EXERCISES FOR MATHEMATICS 246A

FALL 2010

Hatcher's book is the default source for references.

I. Foundational material

I.1 : Categories and functors

1. Definition. A morphism $f : A \to B$ in a category is a *monomorphism* if for all $g, h : C \to A$ we have that $f \circ h = f \circ g$ only if $h = g$. Dually, a morphism $f : A \to B$ in a category is an *epimorphism* if for all $u, v : B \to D$ we have that $u \circ f = v \circ f$ only if $u = v$.

(a) Prove that a monomorphism in the category **Set** is $1 - 1$ and an epimorphism in **Set** is onto. [*Hint:* Prove the contrapositives.]

(b) Prove that in the category of Hausdorff topological spaces (and continuous maps) a morphism $f : A \to B$ is an epimorphism if $f(A)$ is dense in B.

(c) Prove that the composite of two monomorphisms is a monomorphism and the composite of two epimorphisms is an epimorphism.

(d) A morphism $r : X \to Y$ in a category is called a *retract* if there is a morphism $q : Y \to X$ such that $qr = \mathrm{id}_X$. For example, in the category of sets or topological spaces the diagonal map $d_X : X \to X \times X$ is a retract with $q =$ projection onto either factor. Prove that every retract is a monomorphism.

(e) A morphism $p : A \to B$ in a category is called a *retraction* if there is a morphism $s : B \to A$ such that $q \circ r = \mathrm{id}_B$. For example, if r and q are as in (d) then q is a retraction. Prove that every retract is a monomorphism and every retraction is an epimorphism.

2. Let **A** be a category, and let $f : A \to B$ be a morphism in **A** such that

$$\mathsf{Morph}\,(f, C) : \mathsf{Morph}\,(B, C) \to \mathsf{Morph}\,(A, C)$$

is an isomorphism for all objects C in **A**. Prove that f is an isomorphism. [*Hint:* Choose $C = B$ or A and consider the preimages of the identity elements.] Also prove the (relatively straightforward) converse.

3. An object **0** is called an *initial object* in the category **A** if for each object A in **A** there is a unique morphism $\mathbf{0} \to A$. An object **1** is a *terminal object* in **A** if for each object A there is a unique morphism $A \to \mathbf{1}$.

(a) Prove that the empty set is initial and every one point set is terminal in **Set**.

(b) Prove that a zero-dimensional vector space is both initial and terminal in the category **Vec**–F of vector spaces over a field F.

(c) Prove that every two initial objects in a category $\mathbf{A}$ are uniquely isomorphic (there is a unique isomorphism from one to the other), and similarly for terminal objects.

(d) If $\mathbf{A}$ contains an object Z that is both initial and terminal (a *null object*), prove that for each pair of objects A, B in $\mathbf{A}$ there is a unique morphism $A \to B$ that factors as $A \to Z \to B$. Also, if W is any other such object, prove that this composite equals the composite $A \to W \to B$. [*Hint:* Consider the unique maps from W to Z and vice versa.]

4. Prove that a covariant functor takes retracts to retracts and retractions to retractions. State the corresponding result for contravariant functors.

5. If E is a terminal object in the category $\mathbf{A}$ and $f : E \to X$ is a morphism in $\mathbf{A}$, prove that f is a monomorphism (in fact, something stronger is true—what is it?).

6. Let $\mathbf{A} = (\mathbb{N}^+, \mathsf{Morph}, \varphi)$, where $\mathbb{N}^+$ denotes the positive integers, $\mathsf{Morph}\,(p, q)$ denotes all $p \times q$ matrices with integer coefficients, and

$$\varphi : \mathsf{Morph}\,(p, q) \times \mathsf{Morph}\,(q, r) \to m(p, r)$$

is matrix multiplication. Verify that $\mathbf{A}$ is a category.

7. If f is a morphism in a category $\mathbf{A}$, a morphism g (in the same category) is called a *quasi-inverse* for f if and only if $f \circ g \circ f = f$. Prove that every morphism that has a quasi-inverse is itself the quasi-inverse of some morphism in the category.

8. In the category of sets, show that the Axioms of Choice implies that every mapping has a quasi-inverse. Also, in the matrix category of Exercise 6, show that every matrix has a quasi-inverse. [*Hint:* Look at the associated linear transformations, and choose bases in a suitable manner.]

NOTE. In fact, there are canonical choices of quasi-inverses. See the following Wikipedia articles for further information on generalizations of matrix inverses:

`http://en.wikipedia.org/wiki/Moore-Penrose_inverse`

`http://en.wikipedia.org/wiki/Group_inverse`

`http://planetmath.org/encyclopedia/DrazinInverse.html`

9. Suppose that $\mathbf{C}$ is a category in which every map has a quasi-inverse. Prove that every monomorphism in $\mathbf{C}$ is a retract. Using this, give examples of mappings in the category of topological spaces (and continuous mappings) which do not have quasi-inverses.

10. Let $\mathbf{A}$ and $\mathbf{B}$ be small categories. Prove that one can define a *product category* $\mathbf{A} \times \mathbf{B}$ whose objects are given by ordered pairs (X, Y), where X and Y are objects of $\mathbf{A}$ and $\mathbf{B}$ respectively, whose morphisms are given by ordered pairs (f, g) of morphisms f in $\mathbf{A}$ and g in $\mathbf{B}$, and whose domain, codomain and composition operations are given as follows:

$$\begin{aligned} \mathrm{Domain}(f, g) &= \big(\mathrm{Domain}(f), \mathrm{Domain}(g)\big) \\ \mathrm{Codomain}(f, g) &= \big(\mathrm{Codomain}(f), \mathrm{Codomain}(g)\big) \\ (f_1, g_1) \circ (f_0, g_0) &= (f_1 \circ f_0, g_1 \circ g_0) \end{aligned}$$

Prove that $\mathbf{A} \times \mathbf{B}$ with these definitions of objects, morphisms, domains, codomains and composition forms a category, and show that "projections onto the first and second coordinates" define covariant functors from this category into $\mathbf{A}$ and $\mathbf{B}$ respectively.

11. Suppose that we are in a category $\mathbf{C}$ with morphisms $f : X \to Y$ and $g : Y \to Z$. Prove that if any two of f, g and $g \circ f$ are isomorphisms, then so is the third.

12. Let $\mathbf{IC_0}$ be the category whose objects are open intervals in the real line and whose morphisms are continuous mappings, and let $\mathbf{IC_1}$ be the subcategory with the same objects, but whose morphisms are maps with continuous first derivatives. Give an example of a morphism in $\mathbf{IC_1}$ which is an isomorphism in $\mathbf{IC_0}$ but not in $\mathbf{IC_1}$ (hence subcategories are not necessarily closed under taking inverses).

13. Let $\{X_\alpha\}$ be an indexed family of objects in a category $\mathbf{C}$. Then a *categorical product* of the X_α is given by an object P and morphisms $p_\alpha : P \to X_\alpha$ such that for each indexed family of maps f_α from a fixed object Y into the objects X_α, there is a unique $f : Y \to P$ such that $p_\alpha \circ f = f_\alpha$ for all α. — All the standard examples of product constructions turn out to have this property.

(a) Prove that if (P, p_α) and (Q, q_α) are categorical products, then there is a unique isomorphism $h : Q \to P$ such that $q_\alpha = p_\alpha \circ h$ for all α. [*Hint:* The only morphism φ from P to itself satisfying $p_\alpha = p_\alpha \circ \varphi$ for all α is the identity.]

(b) Formulate the dual notion of coproduct in a category (a product in the opposite category), and state the dual of the conclusion in (a).

(c) Show that the (external) direct sum is both a product and coproduct in $\mathbf{VEC}_{\mathbb{F}}$ for finite families of vector spaces, and show that the coproduct can be viewed as a proper subspace of the product for infinite families.

14. Let **FLD** be the category of (commutative) fields with morphisms given by field homomorphisms. Show that the category **FLD** does not have products. [*Hints:* Suppose we could construct a product $\mathbb{A}$ of the complex numbers with itself in this category, and consider the morphisms from $\mathbb{C}$ to itself given by the identity and complex conjugation. Recall that every homomorphism of fields is injective.]

15. Let **TOP** be the category of topological spaces and continuous mappings. Show that there is a *homotopy category* **HTP** whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps from one space to another. [*Hint:* The key thing to note is that one has identities and a decent well-defined notion of composition in **HTP**.]

16. We have mentioned that the reason for specifying codomains as part of the structure for morphisms is that functors to not necessarily preserve the injectivity of mappings. Illustrate this for the fundamental group functor $\pi_1(X, x)$ on pointed topological spaces by giving an example of a continuous map of pointed spaces $f : (X, x) \to (Y, y)$ such that f is injective but f_* is surjective and not injective, and also give an example of a continuous map of pointed spaces $f : (X, x) \to (Y, y)$ such that f is surjective but f_* is injective and not surjective.

I.2 : Barycentric coordinates and polyhedra

1. Suppose that P is a polyhedron which has a simplicial decomposition $\mathbf{K}$ with N vertices. Prove that P is homeomorphic to a subset of the simplex Δ_N such that the simplices in $\mathbf{K}$ correspond to sub-simplices of Δ_n.

2. Suppose that $(P, \mathbf{K})$ is a simplicial complex, and let $\mathbf{L}$ be a subcollection of $\mathbf{K}$ which is closed under taking faces. If Q is the union of all the simplices in $\mathbf{L}$, prove that $(Q, \mathbf{L})$ is a polyhderon.

> **Definition.** Let X be a metrizable topological space, let n be a nonnegative integer, and let $x \in X$. Then x is said to be an n-fold branch point of X if there is an open neighborhood base $U_1 \supset U_2 \cdots$ of x in X such that each U_k is connected, each deleted neighborhood $U_k - \{x\}$ has exactly n components, and if $m < k$ then the inclusion mappings $U_k - \{x\} \subset U_m - \{x\}$ induce 1–1 correspondences between the connected components of these spaces (hence different components of $U_k - \{x\}$ map to different components of $U_m - \{x\}$); see Example 5 on page 11 of the notes for the precise definition of the map of connected components associated to a continuous function $f : X \to Y$.

3. (*a*) Let X and x be as above. Explain why x is a 0-fold branch point of X if and only if x is isolated in X (in other words $\{x\}$ is open).

(*b*) Suppose that x is an n-fold branch point of X. Prove that for every sufficiently small open neighborhood V of x, the deleted neighborhood $V - \{x\}$ contains at least n connected components.

(*c*) Suppose that $(P, \mathbf{K})$ is a connected 1-dimensional polyhedron in some $\mathbb{R}^n$ such that every vertex of $\mathbf{K}$ is contained in a 1-simplex. Prove that for each $x \in P$ there is some positive integer n such that x is an n-fold branch point of P. [*Hint:* Why can we take $n = 2$ if x is not a vertex? If x is a vertex, then x lies on some finite number of 1-simplices.]

(*d*) Suppose that x is an n-fold branch point of X and $m \neq n$ is another nonnegative integer. Prove that x cannot be an m-fold branch point of X. [*Hint:* Use (*b*).]

(*e*) Use the preceding two parts of the exercise to show that if $(P, \mathbf{K})$ satisfies the conditions in (*c*) then for each $x \in X$ there is a unique positive integer n_x such that x is an n_x-fold branch point of x. Also, explain why the set $V_n(P)$ of n-fold branch points is finite if and only if $n \neq 2$.

(*f*) Let $X \subset \mathbb{R}^2$ be the union of the circles of radius $1/n$ centered at the points $(0, 1/n)$, where n is a positive integer. Show that there is no $n \geq 0$ such that n is an n-fold branch point of the origin. [*Hint:* For each $M > 0$ show that there is some open neighborhood U_M of $(0, 0)$ such that if $V \subset U_M$ then $V - \{x\}$ contains at least M components.]

4. (*a*) Suppose that $(P, \mathbf{K})$ and $(Q, \mathbf{L})$ are connected 1-dimensional polyhedra in some $\mathbb{R}^n$ such that every vertex in eiher polyhedron is contained in a 1-simplex, and let $f : P \to Q$ be a homeomorphism. Prove that for all positive integers n the map h sends $V_n(P)$ to $V_n(Q)$. In particular, show that if $n \neq 2$ then $V_n(P)$ and $V_n(Q)$ have the same numbers of elements and that $V_2(P)$ and $V_2(Q)$ have the same (finite) numbers of components.

(*b*) Using the notion of n-fold branch points, show that there are at least **7** homeomorphism types represented by the standard hexadecimal digits as written below (in sans-serif type):

0 1 2 3 4 5 6 7 8 9 A B C D E F

Are new homeomorphism types added if we consider the remaining letters of the alphabet? Explain.

(*c*) As noted in the next to last paragraph on page 358 of Munkres, the Figure 8 and Figure Theta spaces, corresponding to 8 and θ respectively, have the same homotopy type, but neither is a deformation retract of the other, and in fact neither is homeomorphic to a subspace of the other. Prove the last assertion in the preceding sentence. [*Hint:* Suppose more generally that we have 1-dimensional polyhedra P and Q such that P is homeomorphic to a subset of Q, and let $x \in P$. Modify earlier arguments to show that $V_n(X;P) \leq V_n(x,Q)$, and explain why this shows that the Figure Eight cannot be a subset of the Figure Theta and vice versa by describing the sets V_n(Figure Eight) and V_n(Figure Theta) for $n > 2$.]

5. Let $(P, \mathbf{K})$ be a 1-dimensional complex satisfying the conditions in previous exercises. Prove that $V_2(P)$ is an open subset with finitely many connected (equivalently, arc/path) components, prove that each of these components is homeomorphic to an open interval, and prove that the closure of each component is homeomorphic to a closed interval.

> **Note.** Using this result it is not difficult to prove the following statement, which is often called the **Hauptvermutung for 1-complexes**: *If* $(P, \mathbf{K})$ *and* $(Q, \mathbf{L})$ *are 1-dimensional simplicial complexes such that* P *and* Q *are homeomorphic, then there are* linear subdivisions *(as defined in the next section)* $\mathbf{K}_1$ *of* $\mathbf{K}$ *and* $\mathbf{L}_1$ *of* $\mathbf{L}$ *such that* $(P, \mathbf{K}_1)$ *and* $(Q, \mathbf{L}_1)$ *are isomorphic simplicial complexes.* — Although the proof is somewhat lengthy and inelegant, it can be done only using the methods and results described above. — The history of such statements dates back to at least 1908, when E. Steinitz and H. Tietze raised the question of whether this holds for polyhedra of arbitrary dimensions in connection with the constructions for simplicial homology groups in Unit III of these notes. Studies of the Hauptvermutung and related issues have had an enormous impact on geometric topology, and a fairly comprehensive bibliography is given on the Hauptvermutung website `http://www.maths.ed.ac.uk/~aar/haupt`; one other important reference is the following paper of E. M. Brown: *The Hauptvermutung for* 3*-complexes*, Transactions of the American Mathematical Society Vol. **144** (1969), 173–196. — To summarize known results, the Hauptvermutung is true for complexes of dimension ≤ 3 and false in all higher dimensions. In fact, for **every** simplicial complex $(P, \mathbf{K})$ of dimension ≥ 5, there is another complex $(Q, \mathbf{L})$ of the same dimension such that P and Q are homeomorphic but $\mathbf{K}$ and $\mathbf{L}$ do not have isomorphic subdivisions.

6. A simplicial complex $(P, \mathbf{K})$ is said to be a *star complex* if there is some vertex v of $\mathbf{K}$ such that every maximal simplex σ of $\mathbf{K}$ has v as one of its vertices. Prove that if $(P, \mathbf{K})$ is a star complex, then P is contractible (and in fact $\{v\}$ is a deformation retract of P.

I.3 : Subdivisions

1. Suppose that $(P, \mathbf{K})$ is a simplicial complex of dimension ≥ 1. Prove that P has infinitely many different simplicial decompositions, and in fact, if M is an arbitrary positive number then there is a simplicial decomposition of P with more than M vertices.

2. (*a*) Suppose that $(P, \mathbf{K})$ is a polyhedron and $(Q, \mathbf{L})$ is a subpolyhedron. If U is an open neighborhood of Q in P, prove at there is some $r > 0$ such that in the r^{th} barycentric subdivision, every simplex of $B^r(\mathbf{K})$ which contains points of Q is a subset of U.

(*b*) Using the preceding and the methods and results from Section II.9 of Eilenberg and Steenrod, prove that there is an open set V such that $Q \subset V \subset \overline{V}$ and Q is a strong deformation retract of both V and $\overline{V}$.

3. (*a*) Prove that $\mathbb{R}^n$ contains an infinite sequence of points such that any $n+1$ points in the set are affinely independent.

(*b*) Let A be a simplex with vertices v_i, and let $f : A \to \mathbb{R}^n$ be the affine-linear map

$$f\left(\sum_i t_i\, v_i\right) \quad = \quad \sum_i t_i\, w_i$$

for $w_i \in \mathbb{R}^n$. Prove that f is an isomorphism of simplices preserving barycentric coordinates if the vectors w_i are affinely independent.

(*c*) Using the preceding observations, prove that if $(P, \mathbf{K})$ is a simplicial complex of dimension n, then it is isomorphic to a polyhedron in $\mathbb{R}^{2n+1}$. — Later in this course we shall give examples of 1-dimensional complexes which cannot be even topologically embedded in $\mathbb{R}^2$.