Computation of cellular homology

This is a more spread out version of the proof in the notes. As in the latter, we can use induction to reduce everything to showing that $H_q(X) = H_q(C_*(X, \mathcal{E}))$ for $q = n-1, n$ where $n = \dim X$.

The n-dimensional case

We have

$$\begin{tikzcd}
 & & & H_{n-1}(X_{n-2}) = 0 \arrow[d] \\
0 \arrow[r] & H_n(X_n) \arrow[r, "1-1", "j_*"'] & H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & H_{n-1}(X_{n-1}) \arrow[d, "J_*"] \\
 & & & H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd}$$

exact row + column.

By exactness $H_n(X) \cong \mathrm{Im}\, j_*$ equals Kernel ∂_n which $=$ Kernel $J_* d_n$ since J_* is 1-1

But RHS $=$ Ker d_n.

The [harder] $(n-1)$-dimensional case.

We now have the following diagram with exact rows and columns:

$$
\begin{tikzcd}
& 0 \arrow[d] \arrow[r, phantom, "= H_{n-1}(X_{n-2}) ="] & 0 \arrow[d] & \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & H_{n-1}(X_{n-1}) \arrow[r, "k_*"] \arrow[d, "j_*"] & H_{n-1}(X_n) \arrow[r] \arrow[d, "q_*"'] & 0 = H_{n-1}(X_n, X_{n-1}) \arrow[d] \\
& H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "p_*"] & H_{n-1}(X_n, X_{n-2}) \arrow[r] & 0 = \text{same}
\end{tikzcd}
$$

(so j_* + q_* are 1-1, k_* and p_* are onto).

As in the preceding discussion, $\mathrm{Im}\, j_* = \mathrm{Ker}\, d_{n-1}$.

Hence $\mathrm{Kernel}\, d_{n-1} / \mathrm{Image}\, d_n = \mathrm{Image}\, j_* / j_*[\mathrm{Im}\, \partial_n]$

$\cong H_{n-1}(X_{n-1}) / \mathrm{Image}\, \partial_n$ since j_* is 1-1.

But now the exactness of the top row $\Longrightarrow$ k_* induces an isomorphism from the latter quotient to $H_{n-1}(X_n)$, which is what we wanted to prove.