The cellular chain complex for $\mathbb{R}P^n$

This is a more algebraic and detailed derivation for Example 2.42 on page 144 of Hatcher.

We start with a cell complex structure for S^n whose cells $e^k_\pm$ are the upper and lower hemispheres in S^k, where $0 \leq k \leq n$. Let $C_*(S^n)$ be the associated cellular chain complex whose homology is $H_*(S^n)$; if we restrict to $D^n_+ \subseteq S^n$ we get a cellular chain complex for $H_*(D^n_+)$ such that

if $k < n$, $C_k(D^n_+)$ is free abelian on generators $e^k_\pm$

$k = n$. $C_n(D^n_+)$ is free abel. on e^n_+.

Now let $T: S^n \to S^n$ be the antipodal map

$$T(v) = -v.$$

Then T yields an automorphism of the cell complex structure for S^n, sending e^k_+ to e^k_- and vice versa. We get the cellular chain complex $C_*(\mathbb{RP}^n)$ by taking the quotient $C_*(S^n)/x \sim T_\# x$, where $T_\#$ is the associated automorphism of the cellular chain group $C_*(S^n)$.

Assume our free generators $e^k_\pm$ are chosen so that $T_\#$ interchanges them.

<u>CLAIM</u> Up to sign, the differentials in $C_*(S^n)$ are given by

$$d_k e^k_+ = e^{k-1}_+ + (-1)^k e^{k-1}_-$$

$$d_k e^k_- = e^{k-1}_- + (-1)^k e^{k-1}_+$$

[Note that the first equation implies the second.]

PROOF OF CLAIM Go by induction on k. If $k=0$ the boundaries all vanish for dimensional reasons, and if $k=1$ we can check the identities directly. — Suppose we know the formula for $k-1 \geq 1$.

Let's look instead at $C_*(D^n_+)$; as noted above, it will suffice to compute de^k_+ in here since $C_*(D^n_+)$ is a chain subcomplex of $C_*(S^n)$ [D^n_+ is a topological subcomplex of S^n]. The advantage of doing this is that we know $H_{k-1}(D^n_+) = 0 = H_k(D^n_+)$ in advance.

In any case, we know that

$$de^k_+ = a\,e^{k-1}_+ + b\,e^{k-1}_-, \text{ some } a, b \in \mathbb{Z}.$$

To get some information on a and b, note that we must have $d(\text{R.H.S.}) = 0$

By induction $d(\text{RHS})$ equals

$$\left[a + (-1)^{k-1} b\right] e_+^{k-1} + \left[b + (-1)^{k-1} a\right] e_-^{k-1}.$$

Hence the cycles in $C_{k-1}(D_+^n)$ are all $a e_-^{k-1} + b e_-^{k-1}$ s.t. $a + (-1)^{k-1} b = 0$ or equivalently $b = (-1)^k a$. A generator is given by $\pm \left(e_+^{k-1} + (-1)^k e_-^{k-1}\right)$.

Since $H_{k-1}(D_+^k) = 0$, these generators lie in the boundary map's image. This happens $\Longleftrightarrow$ $d_k e_+^k$ is equal to $\pm \left(e_+^{k-1} + (-1)^k e_-^{k-1}\right)$. ■

Note that one obtains the same homology for either choice of sign, so we shall assume henceforth that

$$d e_+^k = e_+^{k-1} + (-1)^k e_-^{k-1}.$$

Application to $\mathbb{R}P^n$

The two generators $e^k_\pm$ for $C_*(S^n)$ collapse to the single generator e^k for $C_*(\mathbb{R}P^n)$, and if we take quotients and apply the formula on the preceding page we get

$$de^k = [1+(-1)^k]e^{k-1} = \begin{cases} 2e^{k-1} & k \text{ even} \\ 0 & k \text{ odd.} \end{cases}$$

As in Hatcher, this yields the following:

If n is odd then $H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } n \\ \mathbb{Z}_2 & \text{if } 1 \le k < n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$

If n is even then $H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}_2 & \text{if } 1 \le k < n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$