

# KRONECKER INDEX DUALITY

Setting:  $\mathbb{F}$  is a field. If  $V$  is a vector space over  $\mathbb{F}$  then  $V^* = \text{Lin}_{\mathbb{F}}(V, \mathbb{F})$ . This extends to a contravariant functor by the following:  
Given  $T: V \rightarrow W$  linear, let  $T^*: W^* \rightarrow V^*$  be defined by  $T^*f(v) = f(Tv)$ .

Now suppose we have (over  $\mathbb{F}$ )

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

such that  $d_n \circ d_{n+1} = 0$ . As usual let

$$H_n = \text{Ker } d_n / \text{Im } d_{n+1}, \quad H^n = \text{Ker } d_{n+1}^* / \text{Im } d_n^*$$

The Kronecker index pairing is the bilinear map

$$\kappa: H^n \otimes H_n \longrightarrow \mathbb{F}$$

defined by  $\kappa([f] \otimes [a]) = f(a)$ , where  $f$  and  $a$  are arbitrary representatives (checking this is well-defined can be done as in the course notes).

Main Theorem If  $\kappa^\#: H^n \rightarrow (H_n)^*$  is

$$\text{defined by } \kappa^\#(u) [b] = \kappa(u \otimes b)$$

$\begin{matrix} \circ \\ H^n & \in H_n \end{matrix}$

then  $\kappa^\#$  is an isomorphism.

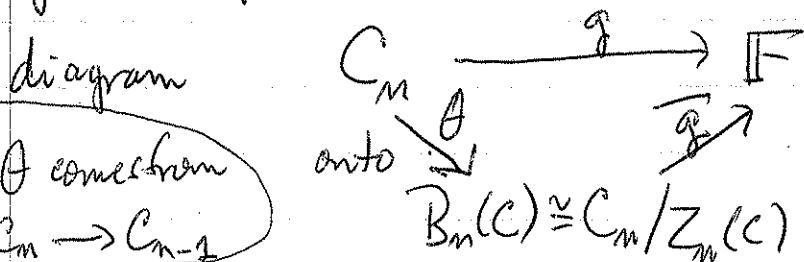
Direct proof that  $\tau^\#$  is onto

Given  $\varphi: H_m \rightarrow F$  in  $H_m^*$ , consider the composite  $\varphi': Z_m(C) \rightarrow H_m \rightarrow F$ . By definition,  $\varphi' \mid B_{m+1}(C) = 0$  (boundaries).  
 (cycles) ONTO

Extend  $\varphi'$  to all of  $C_m$  [note: this can be done even if  $F$  is just a principal ideal domain], call this map  $\psi$ . Then  $\psi \mid B_{m+1}(C) = 0$  and thus  $\psi = 0$  on  $\text{image } d_{m+1}$ , so that  $\psi d_{m+1} = 0$  and hence  $\psi$  is a cocycle. By construction, if  $a$  is a cycle in  $C_m$ , then  $\psi(a) = \tau([ \psi ] \otimes [ a ])$ , so  $\varphi = \tau^\#([ \psi ])$ .

Direct proof that  $\tau^\#$  is 1-1

Suppose we are given  $g: C_m \rightarrow F$  such that  $g(a) = 0$  for all  $a \in Z_m(C)$ . Then one has a comm. cocycle



Extend  $\bar{g}$  to  $\bar{h}: C_{m-1} \rightarrow F$ .

$\theta$  comes from  $d_m^*: C_m \rightarrow C_{m-1}$

[boxed statement not necessarily OK for PID, but is for a field!]

By construction, if  $x \in C_m$  then  $g(x) = \bar{g}(\theta x) = \bar{h}(d_m x) = (\bar{h} \circ d_m)(x)$ . Hence  $[g] = [\bar{h}d] = 0$  in  $H^m$ .

(\*) Since  $B_{m+1} \subset Z_m$ ,  $g \mid Z_m = 0 \Rightarrow g d_{m+1} = 0 \Rightarrow g$  is a cocycle.