Preface

Perhaps the simplest motivation for algebraic topology is the following basic question:

If m and n are distinct positive integers, is $\mathbb{R}^m$ ever homeomorphic to $\mathbb{R}^n$?

Results from point set topology imply the answer is **NO** if one of m and n is equal to 1. If a homeomorphism $h : \mathbb{R}^m \to \mathbb{R}$ existed then for each $\mathbf{x} \in \mathbb{R}^m$ we could conclude that $\mathbb{R}^n - \{\mathbf{x}\}$ is homeomorphic to $\mathbb{R} - \{h(\mathbf{x})\}$. Since $\mathbb{R}^m - \{\mathbf{x}\}$ is connected for all $\mathbf{x} \in \mathbb{R}$ if $m > 1$ while $\mathbb{R} - \{t\}$ is not connected for any choice of $t \in \mathbb{R}$, it follows that $\mathbb{R}^m - \{\mathbf{x}\}$ is never homeomorphic to $\mathbb{R} - \{t\}$ if $m > 1$ and hence $\mathbb{R}^m$ cannot be homeomorphic to $\mathbb{R}$. Similarly, results on fundamental groups imply that for all relevant choices of $\mathbf{x}$ the set $\mathbb{R}^m - \{\mathbf{x}\}$ is simply connected if $m > 2$ while $\mathbb{R}^2 - \{\mathbf{x}\}$ has an infinite cyclic fundamental group, so we also know that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^2$ provided $m > 2$. One basic goal of an introductory course in algebraic topology is to show that $\mathbb{R}^m$ is never homeomorphic to $\mathbb{R}^n$ if $m \neq n$.

The idea behind proving such results is to define certain abelian groups which give an **algebraic picture** of a given topological space; in particular, if two topological spaces are homeomorphic, then their associated groups will be algebraically isomorphic. Unfortunately, the definitions for these **homology groups** are less straightforward than the definition of the fundamental group, and much of the work in this course involves the construction of such groups and the proofs that they have good formal properties.

In analogy with standard results for fundamental groups, the homology groups of two spaces will be isomorphic if the spaces satisfy a condition that is somewhat weaker than the existence of a homeomorphism between them; namely, an the groups are isomorphic if the two spaces have the same *homotopy type* as defined on page 363 of the book by Munkres cited below.

Since the constructions for the associated groups are somewhat complicated, it is natural to expect that they should be useful for more than simply answering the homeomorphism question for Euclidean spaces. In particular, one might ask if these groups (and a course in algebraic topology) can shed new light on some questions left open in undergraduate or beginning graduate courses in mathematics.

1. The material in introductory graduate level courses does not really give much insight into the popular characterization of topology as a "rubber sheet geometry." In other words, topology is generally viewed as the study of properties that do not change under various sorts of bending or stretching operations. Some aspects of this already appear in the study of fundamental groups, and one objective of this course is to develop these ideas much further.

2. As a refinement of the problem at the beginning of this preface, one can ask if there is some topological criterion which characterizes the algebraic notion of n–dimensionality, at least for spaces that are relatively well-behaved.

3. An algebraic topology course should also yield better insight into several issues that arise in undergraduate courses, including (a) the Fundamental Theorem of Algebra, (b) various facts about planar and nonplanar networks, (c) insides and outsides of plane curves and closed surfaces in 3–dimensional space, and (d) Euler's Formula for "nice" polyhedra in $\mathbb{R}^3$; namely, if P is a polyhedron bounding a convex body in $\mathbb{R}^3$, then the numbers V, E and F of vertices, edges and faces satisfy the equation $E + 2 = V + F$.

4. Some of the basic results on the topology of the edge-path graphs in Mathematics 205B should be placed into a broader context. In particular, the notion of Euler characteristic should be extended to a larger class of spaces.

5. If time permits, another goal will be to give a unified approach to certain results in multivariable calculus involving the ∇ operator, Green's Theorem, Stokes' Theorem and the Divergence Theorem, and to formulate analogs for higher dimensions.

Throughout the course we shall use the following text for the basic graduate topology courses as a reference for many topics and definitions:

> **J. R. Munkres.** *Topology* (Second Edition), *Prentice-Hall, Saddle River NJ*, 2000. ISBN: 0–13–181629–2.

The official text for this course is the following book:

> **A. Hatcher.** *Algebraic Topology* (Third Paperback Printing), *Cambridge University Press, New York NY*, 2002. ISBN: 0–521–79540–0.

This book can be legally downloaded from the Internet at no cost for personal use, and here is the link to the online version:

`www.math.cornell.edu/~hatcher/AT/ATpage.html`

Here are four other references. The first is a book that has been used as a text in the past, the second is a fairly detailed history of the subject during its formative years, and the last two are classic (but not outdated) books; the first also has detailed historical notes.

> **J. W. Vick.** *Homology Theory*. (Second Edition). *Springer–Verlag, New York etc.*, 1994. ISBN: 3–540–94126–6.

> **J. Dieudonné.** *A History of Algebraic and Differential Topology* (1900 − 1960). *Birkhäuser Verlag, Zurich etc.*, 1989. ISBN: 0–817–63388–X.

> **S. Eilenberg and N. Steenrod.** *Foundations of Algebraic Topology*. (Second Edition). *Princeton University Press, Princeton NJ*, 1952. ISBN: 0–691–07965–X.

> **E. H. Spanier.** *Algebraic Topology*, *Springer–Verlag, New York etc.*, 1994.

The `amazon.com` sites for Hatcher's and Spanier's books also give numerous other texts in algebraic topology that may be useful. Finally, there are two other books by Munkres that we shall quote repeatedly throughout these notes. The first will be denoted by [MunkresEDT] and the second by [MunkresAT]; if we simply refer to "Munkres," it will be understood that we mean the previously cited book, *Topology* (Second Edition).

> **J. R. Munkres**. *Elementary differential topology.* (Lectures given at Massachusetts Institute of Technology, Fall, 1961. Revised edition. Annals of Mathematics Studies, No. 54.) *Princeton University Press, Princeton, NJ*, 1966. ISBN: 0–691–09093–9.

> **J. R. Munkres.** *Elements of Algebraic Topology*. *Addison-Wesley, Reading, MA*, 1984. (Reprinted by Westview Press, Boulder, CO, 1993.) ISBN: 0–201–62728–0.

One important feature of homology groups is that if $f : X \to Y$ is a continuous mapping of topological spaces, then there is an associated homomorphism f_* from the homology groups of X to the homology groups of Y; this is again similar to the situation for fundamental groups of pointed spaces, and it plays an important role in addressing the issues listed above. In fact, algebraic topology turns out to be an effective means for analyzing the following central problem:

> Given two "reasonably well-behaved" spaces X and Y, describe the homotopy classes of continuous mappings from X to Y.

In general, the descriptions of the homotopy classes can be quite complicated, and only a few cases of such problems can be handled using the methods of a first course, but we shall mention a few special cases at various points in the course.

Many of the basic properties of homology groups and homomorphisms are best stated using the formalisms of **Category Theory**, and many of the constructions and theorems in algebraic topology are best stated within the framework of **Homological Algebra**. We shall develop these subjects in the course to the extent that we need them.

Prerequisites

The name "algebraic topology" suggests that the subject uses input from both algebra and topology, and this is in fact the case; since topology began as a branch of geometry, it is also reasonable to expect that some geometric input is also required. Our purpose here is to summarize the main points from prerequisite courses that will be needed. Additional background material which is usually not covered explicitly in the prerequisites will be described in the first unit of these notes.

Set theory

Everything we shall need from set theory is contained in the following online directory:

`http://math.ucr.edu/~res/math144`

In particular, a fairly complete treatment is contained in the documents `setsnotes`n`.pdf`, where $1 \leq n \leq 8$.

There are two features of the preceding that are somewhat nonstandard. The first is the definition of a function from a set A to another set B. Generally this is given formally by the graph, which is a subset $G \subset A \times B$ such that for each $a \in A$ there is a unique $b \in B$ such that $(a, b) \in G$. Our definition of function will be a **triple** $f = (A, G, B)$, where $G \subset A \times B$ satisfies the condition in the preceding sentence. The reason for this is that we must specify the target set or **codomain** of the function explicitly; in fact, the need to specify the codomain has already arisen at least implicitly in prerequisite graduate topology courses, specifically in the definition of the fundamental group. A second nonstandard feature is the concept of **disjoint union** or **sum** of an indexed family $\{ X_\alpha \}$ of sets. The important features of the disjoint sum, which is written $\amalg_\alpha X_\alpha$, are that it is a union of subsets Y_α which are canonically in 1–1 correspondence with the sets X_α and that $Y_\alpha \cap Y_\beta = \emptyset$ if $\alpha \neq \beta$. Another source of information on such objects is Unit V of the online notes for Mathematics 205A which are cited below.

Topology

This course assumes familiarity with the basic material in graduate level topology courses through the theory of fundamental groups and covering spaces (in other words, the material in Mathematics 205A and 205B). Everything we need from the first of these courses can be found in the following online directory:

`http://math.ucr.edu/~res/math205A`

In particular, the files `gentopnotes2008.*` contain a fairly complete set of lecture notes for the course. This material is based upon the textbook by Munkres cited in the Preface. Two major differences between the notes and Munkres appear in Unit V. The discussion of quotient topologies is somewhat different from that of Munkres, and in analogy with the previously mentioned discussion of set-theoretic disjoint sums there is a corresponding construction of disjoint sum for an indexed family of topological spaces.

There is a similar directory for the second course, which deals with the basic notions of homotopy, fundamental groups and covering spaces:

`http://math.ucr.edu/~res/math205B`

There is no self-contained set of notes in this directory, but there are comments and (additional) exercises to supplement the text and indicate the sections which will be needed for this course. In addition to the sections mentioned in these references, it might be worthwhile to look also at the supplementary exercises for Chapter 13.

Chapter 14 of Munkres is not a required part of the material covered in the Department's qualifying exam, but it is sometimes covered in 205B and at some points in this course the topics covered in Chapter 14 are relevant, and we shall mention them when it seems appropriate or useful.

Algebra

As in the later parts of Munkres, we shall assume some familiarity with certain topics in group theory. Nearly everything we need is in Sections 67 – 69 of Munkres, but we shall also need the following basic result:

STRUCTURE THEOREM FOR FINITELY GENERATED ABELIAN GROUPS. *Let G be a finitely generated abelian group (so every element can be written as a monomial in integral powers of some finite subset $S \subset G$). Then G is isomorphic to a direct sum*

$$(H_1 \oplus \cdots \oplus H_b) \oplus (K_1 \oplus \cdots \oplus K_s)$$

where each H_i is infinite cyclic and each K_j is finite of order t_j such that t_{j+1} divides t_j for all j. — For the sake of uniformity set $t_j = 1$ if $j > s$. Then two direct sums as above which are given by $(b; t_1, \cdots)$ and $(b'; t'_1, \cdots)$ are isomorphic if and only if $b = b'$ and $t_j = t'_j$ for all j.

A proof of this fundamental algebraic result may be found in Sections II.1 and II.2 of the following standard graduate algebra textbook:

> **T. Hungerford.** *Algebra.* (Reprint of the 1974 original edition, Graduate Texts in Mathematics, No. 73.) *Springer-Verlag, New York–Berlin–etc.*, 1980. ISBN: 0–387–90518–9.

Material from standard undergraduate linear algebra courses will also be used as needed.

Analysis

We shall assume the basic material from an upper division undergraduate course in real variables as well as material from a lower division undergraduate course in multivariable calculus through the theorems of Green and Stokes as well as the 3-dimensional Divergence Theorem. The classic text by W. Rudin (*Principles of Mathematical Analysis*, Third Edition) is an excellent reference for real variables, and the following multivariable calculus text contains more information on the that subject than one can usually find in the usual 1500 page calculus texts (the book is not perfect, but especially at the graduate level it is useful as a background reference).

> **J. E. Marsden and A. J. Tromba.** *Vector Calculus* (Fifth Edition), *W. H. Freeman & Co., New York NY*, 2003. ISBN: 0–7147–4992–0.

I. Foundational and Geometric Background

Aside from the formal prerequisites, algebraic topology relies on some background material from other subjects that is generally not covered in prerequisites. In particular, two concepts from the foundations of mathematics, namely **categories** and **functors**, play a central role in formulating the basic concepts of algebraic topology. Furthermore, since algebraic topology places heavy emphasis on spaces that can be constructed from certain fundamental building blocks, some relatively elementary but fairly detailed properties of the latter are indispensable. The purpose of this unit is to develop enough of category theory so that we can use it to formulate things efficiently and to describe the topological and geometric properties of a class of well-behaved spaces called **polyhedra** that will be needed in the course.

I.1 : Categories and functors

(Hatcher, § 2.3)

If mathematics is the study of abstract systems, then category theory may be viewed as an abstract formal setting for working with such systems. In fact, the theory was originally developed by S. Eilenberg (1919–1998) and S. MacLane (1909–2005) in the 1940s to provide an effective conceptual framework for handling various constructions and phenomena related to algebraic topology (including some from the theory of groups). The formal definition may be viewed as a generalization of familiar properties of ordinary set-theoretic functions. There is a great deal of overlap between the discussion here and the file `categories.pdf` in the `math205A` directory. There is a classic book by P. Freyd (1936–), *Abelian Categories: An Introduction to the Theory of Functors*, which is still an extremely readable introduction to category theory and its role in abstract algebra, and it is available online at the following site:

`http://www.emis.de/journals/TAC/reprints/articles/3/tr3/pdf`

Definition. A CATEGORY is a system $\mathbf{C}$ consisting of

(a) a class $\mathsf{Obj}\,(\mathbf{C})$ of sets called the **objects** of $\mathbf{C}$,

(b) for each ordered pair of objects X and Y an associated set $\mathsf{Morph}_{\mathbf{C}}(X, Y)$ called the **morphisms** from X to Y,

(c) for each ordered triple of objects X, Y and Z, an associated map called a *composition pairing* $\varphi : \mathsf{Morph}_{\mathbf{C}}(X, Y) \times \mathsf{Morph}_{\mathbf{C}}(Y, Z) \longrightarrow \mathsf{Morph}_{\mathbf{C}}(X, Z)$, whose value for (f, g) is generally written $g \circ f$, such that the following hold:

(1) The sets $\mathsf{Morph}_{\mathbf{C}}(X, Y)$ and $\mathsf{Morph}_{\mathbf{C}}(Z, W)$ are disjoint unless $X = Z$ and $Y = W$.

(2) For each object X there is a unique identity morphism $1_X = \mathrm{id}_X \in \mathsf{Morph}_{\mathbf{C}}(X, X)$ such that for each $f \in \mathsf{Morph}_{\mathbf{C}}(X, Y)$ and $g \in \mathsf{Morph}_{\mathbf{C}}(Z, X)$ we have $f \circ 1_X = f$ and $1_X \circ g = g$.

(3) The composition pairings satisfy an associative law; *i.e.*, if $f \in \mathsf{Morph}_{\mathbf{C}}(X, Y)$, $g \in \mathsf{Morph}_{\mathbf{C}}(Y, Z)$, and $h \in \mathsf{Morph}_{\mathbf{C}}(Z, W)$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

By the assumptions, for each $f \in \mathsf{Morph}_{\mathbf{C}}(X,Y)$ the objects X and Y are uniquely determined, and they are called the **domain** and **codomain** of f respectively. When working within a given category we generally use familiar notation like $f : X \to Y$ to indicate that $f \in \mathsf{Morph}_{\mathbf{C}}(X,Y)$.

As in set theory, at some points one must take care to avoid difficulties with classes that are "too large" to be sets (for example, we cannot discuss the set of all sets), but in practice it is always possible to circumvent such problems by careful choices of definitions and wordings (for example, using the theory of *Grothendieck universes*), so we shall generally not dwell on such points.

Examples of categories

By the remarks preceding the definition of a category, it is clear that we have a category **SETS** whose objects are given by all sets, whose morphisms are set-theoretic functions from one set to another (with the conventions mentioned in the Prerequisites!), and whose composition is merely ordinary composition of mappings. Here are some further examples:

1. Given a field $\mathbb{F}$, there is the category $\mathbf{VEC}_{\mathbb{F}}$ whose objects are vector spaces, whose morphisms are $\mathbb{F}$-linear transformations, and whose composition is ordinary composition. The important facts here are that the identity on a vector space is a linear transformation, and the composite of two linear transformations is a linear transformation.

2. There is also a category **GRP** whose objects are groups and whose morphisms are group homomorphisms (with the usual composition). Once again, the crucial properties needed to check the axioms for a category are that identity maps are homomorphisms and the composite of two homomorphisms is a homomorphism.

3. Within the preceding example, there is the subcategory **ABGRP** whose objects are abelian groups, with the same morphisms and compositions. In this category, the set of morphisms from one object to another has a natural abelian group structure given by pointwise addition of functions, and the resulting abelian group of homomorphisms is generally denoted by $\mathrm{Hom}(X,Y)$.

4. More generally, if **C** is a category, then a subcategory **A** is a collection of morphisms and objects which is closed under (i) taking domains and codomains of objects, (ii) taking identity morphisms of objects, (iii) taking composites of morphisms. It is said to be a **full subcategory** if for each pair of objects X and Y in **A** we have $\mathsf{Morph}_{\mathbf{A}}(X,Y) = \mathsf{Morph}_{C}(X,Y)$. It follows that **ABGRP** is a full subcategory of **GRP**. On the other hand, if we let $\mathbf{GRP_{1-1}}$ be the category whose objects are groups and and whose morphisms are *injective* homomorphisms, then $\mathbf{GRP_{1-1}}$ is a subcategory of **GRP** but it is not a full subcategory.

5. If P is a partially ordered set with ordering relation $\leq$, then one has an associated category whose objects are the elements of P and such that $\mathsf{Morph}\,(x,y)$ consists of a single point if $x \leq y$ and is empty otherwise. This is an example of a **small** category in which the class of objects is a set.

6. One can also use partially ordered sets to define a category **POSETS** whose objects are partially ordered sets and whose morphisms are monotonically nondecreasing functions from one partially ordered set to another; as in most other cases, composition has its usual meaning.

7. If G is a group, then G also defines a small category as follows: There is exactly one object, the morphisms of this object to itself are given by the elements of G, and composition is given by the multiplication in G.

8. There is a category **TOP** whose objects are topological spaces, whose morphisms are continuous maps between topological spaces, and whose composition is the usual notion. Again, the crucial properties needed to verify the axioms for a category are that identity maps are continuous and composites of continuous maps are also continuous.

9. There are also categories whose objects are topological spaces and whose morphisms are **open** maps or **closed** maps. The categories with various types of morphisms are distinct. Of course, it is also possible to take combinations of such conditions and obtain structures like the category of spaces with *continuous open mappings* as the morphisms.

10. More generally, given any class of continuous mappings that is closed under taking identities and compositions, one can define a category of topological spaces with such maps as the morphisms. Two examples are maps that are **proper** (inverse images of compact subsets are compact) or **light** (inverse images of points are discrete sets; see Exercise II.3.4 in `gentopexercises2008.pdf` from the `math205A` directory for more on the latter).

11. One also has a category **MET–UNIF** whose objects are metric spaces and whose morphisms are **uniformly continuous** mappings (with the usual composition).

12. Similarly, there is the category **MET–LIP** whose objects are metric spaces and whose morphisms are **Lipschitz** mappings: *i.e.*, there is a constant M such that

$$d\big(\,f(x_1),\,f(x_2)\,\big) \quad \leq \quad M \cdot d(x,y)$$

for all x and y in the domain (such an inequality is called a *Lipschitz condition*). Standard results of (abstract) multivariable calculus show that if K is a compact convex set and $f : K \to \mathbb{R}^m$ extends to a function on an open neighborhood W of K whose coordinates have continuous first partial derivatives, then f satisfies a Lipschitz condition.

13. Still further in the same direction, there is the category **MET–ISO** whose objects are metric spaces and whose morphisms are **isometries** (but not necessarily surjective).

14. (A fundamentally important general construction.) Given an arbitrary category $\mathbf{C}$, one has the **dual** or **opposite** category $\mathbf{D} = \mathbf{C}^{\mathbf{OP}}$ with the same objects as $\mathbf{C}$, but with $\mathsf{Morph}_{\mathbf{D}}(X,Y) = \mathsf{Morph}_{\mathbf{C}}(Y,X)$ (note the reversal!) and the composition pairing $*$ defined by $g * f = f \circ g$. Note that if $\mathbf{D} = \mathbf{C}^{\mathbf{OP}}$ then $\mathbf{C} = \mathbf{D}^{\mathbf{OP}}$.

In most of the preceding examples of categories, there is a fundamental notion of **isomorphism**, and in fact one can formulate this abstractly for an arbitrary category:

Definition. Let $\mathbf{C}$ be a category, and let X and Y be objects of $\mathbf{C}$. A morphism $f : X \to Y$ is an *isomorphism* if there is a morphism $g : Y \to X$ (an inverse) such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

This generalizes notions like an invertible linear transformation, a group isomorphism, and a homeomorphism of topological spaces.

PROPOSITION 1. *Suppose that $f : X \to Y$ is an isomorphism in a category $\mathbf{C}$ and g and h are inverses to f. Then $h = g$.*

Proof. Consider the threefold composite $h \circ f \circ g$. Since $h \circ f = 1_X$, this is equal to g, and since $f \circ g = 1_Y$, it is also equal to h.■

Functors

The examples of categories illustrate a basic principle in modern mathematics: Whenever one defines a type of mathematical system, there is usually a corresponding type of morphism for such systems (and in some cases there are several reasonable choices for morphisms). Since a category is an example of a mathematical system, it is natural to ask whether there is a corresponding notion of morphisms in this case too. In fact, there are two concepts of morphism that turn out to be important and useful. We shall start with the simpler one.

Definition. Let **C** and **D** be categories. A *covariant functor* assigns (i) to each object X of **C** an object $T(X)$ of **D**, (ii) to each morphism $f : X \to Y$ in **C** a morphism $T(f) : T(X) \to T(Y)$ in **D** such that the following hold:

(1) For each object X in **C** we have $T(1_X) = 1_{T(X)}$.

(2) For each pair of morphisms f and g in **C** such that $g \circ f$ is defined, we have $T(g \circ f) = T(g) \circ T(f)$.

HISTORICAL TRIVIA. Eilenberg and MacLane "borrowed" the word **category** from the philosophical writings of the 18^{th} century German philosopher I. Kant (1724–1804) and the word **functor** from the philosophical writings of the 20^{th} century German-American philosopher R. Carnap (1891–1970), who was strongly influenced by Kant's writings on the philosophy of science.

Examples of covariant functors

Numerous constructions from undergraduate and elementary graduate courses can be interpreted as functors; in many cases this does not shed much additional light on the objects constructed, but in other cases the concept does turn out to be extremely useful.

1. Given a category **C**, there is the **identity functor** from **C** to itself, which takes all objects and morphisms to themselves.

2. Given a category **C** and a (possibly different) nonempty category **D**, for each object A of **D** there is a **constant functor** k_A from **C** to **D** which sends every object of **C** to A and every morphism to the identity morphism 1_A.

3. In categories where the objects are given by sets with some extra structure and the morphisms are ordinary functions with additional properties, there are **forgetful functors** which take objects to the underlying sets and morphisms to the underlying mappings of sets. For example, there are forgetful functors from $\mathbf{VEC}_{\mathbb{F}}$, **GRP**, **POSETS**, and **TOP** to **SETS**. Likewise, there is an obvious forgetful functor from **MET–UNIF** to **TOP** which takes a metric space to its underlying topological space and simply views a uniformly continuous mapping as a continuous mapping.

4. There is a **power set functor** P_* on the category **SETS** defined as follows: The set $P_*(X)$ is just the set of all subsets (also known as the power set), and if $f : X \to Y$ is a set-theoretic function, then $P_*(f) : P_*(X) \to P_*(Y)$ takes an element $A \in P(X)$ — which by definition is just a subset of X — to its image $f[A] \subset Y$. A short argument is needed to verify this construction actually defines a covariant functor, but it is elementary. First, we need to check that for every set X we have $P_*(1_X) = 1_{P(X)}$; this follows because $1_X[A] = A$

for all $A \subset X$. Next, we must check that $P_*(g \circ f) = P_*(g) \circ P_*(f)$ for all composable f and g. But this is a consequence of the elementary identity $g[\,f[A]\,] \;=\; g \circ f[A]$.

5. If we are given two partially ordered sets and a weakly order-preserving mapping f from the first to the second such that $u \leq v$ implies $f(u) \leq f(v)$, then f may be interpreted as a covariant functor on the associated categories.

6. If we are given two groups and a homomorphism f from the first to the second, then f may be interpreted as a covariant functor on the associated categories.

7. Finally, we shall give a more substantial example that played a central role in Mathematics 205B. Define a new category $\mathbf{TOP}_*$ of *pointed topological spaces* whose objects are pairs (X, y), where X is a topological space and $y \in X$; the point y is said to be the *basepoint* of the pointed space. A morphism $f : (X, y) \to (Z, w)$ in this category will be a continuous mapping from X to Z (usually also denoted by f) which maps y to w (*i.e.*, a **basepoint preserving** continuous mapping). The fundamental group $\pi_1(X, y)$ then has a natural interpretation as a covariant functor from $\mathbf{TOP}_*$ to **GRP**, for if f is a morphism of pointed spaces, then then one has an associated homomorphism f_* from $\pi_1(X, y)$ to $\pi_1(Z, w)$, and these have the required properties that $1_{(X,y)*}$ is the identity and $(g \circ f)_* = g_* \circ f_*$.

Contravariant functors and examples

Experience shows there are many instances in which it is useful to work with functors that **reverse** the order of function composition; such objects are called *contravariant functors*.

Definition. Let **C** and **D** be categories. A *contravariant functor* assigns (i) to each object X of **C** an object $U(X)$ of **D**, (ii) to each morphism $f : X \to Y$ in **C** a morphism $U(f) : U(Y) \to U(X)$ in **D** (note that the domain and codomain are the *opposites* of those in the covariant case!) such that the following hold:

(1) For each object X in **C** we have $U(1_X) = 1_{U(X)}$.

(2) For each pair of morphisms f and g in **C** such that $g \circ f$ is defined, we have $U(g \circ f) = U(f) \circ U(g)$.

The simplest examples of contravariant functors are given by the *pseudo-identity functors*, which map the objects and morphisms in the category **C** to their obvious counterparts in the opposite category $\mathbf{C}^{\mathbf{OP}}$. In fact, there is an obvious correspondence between contravariant functors from **C** to **D** and covariant functors from **C** to $\mathbf{D}^{\mathbf{OP}}$, or equivalently covariant functors from $\mathbf{C}^{\mathbf{OP}}$ to **D**. The best way to motivate the definition is to give some less trivial examples.

1. Let **C** be the category of all vector spaces over some field, and consider the construction which associates to each vector space its *dual space* V^* of linear mappings from V to the scalar field F. There is a simple way of defining a corresponding construction for morphisms; if $L : V \to W$ is a linear transformation, consider the linear transformation $L^* : W^* \to V^*$ whose value on a linear functional $h : W \to F$ is given by $L^*(h) = h \circ L$, which is a linear functional on V. Standard results in linear algebra show that L^* is a linear transformation, that L^* is an identity map if L is an identity map, and if L is a composite $L_1 \circ L_2$, then we have $L^* = L_2^* \circ L_1^*$.

2. There is a contravariant power set functor P^* on the category **SETS** defined as follows: The set $P^*(X)$ is just the set of all subsets, but now if $f : X \to Y$ is a set-theoretic function, then $P^*(f) : P^*(Y) \to P^*(X)$ takes an element $B \in P(Y)$ — which by definition is just a subset of Y — to its **inverse image** $f^{-1}[B] \subset X$. As in the case of P_*, a short elementary argument is needed to verify this construction actually defines a contravariant functor. The construction preserves identity maps because $1_X^{-1}[B] = B$ for all $B \subset X$, and the identity $P^*(g \circ f) = P^*(f) \circ P^*(g)$ is essentially a restatement of the elementary identity $f^{-1}[\,g^{-1}[B]\,] = (g \circ f)^{-1}[B]$.

3. Example 2 actually yields a little more. Define a *Boolean algebra* to be a set with two binary operations $\cap$ and $\cup$, a unary operation $x \to x'$, and special elements 0 and 1 such that the system satisfies the usual properties for unions, intersections, and complementation for the algebra $P(X)$ of subsets of a set X, where 0 corresponds to the empty set and 1 corresponds to X. One then has an associated category **BOOL–ALG** whose objects are Boolean algebras and whose morphisms preserve unions, intersection, complementation, and the special elements. Obviously each power set $P(X)$ is a Boolean algebra, and in fact P^* defines a contravariant functor from **SETS** to **BOOL–ALG**. — In contrast, the covariant functor P_* does NOT define such a functor because $P_*(f)$ does not preserves intersections even though it does preserve unions (for example, we can have $f[A] \cap f[B] \neq \emptyset$ when $A \cap B = \emptyset$).

4. The desirability of having both contravariant and covariant functors is illustrated by the following examples. Given a category **C**, modulo foundational questions we can informally view the set $\mathsf{Morph}_{\mathbf{C}}(X, Y)$ of morphisms from X to Y as a function of two variables on **C**. What happens if we hold one of these variables constant to get a single variable construction? — Suppose first that we hold X constant and set $A_X(Y) = \mathsf{Morph}_{\mathbf{C}}(X, Y)$. Then we can make A_X into a covariant functor aas follows: Given a morphism $g : Y \to Z$, let $A_X(g)$ take $f \in A_X(Y) = \mathsf{Morph}_{\mathbf{C}}(X, Y)$ to the composite $g \circ f$. The axioms for a category then imply that $A_X(1_Y)$ is the identity and that $A_X(h \circ g) = A_X(h) \circ A_X(g)$ if g and h are composable. — Now suppose that we hold Y constant and set $B_Y(X) = \mathsf{Morph}_{\mathbf{C}}(X, Y)$. Then we can make B_Y into a **contravariant** functor as follows: Given a morphism $k : W \to X$, let $B_Y(g)$ take $f \in B_Y(X) = \mathsf{Morph}_{\mathbf{C}}(X, Y)$ to the composite $f \circ k$. The axioms for a category then imply that $B_Y(1_X)$ is the identity and that $B_Y(k \circ h) = B_Y(h) \circ B_Y(k)$ if h and k are composable.

5. Given a topological space X, let $\mathbf{CComp}(X)$ and $\mathbf{AComp}(X)$ denote the sets of components and arc components of X respectively. These constructions extend to covariant functors from the category of topological spaces and continuous maps to the category of sets because a continuous map $f : X \to Y$ sends a component or arc component of X into a single component or arc component of Y. Similarly, there is a notion of *quasicomponents* for topological space such that each connected component lies in a quasicomponent (but the converse might not hold). This notion is described in Exercise 10 on page 163 of Munkres (see also Exercise 4 on page 236). One can show that if $f : X \to Y$ is continuous, the image of a quasicomponent of X is contained in a quasicomponent of Y (prove this!), it follows that one has a third functor $\mathbf{QComp}(X)$.

6. In the preceding example, suppose that **C** is the category of topological spaces and continuous mappings, and let Y be the real numbers with the usual topology. In this case the contravariant functor B_Y has the algebraic structure of a commutative ring with unit given by pointwise multiplication of continuous real valued functions, and if $f : W \to X$ is continuous then $B_Y(f)$ is in fact a homomorphism of commutative rings with unit. Therefore, if we define a category of continuous rings with unit (whose morphisms are unit preserving homomorphisms), it follows that B_Y defines a functor from topological spaces and continuous mappings to commutative rings with unit. — In contrast, there is no comparable structure for the covariant functor B_X if X is the real numbers.

Properties of functors

One of the most important properties of functors is that they send isomorphic objects in one one category to isomorphic objects in the other.

PROPOSITION 2. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories, let* $T : \mathbf{C} \to \mathbf{D}$ *be a (covariant or contravariant) functor, and let* $f : X \to Y$ *be an isomorphism in* $\mathbf{C}$*. Then* $T(f)$ *is an isomorphism in* $\mathbf{D}$*. Furthermore, if* g *is the inverse to* f*, then* $T(g)$ *is the inverse to* $T(f)$*.*

Proof. CASE 1. Suppose the functors are covariant. Then we have

$$1_{T(X)} = T(1_X) = T(g \circ f) = T(g) \circ T(f)$$

$$1_{T(Y)} = T(1_Y) = T(f \circ g) = T(f) \circ T(g)$$

and hence $T(g)$ is inverse to $T(f)$.

CASE 2. Suppose that the functors are contravariant. Then we have

$$1_{T(X)} = T(1_X) = T(g \circ f) = T(f) \circ T(g)$$

$$1_{T(Y)} = T(1_Y) = T(f \circ g) = T(g) \circ T(f)$$

and hence $T(g)$ is inverse to $T(f)$.■

The next result states that a composite of two functors is also a functor.

PROPOSITION 3. *Suppose that* $\mathbf{C}$, $\mathbf{D}$ *and* $\mathbf{E}$ *are categories and that* $F : \mathbf{C} \to \mathbf{D}$ *and* $G : \mathbf{D} \to \mathbf{E}$ *are functors (in each case, the functor may be covariant or contravariant). Then the composite* $G \circ F$ *also defines a functor; this functor is* covariant *if* F *and* G *are both covariant or contravariant, and it is* contravariant *if one of* F, G *is covariant and the other is contravariant.*■

This result has a curious implication:

COROLLARY 4. *There is a "category of small categories"* **SMCAT** *whose objects are small categories and whose morphisms are covariant functors from one small category to another.*■

SEMANTIC TRIVIA. (For readers who are familiar with contravariant and covariant tensors.) In the applications of linear algebra to differential geometry and topology, one often sees objects called *contravariant tensors* and *covariant tensors*, and for finite-dimensional vector spaces these are given by finitely iterated tensor products $V \otimes \cdots \otimes V$ of V with itself in the contravariant case and similar objects involving V^* in the covariant case; for our purposes it will suffice to say that if U and W are vector spaces with bases $\{ \mathbf{u}_i \}$ and $\{ \mathbf{w}_j \}$ respectively, then their tensor product $U \otimes W$ is a vector space having a basis of the form $\{ \mathbf{u}_i \otimes \mathbf{w}_j \}$ where i and j are allowed to vary independently (hence the dimension of $U \otimes W$ is $[\dim U] \cdot [\dim W]$). Since the identity functor on the category of vector spaces is covariant and the dual space functor is covariant, at first it might seem that something is the opposite of what it should be. However, the classical tensor notation refers to the manner in which the **coordinates** transform; now coordinates for a vector space may be viewed linear functionals on that space, or equivalently as elements of the dual space, which is contravariant. Therefore individual coordinates on $V \otimes \cdots \otimes V$ correspond to elements of the *dual space* of the latter, and in fact the construction which associates the space $(V \otimes \cdots \otimes V)^*$

to V defines a *contravariant* functor on the category of finite-dimensional vector spaces over the given scalars; likewise, the construction which associates the space $(V^* \otimes \cdots \otimes V^*)^*$ to V defines a **covariant** functor on the category of finite-dimensional vector spaces over the given scalars.

Natural transformations

The final concept in category theory to be considered here is the notion of **natural transformation** from one functor to another. In fact, the motivation for category theory in the work of Eilenberg and MacLane was a need to discuss "natural mappings" in a mathematically precise manner. There are actually two definitions, depending whether both functors under consideration are covariant or contravariant.

Definition. Let **C** and **D** be categories, and let F and G be covariant functors from **C** to **D**. A *natural transformation* θ from F to G associates to each object X in **C** a morphism $\theta_X : F(X) \to G(X)$ such that for each morphism $f : X \to Y$ we have $\theta_Y \circ F(f) = G(f) \circ \theta_X$.

The morphism identity is often expressed graphically by saying the the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \big\downarrow \theta_X & & \big\downarrow \theta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

is a **commutative diagram**. The idea is that all paths of arrows from one object-vertex to another yield the same function.

The definition of a natural transformation of contravariant functors is similar.

Definition. Let **C** and **D** be categories, and let T and U be contravariant functors from **C** to **D**. A *natural transformation* θ from F to G associates to each object X in **C** a morphism $\theta_X : T(X) \to U(X)$ such that for each morphism $f : X \to Y$ we have $\theta_X \circ T(f) = U(f) \circ \theta_Y$.

Here is the corresponding commutative diagram:

$$\begin{array}{ccc} T(Y) & \xrightarrow{T(f)} & T(X) \\ \big\downarrow \theta_Y & & \big\downarrow \theta_X \\ U(Y) & \xrightarrow{U(f)} & U(X) \end{array}$$

Once again we need to give some decent examples

1. Given any functor $T : \mathbf{C} \to \mathbf{D}$, there is an obvious identity transformation j^T from T to itself; specifically, j^T_X is the identity map on $T(X)$.

2. Let **C** be one of the categories as above for which reasonable products and diagonal maps can be defined. Then there is a natural diagonal transformation Δ from the identity to the diagonal functor such that for each object X the mapping $\Delta_X : X \to X \times X$ is the diagonal map.

3. On the category of vector spaces over some field F, one can iterate the dual space functor to obtain a covariant *double dual space functor* $(V^*)^*$. There is a natural transformation $e_V : V \to (V^*)^*$ defined as follows: For each $\mathbf{v} \in V$, let $e_V(\mathbf{v}) : V^* \to F$ be the linear function given by evaluation at v; in other words, the value of $e_V(\mathbf{v})$ on a linear functional

f is given by $f(\mathbf{v})$. If V is finite-dimensional, this map is an isomorphism (a *natural isomorphism*).

Note that if V is finite-dimensional then V and its dual space V^* are isomorphic, but the isomorphism depends upon some additional data such as the choice of a basis, an inner product, or more generally a nondegenerate bilinear form. In contrast, *the natural isomorphism e_V does not depend upon any such choices.*

4. In the category of sets or topological spaces and continuous mappings, let A be an arbitrary object and define functors L_A and R_A such that $L_A(X) = A \times X$ and $R_A(X) = X \times A$. One can make these into covariant functors by sending the morphism $f : X \to Y$ to $L_A(X) = 1_A \times f$ and $R_A(f) = f \times 1_A$. There is an obvious natural transformation $t : L + A \to R_A$ such that $t_A(X) : A \times X \to X \times A$ sends (a, x) to (x, a) for all $a \in A$ and $x \in X$, and it is an elementary exercise to verify that this is a natural transformation such that each map $t_A(X)$ is an *isomorphism*; in other words, t_A is a natural isomorphism from the functor L_A to the functor R_A.

5. For the morphism examples A_X and B_Y discussed previously, if $h : W \to X$ is a morphism in the category, then it defines a natural transformation $h^* : A_X \to A_W$ which sends $f \in A_X(Y)$ to $f \circ h \in A_W(Y)$; the naturality condition follows from associativity of composition. Similarly, if $g : Y \to Z$ is a morphism then there is a natural transformation $g_* : B_Y \to B_Z$ sending f to $g \circ f$; once again, the key naturality condition follows from the associativity of composition. Furthermore, h^* is a natural isomorphism if h is an isomorphism and g_* is a natural isomorphism if g is an isomorphism,

6. For the arc component, connected component and quasicomponent functors described above, there are natural transformations θ : **AComp** $\to$ **CComp** reflecting the fact that every arc component of a topological space X is contained in a connected component and ψ : **CComp** $\to$ **QComp** reflecting the fact that every conected component of a space is contained in a quasicomponent.

A basic exercise in category theory is to prove the following:

PROPOSITION 5. *There are $1-1$ correspondences between natural transformations from A_X to A_W and morphisms from W to X and between natural transformations from B_Y to B_Z and morphisms from Y to Z.*

Sketch of proof. The main point is to retrieve the function from the natural transformation. Given $\theta : A_X \to A_W$, one does this by considering the image of 1_X, and given $\varphi : B_Y \to B_Z$, one does this by considering the image of 1_Y.∎

Finally, we have the following result on natural isomorphisms (*i.e.*, natural transformations θ such that each map θ_X is an isomorphism):

PROPOSITION 6. *Let $\theta : F \to G$ be a natural transformation such that for each object X the map θ_X is an isomorphism. The there is a natural transformation $\varphi_X : G \to F$ such that for each X the map φ_X is inverse to θ_X.*

Proof. The main thing to check is that the relevant diagrams are commutative; we shall only do the case where F and G are covariant, leaving the other case to the reader. Since $\theta_X \circ \varphi_X$ is the identity on $G(X)$ and $\varphi_X \circ \theta_X$ is the identity on $F(X)$, we have

$$\theta_Y \circ \varphi_Y \circ G(f) \quad = \quad G(f) \quad = \quad g(f) \circ \theta_X \circ \varphi_X \quad = \quad \theta_Y \circ F(f) \circ \varphi_X$$

and if we compose with the inverse θ_X on the left of these expressions we obtain

$$\varphi_Y \circ G(f) \quad = \quad F(f) \circ \varphi_X$$

which is the naturality condition.■

We say that two functors are *naturally isomorphic* if there is a natural isomorphism from one to the other.

Equivalences of categories

One can obviously define an isomorphism of categories to be a covariant functor $T : \mathbf{C} \to \mathbf{D}$ for which there is an inverse covariant functor $U : \mathbf{D} \to \mathbf{C}$ such that the composites $T \circ U$ and $U \circ T$ are the identities on $\mathbf{C}$ and $\mathbf{D}$ respectively. However, for many purposes one needs a less rigid notion of category equivalence.

Definition. A covariant functor $T : \mathbf{C} \to \mathbf{D}$ is a *category equivalence* (or equivalence of categories) if there is a covariant functor $U : \mathbf{D} \to \mathbf{C}$ such that the composites $T \circ U$ and $U \circ T$ are *naturally isomorphic to* the identities on $\mathbf{C}$ and $\mathbf{D}$ respectively.

In particular, if T and U define an equivalence of categories, then every object in $\mathbf{D}$ is isomorphic to an object of the form $T(X)$, and conversely every object in $\mathbf{C}$ is isomorphic to an object of the form $U(A)$.

... to be continued ...