KRONECKER INDEX DUALITY

Setting: $\mathbb{F}$ is a field. If V is a vector space over $\mathbb{F}$ then $V^* = \mathrm{Lin}_{\mathbb{F}}(V, \mathbb{F})$. This extends to a contravariant functor by the following: Given $T: V \to W$ linear, let $T^*: W^* \to V^*$ be defined by $T^*f(v) = f(Tv)$.

Now suppose we have (over $\mathbb{F}$)

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

such that $d_n \circ d_{n+1} = 0$. As usual let

$$H_n = \mathrm{Ker}\, d_n / \mathrm{Im}\, d_{n+1}, \quad H^n = \mathrm{Ker}\, d_{n+1}^* / \mathrm{Im}\, d_n^*.$$

The Kronecker index pairing is the bilinear map

$$\kappa : H^n \otimes H_n \longrightarrow \mathbb{F}$$

defined by $\kappa([f] \otimes [a]) = f(a)$, where f and a are arbitrary representatives (checking this is well-defined can be done as in the course notes).

Main Theorem If $\kappa^\#: H^n \longrightarrow (H_n)^*$ is defined by $\underset{\in H^n}{\kappa^\#(u)}\, \underset{\in H_n}{[b]} = \kappa(u \otimes b)$

then $\kappa^\#$ is an isomorphism.

Direct proof that $\kappa^\#$ is onto

Given $\varphi : H_n \longrightarrow \mathbb{F}$ in H_n^*, consider the composite $\varphi' : Z_n(C) \xrightarrow[\text{ONTO}]{} H_n \longrightarrow \mathbb{F}$. (cycles) By definition $\varphi' | B_{n+1}(C) = 0$ (boundaries)

Extend φ' to all of C_n [note: this can be done even if $\mathbb{F}$ is just a principal ideal domain], call this map ψ. Then $\psi | B_{n+1}(C) = 0$ and thus $\psi = 0$ on image d_{n+1}, so that $\psi d_{n+1} = 0$ and hence ψ is a cocycle. By construction, if a is a cycle in C_n, then $\psi(a) = \kappa([\psi] \otimes [a])$, so $\varphi = \kappa^\#([\psi])$.

Direct proof that $\kappa^\#$ is 1-1

Suppose we are given cocycle $g : C_n \longrightarrow \mathbb{F}$, such that $g(a) = 0$ for all $a \in Z_n(C)$.* Then one has a comm. diagram

$$
\begin{tikzcd}
C_n \arrow[rr, "g"] \arrow[dr, "\theta"', "\text{onto}"] & & \mathbb{F} \\
& B_n(C) \cong C_n / Z_n(C) \arrow[ur, "\bar{g}"'] &
\end{tikzcd}
$$

θ comes from $d_n : C_n \to C_{n-1}$

$\boxed{\text{Extend } \bar{g} \text{ to } \bar{h} : C_{n-1} \longrightarrow \mathbb{F}.}$

[boxed statement not necessarily OK for PID, but is for a field!]

By construction, if $x \in C_n$ then $g(x) = \bar{g}(\theta x) = \bar{h}(d_n x) = (\bar{h} \circ d_n)(x)$. Hence $[g] = [\bar{h} d] = 0$ in H^n.

(*) Since $B_{n+1} \subseteq Z_n$, $g | Z_n = 0 \Longrightarrow g d_{n+1} = 0 \Longrightarrow g$ is a cocycle.