Connecting homomorphisms

Start with $0 \to A_* \to B_* \to B_*/A_* \to 0$

Define $\partial : H_q(B/A) \to H_{q-1}(A)$ as follows:

$u \in H_q(B/A)$ represented by $x \in B_q/A_q$ with $d^{B/A}x = 0$. Lift x to $\bar{X}$ in B_q, so $j\bar{X} = x$. Now

$$j\, d^B \bar{X} = d^{B/A} j \bar{X} = d^{B/A} x = 0, \text{ so}$$

$$d^B \bar{X} = i\bar{Y} \text{ for some } \bar{Y} \in A_{q-1}.$$

<u>CLAIM</u> $d^A \bar{Y} = 0$; since i is 1–1, it is enough to show $i d^A \bar{Y} = 0$. But

$$i d^A \bar{Y} = d^B i Y = d^B d^B \bar{X} = 0.$$

We want to set $\partial(u) = [Y] \in H_{q-1}(A)$.

Need to show ∂ is

(a) well-defined

(b) a module homomorphism

(c) natural in an appropriate sense.

(a) Show ∂ is well-defined

Suppose that x' also represents u, and suppose that X' is arbitrary such that $jX' = x'$. Then $x' - x = d^{B/A} z$ for some z. If W is such that $jW = z$, then

$$X' - X = d^B W + \text{something in Kernel } j$$
$$= d^B W + iU, \text{ some } U.$$

Now $dX' = iY'$, $dX = iY$ for some Y, Y' so that

$$iY' - iY = dX' - dX = \underset{0}{dd}W + diU$$
$$= idU, \text{ so}$$

$i(Y' - Y) = i(dU)$ and since i is 1-1 this means $Y' - Y = dU$, so that $[Y] = [Y']$ and the map is well defined.

(b) Show ∂ is a module homomorphism

Given $u_1 + u_2$, choose representatives $x_1 + x_2$, $X_1 + X_2$. Then the formula for $\partial(u_1 + u_2)$ yields $[Y_1 + Y_2] = [Y_1] + [Y_2] = \partial(u_1) + \partial(u_2)$, for

$u_1 + u_2 = [x_1 + x_2]$ and $x_1 + x_2 = j(\bar{X}_1 + \bar{X}_2)$.
Likewise, if r is in the ground ring R, then given $u \rightsquigarrow x \rightsquigarrow \bar{X}$, we have the choices

$$ru \rightsquigarrow rx \rightsquigarrow r\bar{X} \text{ and hence}$$

$$\partial(ru) = [r\bar{Y}] = r[\bar{Y}] = r\,\partial(u).$$

(c) Show ∂ is natural for diagrams

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r] \arrow[d, "f^A"] & B_* \arrow[r] \arrow[d, "f^B"] & B_*/A_* \arrow[r] \arrow[d, "f^{B/A}"] & 0 \\
0 \arrow[r] & A'_* \arrow[r] & B'_* \arrow[r] & B'_*/A'_* \arrow[r] & 0
\end{tikzcd}$$

$\partial f_*^{B/A}(u)$ is represented by choosing

$x \in B_q/A_q$, $d^{B/A}x = 0$

pick $\xi \in B'_q$ ~~$/A'_q$~~ s.t. $\xi \longrightarrow f^{B/A}(x)$

take $d\xi \in B'_{q-1}$, which lies in the image of A'_{q-1}.

This can be done by letting $\xi = f^B\bar{X}$, where $\bar{X} \longrightarrow x$, and then the class we get is $d^B f^B \bar{X}$.

On the other hand, $f_*^A \partial(u)$ is obtained by choosing $x \in B_q / A_q$, $X \longrightarrow x$, and taking $f(d'X)$, which lies in the image of A'_{q-1}. Since $df = fd$, it follows that we get the same class as before, and hence

$$f_*^A \circ \partial = \partial \circ f_*^{B/A}.$$

The long exact homology sequence

This sequence is described in Theorem III.4.3 of the notes with a reference to Hatcher for the proof. The method is frequently called "element chasing" or "diagram chasing." For the sake of completeness we shall give the details.

There are three things to prove:

[1] Exactness at $H_k(A)$
[2] Exactness at $H_k(B)$
[3] Exactness at $H_k(B/A)$.

Each proof has two parts.

PROOF OF [2]

$$H_k(A) \xrightarrow{i_*} H_k(B) \xrightarrow{j_*} H_k(B/A)$$

Kernel $j_* \supseteq$ Image i_*

We are given that $j \circ i = 0$, so in homology we have $0 = j_* \circ i_*$, which implies that Image $i_* \subseteq$ Kernel j_*.

Kernel $j_* \subseteq$ Image i_*

Let $u \in H_k(B)$ be such that $j_*(u) = 0$. Pick $x \in B_k$ such that $d^B x = 0$ and $u = [x]$. Then $j_* u = 0 \Rightarrow jx = d^{B/A} V$, some $V \in B_k/A_k$.

Pick $V' \in B_k$ so that $jV' = V$. How are x and dV' related?

$jx = dV = djV' = jdV'$, so
$j(x - dV') = 0$, so $x - dV' = i(y)$
for some $y \in A_k$. — Now $[x] = i_*[y]$
will follow if we can show that $d^A y = 0$.
But $0 = dy \Leftrightarrow 0 = i\,dy = di\,y$, and
the right hand side is just $d(x - dV') =$
$dx - ddV' = 0 - 0 = 0$, so $dy = 0$ as we
had wanted.

This completes the proof of exactness at $H_k(B)$.

PROOF OF [1] $H_{k+1}(B/A) \xrightarrow{\partial} H_k(A) \xrightarrow{i_*} H_k(B)$

Kernel $i_* \supseteq$ Image ∂

We need to show that $\partial i_* = 0$ in analogy with the proof of [2].

Let $x \in B_{k+1}/A_{k+1}$ be such that $d^{B/A}(x) = 0$, pick $X \in B_{k+1}$ so that $jX = x$, and let $Y \in A_k$ be such that $i(Y) = dX$, so that $\partial[x] = [Y]$. Then $i_*\partial[x] = i_*[Y] = [i(Y)] = [dX] = 0$.

Kernel $i_* \subseteq$ Image ∂

Suppose that $y \in A_k$ satisfies $dy = 0$; Since $i_*[y] = 0$, it follows that $i(y) = dX$ for some $X \in B_{k+1}$. Let $x = j(X) \in B_{k+1}/A_{k+1}$.

CLAIM $d^{B/A}(x) = 0$ and $[y] = \partial[x]$.

But $dx = djX = jdX = ji(y) = 0$, and since $jX = x$ and $dX = i(y)$ it follows that $[y] = \partial[x]$.

PROOF OF [3] $H_k(B) \xrightarrow{j_*} H_k(B/A) \xrightarrow{\partial} H_{k-1}(A)$

Kernel $\partial \supseteq$ Image j_*

As before, we need to show that $\partial \circ j_* = 0$.
Let $z \in B_k$ be such that $d^B(z) = 0$, so
that $j_*[z] = [j(z)]$. By definition,
$\partial[j(z)]$ is given by the class of dz since
$j(z) = z$; but $dz = 0$ implies that
$\partial j_*[z] = [dz] = 0$.

Kernel $\partial \subseteq$ Image j_*

Let $x \in B_k/A_k$ satisfy $d^{B/A}x = 0$.
Then $0 = \partial[x]$ amounts to saying that if we
let $j(X) = x$, then $dX = i\,dY$ for some
$Y \in A_k$.

Now $j(X - iY) = x$ since $j \circ i = 0$,
but $d(X - iY) = dX - idY = idY - idY = 0$.
Therefore $[x] = j_*[X - iY]$ and we are done.

The Five Lemma

This is Proposition III.4.5 in the notes:

$$
\begin{tikzcd}
A \arrow[r, "f_*"] \arrow[d, "a"] & B \arrow[r, "g_*"] \arrow[d, "b"] & C \arrow[r, "h_*"] \arrow[d, "c"] & D \arrow[r, "k_*"] \arrow[d, "d"] & E \arrow[d, "e"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "h'"] & E'
\end{tikzcd}
$$

(the stars are suppressed in the derivation)

Given: Both rows exact
a, b, d, e isomorphisms.

To prove: c is an isomorphism too.

As usual there are two parts — proving that c is 1-1 and proving that c is onto.

We should note that for each part, slightly weaker hypotheses on a, b, d, e will suffice (different weakenings for the separate parts!)

Proof that c is 1-1

Suppose $x \in C$ and $cx = 0$.

Then $0 = h'cx = dhx$, and since d is an isomorphism, we know that $h(x) = 0$ and ~~h~~ $x = gy$ for some $y \in B$.

But now $0 = cgy = g'by$, so $by = f'z$ for some $z \in A'$.

Since a is onto, we have $z = a(w)$ for some $w \in A$.

How are $f(w)$ and y related? $b(f(w) - y) =$

$f'a(w) - by = f'z - by = 0$. Hence $y = f(w)$.

Finally, we have $x = g(y) = gf(w)$, and this is zero because $g \circ f = 0$

Proof that c is onto

Suppose now that $x \in C'$. Since d is an isomorphism, $h'x = dy$ for some $y \in D$.

Now $0 = k'h'x = k'dy = eky$, and since

e is an isomorphism this implies that $ky = 0$, so that $y = hz$ for some $z \in C$.

How are x and cz related?

Apply h' to $x - cz$, obtaining
$h'x - h'cz = dy - dhz = dy - dy = 0$.

So $x - cz = g'(y)$ for some $y \in B'$.

But now $y = by'$ for some y since b is an isomorphism, and thus we have $x - cz = g'by' = cgy'$, so that $x = c(z + gy')$.

Combining these, we see that c is an isomorphism.