that the statement in the preceding sentence implies that f cannot be nullhomotopic. If it were, then there would be a retraction $\rho : K(f) \to S^{2m}$, and θ would have to be in the image of ρ^*. But if $\theta = \rho^*\theta_0$ for some $\theta_0 \in H^*(S^{2m})$, then $\theta_0^2 = 0$ and hence $\theta^2 = 0$, contradicting the conclusions in the preceding paragraph. Hence the only possibility consistent with the latter is that f is not nullhomotopic.■