

## Preface

Perhaps the simplest motivation for algebraic topology is the following basic question:

*If  $m$  and  $n$  are distinct positive integers, is  $\mathbb{R}^m$  ever homeomorphic to  $\mathbb{R}^n$ ?*

Results from point set topology imply the answer is **NO** if one of  $m$  and  $n$  is equal to 1. If a homeomorphism  $h : \mathbb{R}^m \rightarrow \mathbb{R}$  existed then for each  $\mathbf{x} \in \mathbb{R}^m$  we could conclude that  $\mathbb{R}^m - \{\mathbf{x}\}$  is homeomorphic to  $\mathbb{R} - \{h(\mathbf{x})\}$ . Since  $\mathbb{R}^m - \{\mathbf{x}\}$  is connected for all  $\mathbf{x} \in \mathbb{R}^m$  if  $m > 1$  while  $\mathbb{R} - \{t\}$  is not connected for any choice of  $t \in \mathbb{R}$ , it follows that  $\mathbb{R}^m - \{\mathbf{x}\}$  is never homeomorphic to  $\mathbb{R} - \{t\}$  if  $m > 1$  and hence  $\mathbb{R}^m$  cannot be homeomorphic to  $\mathbb{R}$ . Similarly, results on fundamental groups imply that for all relevant choices of  $\mathbf{x}$  the set  $\mathbb{R}^m - \{\mathbf{x}\}$  is simply connected if  $m > 2$  while  $\mathbb{R}^2 - \{\mathbf{x}\}$  has an infinite cyclic fundamental group, so we also know that  $\mathbb{R}^m$  is not homeomorphic to  $\mathbb{R}^2$  provided  $m > 2$ . One basic goal of an introductory course in algebraic topology is to show that  $\mathbb{R}^m$  is never homeomorphic to  $\mathbb{R}^n$  if  $m \neq n$ .

The idea behind proving such results is to define certain abelian groups which give an **algebraic picture** of a given topological space; in particular, if two topological spaces are homeomorphic, then their associated groups will be algebraically isomorphic. Unfortunately, the definitions for these **homology groups** are less straightforward than the definition of the fundamental group, and much of the work in this course involves the construction of such groups and the proofs that they have good formal properties.

In analogy with standard results for fundamental groups, the homology groups of two spaces will be isomorphic if the spaces satisfy a condition that is somewhat weaker than the existence of a homeomorphism between them; namely, the groups are isomorphic if the two spaces have the same *homotopy type* as defined on page 363 of the book by Munkres cited below.

Since the constructions for the associated groups are somewhat complicated, it is natural to expect that they should be useful for more than simply answering the homeomorphism question for Euclidean spaces. In particular, one might ask if these groups (and a course in algebraic topology) can shed new light on some questions left open in undergraduate or beginning graduate courses in mathematics.

1. The material in introductory graduate level courses does not really give much insight into the popular characterization of topology as a “rubber sheet geometry.” In other words, topology is generally viewed as the study of properties that do not change under various sorts of bending or stretching operations. Some aspects of this already appear in the study of fundamental groups, and one objective of this course is to develop these ideas much further.
2. As a refinement of the problem at the beginning of this preface, one can ask if there is some topological criterion which characterizes the algebraic notion of  $n$ -dimensionality, at least for spaces that are relatively well-behaved.

3. An algebraic topology course should also yield better insight into several issues that arise in undergraduate courses, including (a) the Fundamental Theorem of Algebra, (b) various facts about planar and nonplanar networks, (c) insides and outsides of plane curves and closed surfaces in 3-dimensional space, and (d) Euler's Formula for "nice" polyhedra in  $\mathbb{R}^3$ ; namely, if  $P$  is a polyhedron bounding a convex body in  $\mathbb{R}^3$ , then the numbers  $V$ ,  $E$  and  $F$  of vertices, edges and faces satisfy the equation  $E + 2 = V + F$ .
4. Some of the basic results on the topology of the edge-path graphs in Mathematics 205B should be placed into a broader context. In particular, the notion of Euler characteristic should be extended to a larger class of spaces.
5. If time permits, another goal will be to give a unified approach to certain results in multivariable calculus involving the  $\nabla$  operator, Green's Theorem, Stokes' Theorem and the Divergence Theorem, and to formulate analogs for higher dimensions.

Throughout the course we shall use the following text for the basic graduate topology courses as a reference for many topics and definitions:

**J. R. Munkres.** *Topology* (Second Edition), *Prentice-Hall, Saddle River NJ*, 2000. ISBN: 0-13-181629-2.

The official text for this course is the following book:

**A. Hatcher.** *Algebraic Topology* (Third Paperback Printing), *Cambridge University Press, New York NY*, 2002. ISBN: 0-521-79540-0.

*This book can be legally downloaded from the Internet at no cost for personal use, and here is the link to the online version:*

[www.math.cornell.edu/~hatcher/AT/ATpage.html](http://www.math.cornell.edu/~hatcher/AT/ATpage.html)

Here are four other references. The first is a book that has been used as a text in the past, the second is a fairly detailed history of the subject during its formative years, and the last two are classic (but not outdated) books; the first also has detailed historical notes.

**J. W. Vick.** *Homology Theory*. (Second Edition). *Springer-Verlag, New York etc.*, 1994. ISBN: 3-540-94126-6.

**J. Dieudonné.** *A History of Algebraic and Differential Topology* (1900 – 1960). *Birkhäuser Verlag, Zurich etc.*, 1989. ISBN: 0-817-63388-X.

**S. Eilenberg and N. Steenrod.** *Foundations of Algebraic Topology*. (Second Edition). *Princeton University Press, Princeton NJ*, 1952. ISBN: 0-691-07965-X.

**E. H. Spanier.** *Algebraic Topology*, *Springer-Verlag, New York etc.*, 1994.

The [amazon.com](http://amazon.com) sites for Hatcher's and Spanier's books also give numerous other texts in algebraic topology that may be useful. Finally, there are two other books by Munkres that we shall quote repeatedly throughout these notes. The first will be denoted by [MunkresEDT] and the second by [MunkresAT]; if we simply refer to "Munkres," it will be understood that we mean the previously cited book, *Topology* (Second Edition).

**J. R. Munkres.** *Elementary differential topology*. (Lectures given at Massachusetts Institute of Technology, Fall, 1961. Revised edition. *Annals of Mathematics Studies*, No. 54.) *Princeton University Press, Princeton, NJ*, 1966. ISBN: 0-691-09093-9.

**J. R. Munkres.** *Elements of Algebraic Topology*. *Addison-Wesley, Reading, MA*, 1984. (Reprinted by Westview Press, Boulder, CO, 1993.) ISBN: 0-201-62728-0.

## Overview of the course

One important feature of homology groups is that if  $f : X \rightarrow Y$  is a continuous mapping of topological spaces, then there is an associated homomorphism  $f_*$  from the homology groups of  $X$  to the homology groups of  $Y$ ; this is again similar to the situation for fundamental groups of pointed spaces, and it plays an important role in addressing the issues listed above. In fact, algebraic topology turns out to be an effective means for analyzing the following central problem:

Given two “reasonably well-behaved” spaces  $X$  and  $Y$ , describe the homotopy classes of continuous mappings from  $X$  to  $Y$ .

In general, the descriptions of the homotopy classes can be quite complicated, and only a few cases of such problems can be handled using the methods of a first course, but we shall mention a few special cases at various points in the course.

Many of the basic properties of homology groups and homomorphisms are best stated using the formalisms of **Category Theory**, and many of the constructions and theorems in algebraic topology are best stated within the framework of **Homological Algebra**. We shall develop these subjects in the course to the extent that we need them.

## Prerequisites

The name “algebraic topology” suggests that the subject uses input from both algebra and topology, and this is in fact the case; since topology began as a branch of geometry, it is also reasonable to expect that some geometric input is also required. Our purpose here is to summarize the main points from prerequisite courses that will be needed. Additional background material which is usually not covered explicitly in the prerequisites will be described in the first unit of these notes.

### *Set theory*

Everything we shall need from set theory is contained in the following online directory:

<http://math.ucr.edu/~res/math144>

In particular, a fairly complete treatment is contained in the documents `setsnotes $n$ .pdf`, where  $1 \leq n \leq 8$ .

There are two features of the preceding that are somewhat nonstandard. The first is the definition of a function from a set  $A$  to another set  $B$ . Generally this is given formally by the graph, which is a subset  $G \subset A \times B$  such that for each  $a \in A$  there is a unique  $b \in B$  such that  $(a, b) \in G$ . Our definition of function will be a **triple**  $f = (A, G, B)$ , where  $G \subset A \times B$  satisfies the condition in the preceding sentence. The reason for this is that we must specify the target set or **codomain** of the function explicitly; in fact, the need to specify the codomain has already arisen at least implicitly in prerequisite graduate topology courses, specifically in the definition of the fundamental group. A second nonstandard feature is the concept of **disjoint union** or **sum** of an indexed family  $\{ X_\alpha \}$  of sets. The important features of the disjoint sum, which is written  $\coprod_\alpha X_\alpha$ , are that it is a union of subsets  $Y_\alpha$  which are canonically in 1–1 correspondence with the sets  $X_\alpha$  and that  $Y_\alpha \cap Y_\beta = \emptyset$  if  $\alpha \neq \beta$ . Another source of information on such objects is Unit V of the online notes for Mathematics 205A which are cited below.

### *Topology*

This course assumes familiarity with the basic material in graduate level topology courses through the theory of fundamental groups and covering spaces (in other words, the material in Mathematics 205A and 205B). Everything we need from the first of these courses can be found in the following online directory:

<http://math.ucr.edu/~res/math205A>

In particular, the files `gentopnotes2008.*` contain a fairly complete set of lecture notes for the course. This material is based upon the textbook by Munkres cited in the Preface. Two major differences between the notes and Munkres appear in Unit V. The discussion of quotient topologies is

somewhat different from that of Munkres, and in analogy with the previously mentioned discussion of set-theoretic disjoint sums there is a corresponding construction of disjoint sum for an indexed family of topological spaces.

There is a similar directory for the second course, which deals with the basic notions of homotopy, fundamental groups and covering spaces:

<http://math.ucr.edu/~res/math205B>

There is no self-contained set of notes in this directory, but there are comments and (additional) exercises to supplement the text and indicate the sections which will be needed for this course. In addition to the sections mentioned in these references, it might be worthwhile to look also at the supplementary exercises for Chapter 13.

Chapter 14 of Munkres is not a required part of the material covered in the Department's qualifying exam, but it is sometimes covered in 205B and at some points in this course the topics covered in Chapter 14 are relevant, and we shall mention them when it seems appropriate or useful.

### *Algebra*

As in the later parts of Munkres, we shall assume some familiarity with certain topics in group theory. Nearly everything we need is in Sections 67 – 69 of Munkres, but we shall also need the following basic result:

**STRUCTURE THEOREM FOR FINITELY GENERATED ABELIAN GROUPS.** *Let  $G$  be a finitely generated abelian group (so every element can be written as a monomial in integral powers of some finite subset  $S \subset G$ ). Then  $G$  is isomorphic to a direct sum*

$$(H_1 \oplus \cdots \oplus H_b) \oplus (K_1 \oplus \cdots \oplus K_s)$$

where each  $H_i$  is infinite cyclic and each  $K_j$  is finite of order  $t_j$  such that  $t_{j+1}$  divides  $t_j$  for all  $j$ . — For the sake of uniformity set  $t_j = 1$  if  $j > s$ . Then two direct sums as above which are given by  $(b; t_1, \cdots)$  and  $(b'; t'_1, \cdots)$  are isomorphic if and only if  $b = b'$  and  $t_j = t'_j$  for all  $j$ .

A proof of this fundamental algebraic result may be found in Sections II.1 and II.2 of the following standard graduate algebra textbook:

**T. Hungerford.** *Algebra.* (Reprint of the 1974 original edition, Graduate Texts in Mathematics, No. 73.) *Springer-Verlag, New York–Berlin–etc.*, 1980. ISBN: 0-387-90518-9.

Material from standard undergraduate linear algebra courses will also be used as needed.

### *Analysis*

We shall assume the basic material from an upper division undergraduate course in real variables as well as material from a lower division undergraduate course in multivariable calculus through the theorems of Green and Stokes as well as the 3-dimensional Divergence Theorem. The classic text by W. Rudin (*Principles of Mathematical Analysis*, Third Edition) is an excellent reference for real variables, and the following multivariable calculus text contains more information on the that subject than one can usually find in the usual 1500 page calculus texts (the book is not perfect, but especially at the graduate level it is useful as a background reference).

**J. E. Marsden and A. J. Tromba.** *Vector Calculus* (Fifth Edition), *W. H. Freeman & Co., New York NY*, 2003. ISBN: 0-7147-4992-0.

# I. Foundational and Geometric Background

Aside from the formal prerequisites, algebraic topology relies on some background material from other subjects that is generally not covered in prerequisites. In particular, two concepts from the foundations of mathematics, namely **categories** and **functors**, play a central role in formulating the basic concepts of algebraic topology. Furthermore, since algebraic topology places heavy emphasis on spaces that can be constructed from certain fundamental building blocks, some relatively elementary but fairly detailed properties of the latter are indispensable. The purpose of this unit is to develop enough of category theory so that we can use it to formulate things efficiently and to describe the topological and geometric properties of a class of well-behaved spaces called **polyhedra** that will be needed in the course.

## I.1 : Categories and functors

(Hatcher, § 2.3)

If mathematics is the study of abstract systems, then category theory may be viewed as an abstract formal setting for working with such systems. In fact, the theory was originally developed by S. Eilenberg (1919–1998) and S. MacLane (1909–2005) in the 1940s to provide an effective conceptual framework for handling various constructions and phenomena related to algebraic topology (including some from the theory of groups). The formal definition may be viewed as a generalization of familiar properties of ordinary set-theoretic functions. There is a great deal of overlap between the discussion here and the file `categories.pdf` in the `math205A` directory. There is a classic book by P. Freyd (1936–), *Abelian Categories: An Introduction to the Theory of Functors*, which is still an extremely readable introduction to category theory and its role in abstract algebra, and it is available online at the following site:

<http://www.emis.de/journals/TAC/reprints/articles/3/tr3/pdf>

**Definition.** A **CATEGORY** is a system  $\mathbf{C}$  consisting of

- (a) a class  $\text{Obj}(\mathbf{C})$  of sets called the **objects** of  $\mathbf{C}$ ,
- (b) for each ordered pair of objects  $X$  and  $Y$  an associated set  $\text{Morph}_{\mathbf{C}}(X, Y)$  called the **morphisms** from  $X$  to  $Y$ ,
- (c) for each ordered triple of objects  $X$ ,  $Y$  and  $Z$ , an associated map called a *composition pairing*  $\varphi : \text{Morph}_{\mathbf{C}}(X, Y) \times \text{Morph}_{\mathbf{C}}(Y, Z) \longrightarrow \text{Morph}_{\mathbf{C}}(X, Z)$ , whose value for  $(f, g)$  is generally written  $g \circ f$ , such that the following hold:

(1) The sets  $\text{Morph}_{\mathbf{C}}(X, Y)$  and  $\text{Morph}_{\mathbf{C}}(Z, W)$  are disjoint unless  $X = Z$  and  $Y = W$ .

(2) For each object  $X$  there is a unique identity morphism  $1_X = \text{id}_X \in \text{Morph}_{\mathbf{C}}(X, X)$  such that for each  $f \in \text{Morph}_{\mathbf{C}}(X, Y)$  and  $g \in \text{Morph}_{\mathbf{C}}(Z, X)$  we have  $f \circ 1_X = f$  and  $1_X \circ g = g$ .

(3) The composition pairings satisfy an associative law; *i.e.*, if  $f \in \text{Morph}_{\mathbf{C}}(X, Y)$ ,  $g \in \text{Morph}_{\mathbf{C}}(Y, Z)$ , and  $h \in \text{Morph}_{\mathbf{C}}(Z, W)$ , then  $(h \circ g) \circ f = h \circ (g \circ f)$ .

By the assumptions, for each  $f \in \text{Morph}_{\mathbf{C}}(X, Y)$  the objects  $X$  and  $Y$  are uniquely determined, and they are called the **domain** and **codomain** of  $f$  respectively. When working within a given category we generally use familiar notation like  $f : X \rightarrow Y$  to indicate that  $f \in \text{Morph}_{\mathbf{C}}(X, Y)$ .

As in set theory, at some points one must take care to avoid difficulties with classes that are “too large” to be sets (for example, we cannot discuss the set of all sets), but in practice it is always possible to circumvent such problems by careful choices of definitions and wordings (for example, using the theory of *Grothendieck universes*), so we shall generally not dwell on such points.

### *Examples of categories*

By the remarks preceding the definition of a category, it is clear that we have a category **SETS** whose objects are given by all sets, whose morphisms are set-theoretic functions from one set to another (with the conventions mentioned in the Prerequisites!), and whose composition is merely ordinary composition of mappings. Here are some further examples:

1. Given a field  $\mathbb{F}$ , there is the category  $\mathbf{VEC}_{\mathbb{F}}$  whose objects are vector spaces, whose morphisms are  $\mathbb{F}$ -linear transformations, and whose composition is ordinary composition. The important facts here are that the identity on a vector space is a linear transformation, and the composite of two linear transformations is a linear transformation.
2. There is also a category **GRP** whose objects are groups and whose morphisms are group homomorphisms (with the usual composition). Once again, the crucial properties needed to check the axioms for a category are that identity maps are homomorphisms and the composite of two homomorphisms is a homomorphism.
3. Within the preceding example, there is the subcategory **ABGRP** whose objects are abelian groups, with the same morphisms and compositions. In this category, the set of morphisms from one object to another has a natural abelian group structure given by pointwise addition of functions, and the resulting abelian group of homomorphisms is generally denoted by  $\text{Hom}(X, Y)$ .
4. More generally, if  $\mathbf{C}$  is a category, then a subcategory  $\mathbf{A}$  is a collection of morphisms and objects which is closed under (i) taking domains and codomains of objects, (ii) taking identity morphisms of objects, (iii) taking composites of morphisms. It is said to be a **full subcategory** if for each pair of objects  $X$  and  $Y$  in  $\mathbf{A}$  we have  $\text{Morph}_{\mathbf{A}}(X, Y) = \text{Morph}_{\mathbf{C}}(X, Y)$ . It follows that **ABGRP** is a full subcategory of **GRP**. On the other hand, if we let  $\mathbf{GRP}_{1-1}$  be the category whose objects are groups and whose morphisms are *injective* homomorphisms, then  $\mathbf{GRP}_{1-1}$  is a subcategory of **GRP** but it is not a full subcategory.
5. If  $P$  is a partially ordered set with ordering relation  $\leq$ , then one has an associated category whose objects are the elements of  $P$  and such that  $\text{Morph}(x, y)$  consists of a single point if  $x \leq y$  and is empty otherwise. This is an example of a **small** category in which the class of objects is a set.
6. One can also use partially ordered sets to define a category **POSETS** whose objects are partially ordered sets and whose morphisms are monotonically nondecreasing functions from one partially ordered set to another; as in most other cases, composition has its usual meaning.

7. If  $G$  is a group, then  $G$  also defines a small category as follows: There is exactly one object, the morphisms of this object to itself are given by the elements of  $G$ , and composition is given by the multiplication in  $G$ .
8. There is a category **TOP** whose objects are topological spaces, whose morphisms are continuous maps between topological spaces, and whose composition is the usual notion. Again, the crucial properties needed to verify the axioms for a category are that identity maps are continuous and composites of continuous maps are also continuous.
9. There are also categories whose objects are topological spaces and whose morphisms are **open** maps or **closed** maps. The categories with various types of morphisms are distinct. Of course, it is also possible to take combinations of such conditions and obtain structures like the category of spaces with *continuous open mappings* as the morphisms.
10. More generally, given any class of continuous mappings that is closed under taking identities and compositions, one can define a category of topological spaces with such maps as the morphisms. Two examples are maps that are **proper** (inverse images of compact subsets are compact) or **light** (inverse images of points are discrete sets; see Exercise II.3.4 in [gentopexercises2008.pdf](#) from the [math205A](#) directory for more on the latter).
11. One also has a category **MET–UNIF** whose objects are metric spaces and whose morphisms are **uniformly continuous** mappings (with the usual composition).
12. Similarly, there is the category **MET–LIP** whose objects are metric spaces and whose morphisms are **Lipschitz** mappings: *i.e.*, there is a constant  $M$  such that

$$d(f(x_1), f(x_2)) \leq M \cdot d(x, y)$$

for all  $x$  and  $y$  in the domain (such an inequality is called a *Lipschitz condition*). Standard results of (abstract) multivariable calculus show that if  $K$  is a compact convex set and  $f : K \rightarrow \mathbb{R}^m$  extends to a function on an open neighborhood  $W$  of  $K$  whose coordinates have continuous first partial derivatives, then  $f$  satisfies a Lipschitz condition.

13. Still further in the same direction, there is the category **MET–ISO** whose objects are metric spaces and whose morphisms are **isometries** (but not necessarily surjective).
14. (A fundamentally important general construction.) Given an arbitrary category  $\mathbf{C}$ , one has the **dual** or **opposite** category  $\mathbf{D} = \mathbf{C}^{\text{OP}}$  with the same objects as  $\mathbf{C}$ , but with  $\text{Morph}_{\mathbf{D}}(X, Y) = \text{Morph}_{\mathbf{C}}(Y, X)$  (note the reversal!) and the composition pairing  $*$  defined by  $g * f = f \circ g$ . Note that if  $\mathbf{D} = \mathbf{C}^{\text{OP}}$  then  $\mathbf{C} = \mathbf{D}^{\text{OP}}$ .

In most of the preceding examples of categories, there is a fundamental notion of **isomorphism**, and in fact one can formulate this abstractly for an arbitrary category:

**Definition.** Let  $\mathbf{C}$  be a category, and let  $X$  and  $Y$  be objects of  $\mathbf{C}$ . A morphism  $f : X \rightarrow Y$  is an *isomorphism* if there is a morphism  $g : Y \rightarrow X$  (an inverse) such that  $g \circ f = 1_X$  and  $f \circ g = 1_Y$ .

This generalizes notions like an invertible linear transformation, a group isomorphism, and a homeomorphism of topological spaces.

**PROPOSITION 1.** *Suppose that  $f : X \rightarrow Y$  is an isomorphism in a category  $\mathbf{C}$  and  $g$  and  $h$  are inverses to  $f$ . Then  $h = g$ .*

**Proof.** Consider the threefold composite  $h \circ f \circ g$ . Since  $h \circ f = 1_X$ , this is equal to  $g$ , and since  $f \circ g = 1_Y$ , it is also equal to  $h$ . ■



## Functors

The examples of categories illustrate a basic principle in modern mathematics: Whenever one defines a type of mathematical system, there is usually a corresponding type of morphism for such systems (and in some cases there are several reasonable choices for morphisms). Since a category is an example of a mathematical system, it is natural to ask whether there is a corresponding notion of morphisms in this case too. In fact, there are two concepts of morphism that turn out to be important and useful. We shall start with the simpler one.

**Definition.** Let  $\mathbf{C}$  and  $\mathbf{D}$  be categories. A *covariant functor* assigns (i) to each object  $X$  of  $\mathbf{C}$  an object  $T(X)$  of  $\mathbf{D}$ , (ii) to each morphism  $f : X \rightarrow Y$  in  $\mathbf{C}$  a morphism  $T(f) : T(X) \rightarrow T(Y)$  in  $\mathbf{D}$  such that the following hold:

- (1) For each object  $X$  in  $\mathbf{C}$  we have  $T(1_X) = 1_{T(X)}$ .
- (2) For each pair of morphisms  $f$  and  $g$  in  $\mathbf{C}$  such that  $g \circ f$  is defined, we have  $T(g \circ f) = T(g) \circ T(f)$ .

**HISTORICAL TRIVIA.** Eilenberg and MacLane “borrowed” the word **category** from the philosophical writings of the 18<sup>th</sup> century German philosopher I. Kant (1724–1804) and the word **functor** from the philosophical writings of the 20<sup>th</sup> century German-American philosopher R. Carnap (1891–1970), who was strongly influenced by Kant’s writings on the philosophy of science.

### Examples of covariant functors

Numerous constructions from undergraduate and elementary graduate courses can be interpreted as functors; in many cases this does not shed much additional light on the objects constructed, but in other cases the concept does turn out to be extremely useful.

1. Given a category  $\mathbf{C}$ , there is the **identity functor** from  $\mathbf{C}$  to itself, which takes all objects and morphisms to themselves.
2. Given a category  $\mathbf{C}$  and a (possibly different) nonempty category  $\mathbf{D}$ , for each object  $A$  of  $\mathbf{D}$  there is a **constant functor**  $k_A$  from  $\mathbf{C}$  to  $\mathbf{D}$  which sends every object of  $\mathbf{C}$  to  $A$  and every morphism to the identity morphism  $1_A$ .
3. In categories where the objects are given by sets with some extra structure and the morphisms are ordinary functions with additional properties, there are **forgetful functors** which take objects to the underlying sets and morphisms to the underlying mappings of sets. For example, there are forgetful functors from  $\mathbf{VEC}_{\mathbb{F}}$ ,  $\mathbf{GRP}$ ,  $\mathbf{POSETS}$ , and  $\mathbf{TOP}$  to  $\mathbf{SETS}$ . Likewise, there is an obvious forgetful functor from  $\mathbf{MET-UNIF}$  to  $\mathbf{TOP}$  which takes a metric space to its underlying topological space and simply views a uniformly continuous mapping as a continuous mapping.
4. There is a **power set functor**  $P_*$  on the category  $\mathbf{SETS}$  defined as follows: The set  $P_*(X)$  is just the set of all subsets (also known as the power set), and if  $f : X \rightarrow Y$  is a set-theoretic function, then  $P_*(f) : P_*(X) \rightarrow P_*(Y)$  takes an element  $A \in P(X)$  — which by definition is just a subset of  $X$  — to its image  $f[A] \subset Y$ . A short argument is needed to verify this construction actually defines a covariant functor, but it is elementary. First, we need to check that for every set  $X$  we have  $P_*(1_X) = 1_{P(X)}$ ; this follows because  $1_X[A] = A$

for all  $A \subset X$ . Next, we must check that  $P_*(g \circ f) = P_*(g) \circ P_*(f)$  for all composable  $f$  and  $g$ . But this is a consequence of the elementary identity  $g[f[A]] = g \circ f[A]$ .

5. If we are given two partially ordered sets and a weakly order-preserving mapping  $f$  from the first to the second such that  $u \leq v$  implies  $f(u) \leq f(v)$ , then  $f$  may be interpreted as a covariant functor on the associated categories.
6. If we are given two groups and a homomorphism  $f$  from the first to the second, then  $f$  may be interpreted as a covariant functor on the associated categories.
7. Finally, we shall give a more substantial example that played a central role in Mathematics 205B. Define a new category  $\mathbf{TOP}_*$  of *pointed topological spaces* whose objects are pairs  $(X, y)$ , where  $X$  is a topological space and  $y \in X$ ; the point  $y$  is said to be the *basepoint* of the pointed space. A morphism  $f : (X, y) \rightarrow (Z, w)$  in this category will be a continuous mapping from  $X$  to  $Z$  (usually also denoted by  $f$ ) which maps  $y$  to  $w$  (*i.e.*, a **basepoint preserving** continuous mapping). The fundamental group  $\pi_1(X, y)$  then has a natural interpretation as a covariant functor from  $\mathbf{TOP}_*$  to  $\mathbf{GRP}$ , for if  $f$  is a morphism of pointed spaces, then then one has an associated homomorphism  $f_*$  from  $\pi_1(X, y)$  to  $\pi_1(Z, w)$ , and these have the required properties that  $1_{(X, y)*}$  is the identity and  $(g \circ f)_* = g_* \circ f_*$ .

### *Contravariant functors and examples*

Experience shows there are many instances in which it is useful to work with functors that **reverse** the order of function composition; such objects are called *contravariant functors*.

**Definition.** Let  $\mathbf{C}$  and  $\mathbf{D}$  be categories. A *contravariant functor* assigns (i) to each object  $X$  of  $\mathbf{C}$  an object  $U(X)$  of  $\mathbf{D}$ , (ii) to each morphism  $f : X \rightarrow Y$  in  $\mathbf{C}$  a morphism  $U(f) : U(Y) \rightarrow U(X)$  in  $\mathbf{D}$  (note that the domain and codomain are the *opposites* of those in the covariant case!) such that the following hold:

- (1) For each object  $X$  in  $\mathbf{C}$  we have  $U(1_X) = 1_{U(X)}$ .
- (2) For each pair of morphisms  $f$  and  $g$  in  $\mathbf{C}$  such that  $g \circ f$  is defined, we have  $U(g \circ f) = U(f) \circ U(g)$ .

The simplest examples of contravariant functors are given by the *pseudo-identity functors*, which map the objects and morphisms in the category  $\mathbf{C}$  to their obvious counterparts in the opposite category  $\mathbf{C}^{\mathbf{OP}}$ . In fact, there is an obvious correspondence between contravariant functors from  $\mathbf{C}$  to  $\mathbf{D}$  and covariant functors from  $\mathbf{C}$  to  $\mathbf{D}^{\mathbf{OP}}$ , or equivalently covariant functors from  $\mathbf{C}^{\mathbf{OP}}$  to  $\mathbf{D}$ . The best way to motivate the definition is to give some less trivial examples.

1. Let  $\mathbf{C}$  be the category of all vector spaces over some field, and consider the construction which associates to each vector space its *dual space*  $V^*$  of linear mappings from  $V$  to the scalar field  $F$ . There is a simple way of defining a corresponding construction for morphisms; if  $L : V \rightarrow W$  is a linear transformation, consider the linear transformation  $L^* : W^* \rightarrow V^*$  whose value on a linear functional  $h : W \rightarrow F$  is given by  $L^*(h) = h \circ L$ , which is a linear functional on  $V$ . Standard results in linear algebra show that  $L^*$  is a linear transformation, that  $L^*$  is an identity map if  $L$  is an identity map, and if  $L$  is a composite  $L_1 \circ L_2$ , then we have  $L^* = L_2^* \circ L_1^*$ .
2. There is a contravariant power set functor  $P^*$  on the category  $\mathbf{SETS}$  defined as follows: The set  $P^*(X)$  is just the set of all subsets, but now if  $f : X \rightarrow Y$  is a set-theoretic

function, then  $P^*(f) : P^*(Y) \rightarrow P^*(X)$  takes an element  $B \in P(Y)$  — which by definition is just a subset of  $Y$  — to its **inverse image**  $f^{-1}[B] \subset X$ . As in the case of  $P_*$ , a short elementary argument is needed to verify this construction actually defines a contravariant functor. The construction preserves identity maps because  $1_X^{-1}[B] = B$  for all  $B \subset X$ , and the identity  $P^*(g \circ f) = P^*(f) \circ P^*(g)$  is essentially a restatement of the elementary identity  $f^{-1}[g^{-1}[B]] = (g \circ f)^{-1}[B]$ .

3. Example 2 actually yields a little more. Define a *Boolean algebra* to be a set with two binary operations  $\cap$  and  $\cup$ , a unary operation  $x \rightarrow x'$ , and special elements 0 and 1 such that the system satisfies the usual properties for unions, intersections, and complementation for the algebra  $P(X)$  of subsets of a set  $X$ , where 0 corresponds to the empty set and 1 corresponds to  $X$ . One then has an associated category **BOOL-ALG** whose objects are Boolean algebras and whose morphisms preserve unions, intersection, complementation, and the special elements. Obviously each power set  $P(X)$  is a Boolean algebra, and in fact  $P^*$  defines a contravariant functor from **SETS** to **BOOL-ALG**. — In contrast, the covariant functor  $P_*$  does **NOT** define such a functor because  $P_*(f)$  does not preserve intersections even though it does preserve unions (for example, we can have  $f[A] \cap f[B] \neq \emptyset$  when  $A \cap B = \emptyset$ ).
4. The desirability of having both contravariant and covariant functors is illustrated by the following examples. Given a category  $\mathbf{C}$ , modulo foundational questions we can informally view the set  $\text{Morph}_{\mathbf{C}}(X, Y)$  of morphisms from  $X$  to  $Y$  as a function of two variables on  $\mathbf{C}$ . What happens if we hold one of these variables constant to get a single variable construction? — Suppose first that we hold  $X$  constant and set  $A_X(Y) = \text{Morph}_{\mathbf{C}}(X, Y)$ . Then we can make  $A_X$  into a covariant functor as follows: Given a morphism  $g : Y \rightarrow Z$ , let  $A_X(g)$  take  $f \in A_X(Y) = \text{Morph}_{\mathbf{C}}(X, Y)$  to the composite  $g \circ f$ . The axioms for a category then imply that  $A_X(1_Y)$  is the identity and that  $A_X(h \circ g) = A_X(h) \circ A_X(g)$  if  $g$  and  $h$  are composable. — Now suppose that we hold  $Y$  constant and set  $B_Y(X) = \text{Morph}_{\mathbf{C}}(X, Y)$ . Then we can make  $B_Y$  into a **contravariant** functor as follows: Given a morphism  $k : W \rightarrow X$ , let  $B_Y(g)$  take  $f \in B_Y(X) = \text{Morph}_{\mathbf{C}}(X, Y)$  to the composite  $f \circ k$ . The axioms for a category then imply that  $B_Y(1_X)$  is the identity and that  $B_Y(k \circ h) = B_Y(h) \circ B_Y(k)$  if  $h$  and  $k$  are composable.
5. Given a topological space  $X$ , let **CComp**( $X$ ) and **AComp**( $X$ ) denote the sets of components and arc components of  $X$  respectively. These constructions extend to covariant functors from the category of topological spaces and continuous maps to the category of sets because a continuous map  $f : X \rightarrow Y$  sends a component or arc component of  $X$  into a single component or arc component of  $Y$ . Similarly, there is a notion of *quasicomponents* for topological space of a space is contained in a quasicomponent. component or arc component of  $Y$ . Similarly, there is a notion of *quasicomponents* for topological space such that each connected component lies in a quasicomponent (but the converse might not hold). This notion is described in Exercise 10 on page 163 of Munkres (see also Exercise 4 on page 236). One can show that if  $f : X \rightarrow Y$  is continuous, the image of a quasicomponent of  $X$  is contained in a quasicomponent of  $Y$  (prove this!), it follows that one has a third functor **QComp**( $X$ ).
6. In the preceding example, suppose that  $\mathbf{C}$  is the category of topological spaces and continuous mappings, and let  $Y$  be the real numbers with the usual topology. In this case the contravariant functor  $B_Y$  has the algebraic structure of a commutative ring with unit given by pointwise multiplication of continuous real valued functions, and if  $f : W \rightarrow X$  is continuous then  $B_Y(f)$  is in fact a homomorphism of commutative rings with unit.

Therefore, if we define a category of continuous rings with unit (whose morphisms are unit preserving homomorphisms), it follows that  $B_Y$  defines a functor from topological spaces and continuous mappings to commutative rings with unit. — In contrast, there is no comparable structure for the covariant functor  $B_X$  if  $X$  is the real numbers.

### *Properties of functors*

One of the most important properties of functors is that they send isomorphic objects in one category to isomorphic objects in the other.

**PROPOSITION 2.** *Let  $\mathbf{C}$  and  $\mathbf{D}$  be categories, let  $T : \mathbf{C} \rightarrow \mathbf{D}$  be a (covariant or contravariant) functor, and let  $f : X \rightarrow Y$  be an isomorphism in  $\mathbf{C}$ . Then  $T(f)$  is an isomorphism in  $\mathbf{D}$ . Furthermore, if  $g$  is the inverse to  $f$ , then  $T(g)$  is the inverse to  $T(f)$ .*

**Proof.** CASE 1. Suppose the functors are covariant. Then we have

$$1_{T(X)} = T(1_X) = T(g \circ f) = T(g) \circ T(f)$$

$$1_{T(Y)} = T(1_Y) = T(f \circ g) = T(f) \circ T(g)$$

and hence  $T(g)$  is inverse to  $T(f)$ .

CASE 2. Suppose that the functors are contravariant. Then we have

$$1_{T(X)} = T(1_X) = T(g \circ f) = T(f) \circ T(g)$$

$$1_{T(Y)} = T(1_Y) = T(f \circ g) = T(g) \circ T(f)$$

and hence  $T(g)$  is inverse to  $T(f)$ .■

The next result states that a composite of two functors is also a functor.

**PROPOSITION 3.** *Suppose that  $\mathbf{C}$ ,  $\mathbf{D}$  and  $\mathbf{E}$  are categories and that  $F : \mathbf{C} \rightarrow \mathbf{D}$  and  $G : \mathbf{D} \rightarrow \mathbf{E}$  are functors (in each case, the functor may be covariant or contravariant). Then the composite  $G \circ F$  also defines a functor; this functor is covariant if  $F$  and  $G$  are both covariant or contravariant, and it is contravariant if one of  $F$ ,  $G$  is covariant and the other is contravariant.■*

This result has a curious implication:

**COROLLARY 4.** *There is a “category of small categories” **SMCAT** whose objects are small categories and whose morphisms are covariant functors from one small category to another.■*

**SEMANTIC TRIVIA.** (For readers who are familiar with contravariant and covariant tensors.) In the applications of linear algebra to differential geometry and topology, one often sees objects called *contravariant tensors* and *covariant tensors*, and for finite-dimensional vector spaces these are given by finitely iterated tensor products  $V \otimes \cdots \otimes V$  of  $V$  with itself in the contravariant case and similar objects involving  $V^*$  in the covariant case; for our purposes it will suffice to say that if  $U$  and  $W$  are vector spaces with bases  $\{\mathbf{u}_i\}$  and  $\{\mathbf{w}_j\}$  respectively, then their tensor product  $U \otimes W$  is a vector space having a basis of the form  $\{\mathbf{u}_i \otimes \mathbf{w}_j\}$  where  $i$  and  $j$  are allowed to vary independently (hence the dimension of  $U \otimes W$  is  $[\dim U] \cdot [\dim W]$ ). Since the identity functor on the category of vector spaces is covariant and the dual space functor is contravariant, at first it might seem that something is the opposite of what it should be. However, the classical tensor notation refers to the manner in which the **coordinates** transform; now coordinates for a vector space may be viewed linear functionals on that space, or equivalently as elements of the dual space, which is

contravariant. Therefore individual coordinates on  $V \otimes \cdots \otimes V$  correspond to elements of the *dual space* of the latter, and in fact the construction which associates the space  $(V \otimes \cdots \otimes V)^*$  to  $V$  defines a *contravariant* functor on the category of finite-dimensional vector spaces over the given scalars; likewise, the construction which associates the space  $(V^* \otimes \cdots \otimes V^*)^*$  to  $V$  defines a **covariant** functor on the category of finite-dimensional vector spaces over the given scalars.

### *Natural transformations*

The final concept in category theory to be considered here is the notion of **natural transformation** from one functor to another. In fact, the motivation for category theory in the work of Eilenberg and MacLane was a need to discuss “natural mappings” in a mathematically precise manner. There are actually two definitions, depending whether both functors under consideration are covariant or contravariant.

**Definition.** Let  $\mathbf{C}$  and  $\mathbf{D}$  be categories, and let  $F$  and  $G$  be covariant functors from  $\mathbf{C}$  to  $\mathbf{D}$ . A *natural transformation*  $\theta$  from  $F$  to  $G$  associates to each object  $X$  in  $\mathbf{C}$  a morphism  $\theta_X : F(X) \rightarrow G(X)$  such that for each morphism  $f : X \rightarrow Y$  we have  $\theta_Y \circ F(f) = G(f) \circ \theta_X$ .

The morphism identity is often expressed graphically by saying the the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow \theta_X & & \downarrow \theta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

is a **commutative diagram**. The idea is that all paths of arrows from one object-vertex to another yield the same function.

The definition of a natural transformation of contravariant functors is similar.

**Definition.** Let  $\mathbf{C}$  and  $\mathbf{D}$  be categories, and let  $T$  and  $U$  be contravariant functors from  $\mathbf{C}$  to  $\mathbf{D}$ . A *natural transformation*  $\theta$  from  $T$  to  $U$  associates to each object  $X$  in  $\mathbf{C}$  a morphism  $\theta_X : T(X) \rightarrow U(X)$  such that for each morphism  $f : X \rightarrow Y$  we have  $\theta_X \circ T(f) = U(f) \circ \theta_Y$ .

Here is the corresponding commutative diagram:

$$\begin{array}{ccc} T(Y) & \xrightarrow{T(f)} & T(X) \\ \downarrow \theta_Y & & \downarrow \theta_X \\ U(Y) & \xrightarrow{U(f)} & U(X) \end{array}$$

Once again we need to give some decent examples

1. Given any functor  $T : \mathbf{C} \rightarrow \mathbf{D}$ , there is an obvious identity transformation  $j^T$  from  $T$  to itself; specifically,  $j_X^T$  is the identity map on  $T(X)$ .
2. Let  $\mathbf{C}$  be one of the categories as above for which reasonable products and diagonal maps can be defined. Then there is a natural diagonal transformation  $\Delta$  from the identity to the diagonal functor such that for each object  $X$  the mapping  $\Delta_X : X \rightarrow X \times X$  is the diagonal map.
3. On the category of vector spaces over some field  $F$ , one can iterate the dual space functor to obtain a covariant *double dual space functor*  $(V^*)^*$ . There is a natural transformation

$e_V : V \rightarrow (V^*)^*$  defined as follows: For each  $\mathbf{v} \in V$ , let  $e_V(\mathbf{v}) : V^* \rightarrow F$  be the linear function given by evaluation at  $v$ ; in other words, the value of  $e_V(\mathbf{v})$  on a linear functional  $f$  is given by  $f(\mathbf{v})$ . If  $V$  is finite-dimensional, this map is an isomorphism (a *natural isomorphism*).

Note that if  $V$  is finite-dimensional then  $V$  and its dual space  $V^*$  are isomorphic, but the isomorphism depends upon some additional data such as the choice of a basis, an inner product, or more generally a nondegenerate bilinear form. In contrast, *the natural isomorphism  $e_V$  does not depend upon any such choices*.

4. In the category of sets or topological spaces and continuous mappings, let  $A$  be an arbitrary object and define functors  $L_A$  and  $R_A$  such that  $L_A(X) = A \times X$  and  $R_A(X) = X \times A$ . One can make these into covariant functors by sending the morphism  $f : X \rightarrow Y$  to  $L_A(f) = 1_A \times f$  and  $R_A(f) = f \times 1_A$ . There is an obvious natural transformation  $t : L + A \rightarrow R_A$  such that  $t_A(X) : A \times X \rightarrow X \times A$  sends  $(a, x)$  to  $(x, a)$  for all  $a \in A$  and  $x \in X$ , and it is an elementary exercise to verify that this is a natural transformation such that each map  $t_A(X)$  is an *isomorphism*; in other words,  $t_A$  is a natural isomorphism from the functor  $L_A$  to the functor  $R_A$ .
5. For the morphism examples  $A_X$  and  $B_Y$  discussed previously, if  $h : W \rightarrow X$  is a morphism in the category, then it defines a natural transformation  $h^* : A_X \rightarrow A_W$  which sends  $f \in A_X(Y)$  to  $f \circ h \in A_W(Y)$ ; the naturality condition follows from associativity of composition. Similarly, if  $g : Y \rightarrow Z$  is a morphism then there is a natural transformation  $g_* : B_Y \rightarrow B_Z$  sending  $f$  to  $g \circ f$ ; once again, the key naturality condition follows from the associativity of composition. Furthermore,  $h^*$  is a natural isomorphism if  $h$  is an isomorphism and  $g_*$  is a natural isomorphism if  $g$  is an isomorphism,
6. For the arc component, connected component and quasicomponent functors described above, there are natural transformations  $\theta : \mathbf{AComp} \rightarrow \mathbf{CComp}$  reflecting the fact that every arc component of a topological space  $X$  is contained in a connected component and  $\psi : \mathbf{CComp} \rightarrow \mathbf{QComp}$  reflecting the fact that every connected component of a space is contained in a quasicomponent.

A basic exercise in category theory is to prove the following:

**PROPOSITION 5.** *There are 1 – 1 correspondences between natural transformations from  $A_X$  to  $A_W$  and morphisms from  $W$  to  $X$  and between natural transformations from  $B_Y$  to  $B_Z$  and morphisms from  $Y$  to  $Z$ .*

**Sketch of proof.** The main point is to retrieve the function from the natural transformation. Given  $\theta : A_X \rightarrow A_W$ , one does this by considering the image of  $1_X$ , and given  $\varphi : B_Y \rightarrow B_Z$ , one does this by considering the image of  $1_Y$ . ■

Finally, we have the following result on natural isomorphisms (*i.e.*, natural transformations  $\theta$  such that each map  $\theta_X$  is an isomorphism):

**PROPOSITION 6.** *Let  $\theta : F \rightarrow G$  be a natural transformation such that for each object  $X$  the map  $\theta_X$  is an isomorphism. Then there is a natural transformation  $\varphi_X : G \rightarrow F$  such that for each  $X$  the map  $\varphi_X$  is inverse to  $\theta_X$ .*

**Proof.** The main thing to check is that the relevant diagrams are commutative; we shall only do the case where  $F$  and  $G$  are covariant, leaving the other case to the reader. Since  $\theta_X \circ \varphi_X$  is the identity on  $G(X)$  and  $\varphi_X \circ \theta_X$  is the identity on  $F(X)$ , we have

$$\theta_Y \circ \varphi_Y \circ G(f) = G(f) = g(f) \circ \theta_X \circ \varphi_X = \theta_Y \circ F(f) \circ \varphi_X$$

and if we compose with the inverse  $\theta_X$  on the left of these expressions we obtain

$$\varphi_Y \circ G(f) = F(f) \circ \varphi_X$$

which is the naturality condition. ■

We say that two functors are *naturally isomorphic* if there is a natural isomorphism from one to the other.

### *Equivalences of categories*

One can obviously define an isomorphism of categories to be a covariant functor  $T : \mathbf{C} \rightarrow \mathbf{D}$  for which there is an inverse covariant functor  $U : \mathbf{D} \rightarrow \mathbf{C}$  such that the composites  $T \circ U$  and  $U \circ T$  are the identities on  $\mathbf{C}$  and  $\mathbf{D}$  respectively. However, for many purposes one needs a less rigid notion of category equivalence.

**Definition.** A covariant functor  $T : \mathbf{C} \rightarrow \mathbf{D}$  is a *category equivalence* (or equivalence of categories) if there is a covariant functor  $U : \mathbf{D} \rightarrow \mathbf{C}$  such that the composites  $T \circ U$  and  $U \circ T$  are *naturally isomorphic* to the identities on  $\mathbf{C}$  and  $\mathbf{D}$  respectively.

In particular, if  $T$  and  $U$  define an equivalence of categories, then every object in  $\mathbf{D}$  is isomorphic to an object of the form  $T(X)$ , and conversely every object in  $\mathbf{C}$  is isomorphic to an object of the form  $U(A)$ .

## **I.2 : Barycentric coordinates and polyhedra**

(Hatcher, § 2.1)

Drawings to illustrate many of the concepts in this and other sections of the notes can be found in the following document(s):

<http://math.ucr.edu/~res/math246A/algotop1figures01w09.pdf>

A more leisurely and detailed discussion of barycentric coordinates, and more generally the use of linear algebra to study geometric problems, is contained in Section I.4 of the following online document, in which \* is one of the options in the preceding paragraph:

<http://math.ucr.edu/~res/math133/geometrynotes1.pdf>

The file `math133exercises1.pdf` in the same directory has further material on these topics, and pages 13–30 of

<http://math.ucr.edu/~res/progeom/pgnotes02.pdf>

go further into the geometric uses of barycentric coordinates. Another standard reference is Chapter I of the following book:

**J. F. P. Hudson.** *Piecewise Linear Topology.* W. A. Benjamin, New York, 1969.  
(Online: <http://www.maths.ed.ac.uk/~aar/surgery/hudson.pdf>)

An extremely detailed study of the topics in this section appears in the following online book:

<http://www.cis.penn.edu/~jean/gbooks/convexpoly.html>

Finally, Eilenberg and Steenrod also covers the portions this material needed for algebraic and geometric topology in greater detail.

### *Affine independence and barycentric coordinates*

The crucial algebraic information is contained in the following result.

**PROPOSITION 1.** *Suppose that the ordered set of vectors  $\mathbf{v}_0, \dots, \mathbf{v}_n$  lie in some vector space  $V$ . Then the vectors  $\mathbf{v}_1 - \mathbf{v}_0, \dots, \mathbf{v}_n - \mathbf{v}_0$  are linearly independent if and only if every vector  $\mathbf{x} \in V$  has at most one expansion of the form  $t_0\mathbf{v}_0 + \dots + t_n\mathbf{v}_n$  such that  $t_0 + \dots + t_n = 1$ .*

A finite ordered set of vectors satisfying either (hence both) conditions is said to be *affinely independent*. Note that since the second condition does not depend upon the choice of ordering, a set of vectors is affinely independent if and only if for some arbitrary  $j$  the vectors  $\mathbf{v}_i - \mathbf{v}_j$  (where  $i \neq j$ ) is linearly independent. A linear combination in which the coefficients add up to 1 is called an *affine combination*.

**Sketch of proof.** To show the first statement implies the second, use the fact that  $\mathbf{x} - \mathbf{v}_0$  has at most one expansion as a linear combination of  $\mathbf{v}_1 - \mathbf{v}_0, \dots, \mathbf{v}_n - \mathbf{v}_0$ . To prove the reverse implication, show that if  $\mathbf{x} - \mathbf{v}_0$  has more than one expansion as a linear combination of  $\mathbf{v}_1 - \mathbf{v}_0, \dots, \mathbf{v}_n - \mathbf{v}_0$ , then  $\mathbf{x}$  has more than one expansion as an affine combination of  $\mathbf{v}_0, \dots, \mathbf{v}_n$ . ■

**COROLLARY 2.** *If  $S = \{\mathbf{v}_0, \dots, \mathbf{v}_n\}$  is affinely independent, then every nonempty subset of  $S$  is affinely independent.*

This follows immediately from the uniqueness of expansions of vectors as affine combinations of vectors in  $S$ . ■

The coefficients  $t_i$  are called **barycentric coordinates**. If we put physical weights of  $t_i$  units at the respective vertices  $\mathbf{v}_i$ , then the center of gravity for the system will be at the point  $t_0\mathbf{v}_0 + \dots + t_n\mathbf{v}_n$ . If, say,  $n = 2$ , then this center of gravity will be inside the triangle with the given three vertices if and only if each  $t_i$  is positive, and it will be on the triangle defined by these vertices if and only if each  $t_i$  is nonnegative and at least one is equal to zero. A discussion of this physical interpretation in the 2-dimensional case appears in the following online document:

<http://math.ucr.edu/~res/math133/centroids.pdf>

We should note that the discussion in this online reference can be extended to arbitrary (finite) dimensions.

More generally, if  $\mathbf{v}_0, \dots, \mathbf{v}_n$  are affinely independent then the  **$n$ -simplex** with vertices  $\mathbf{v}_0, \dots, \mathbf{v}_n$  is the set of all points expressible as affine combinations such that each coefficient is nonnegative (*i.e.*, **convex combinations**).

Frequently the  $n$ -simplex described above will be denoted by  $\mathbf{v}_0 \cdots \mathbf{v}_n$ . Note that if  $n = 0$ , then a 0-simplex consists of a single point, while a 1-simplex is a closed line segment, a 2-simplex is given by a triangle and the points that lie “inside” the triangle (also called a *solid triangular region*), and a 3-simplex is given by a pyramid with a triangular base (*i.e.*, a *tetrahedron*) together with the points inside this pyramid (also called a *solid tetrahedral region*).

The following definition will also play an important role in our discussions.

**Definition.** If  $\mathbf{v}_0, \dots, \mathbf{v}_n$  form the vertices of a simplex  $\mathbf{v}_0 \cdots \mathbf{v}_n$ , then the **faces** of this simplex are the simplices whose vertices are given by proper subsets of  $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ ; note that such proper subsets are affinely independent by Corollary 2. If a proper subset  $T \subset S$  has  $k + 1$



elements, then we shall say that the simplex  $\Delta(T)$  whose vertices are given by  $T$  is a  $k$ -**face** of the original  $n$ -simplex, which in this notation is equal to  $\Delta(S)$ .

**Definition.** The *standard  $n$ -simplex*  $\Delta_n$  is the set of all points  $(t_0, \dots, t_n) \in \mathbb{R}^{n+1}$  such that  $t_j \geq 0$  for all  $j$  and  $\sum_j t_j = 1$ . Note that the set of unit vectors  $\{\mathbf{e}_0, \dots, \mathbf{e}_n\}$  is affinely independent because the set  $\{\mathbf{e}_1 - \mathbf{e}_0, \dots, \mathbf{e}_{n+1} - \mathbf{e}_0\}$  is linearly independent.

### *Sets with simplicial decompositions*

In calculus textbooks, the derivation of Green's Theorem is often completed only for special sorts of closed regions such as the simplex whose vertices are  $(0,0)$ ,  $(1,0)$  and  $(1,1)$ . One then finds discussions indicating how the general case can be retrieved from special cases by splitting a general region into pieces that are nicely homeomorphic to closed regions of the special type; in particular, there is one such discussion on page 523 of the text by Marsden and Tromba, and it is taken further in the online document with figures for these notes (see Figure I.2.8 in the document `algotop1figures01w09.pdf`).

Here are the formal descriptions.

**Definition.** A subset  $P \subset \mathbb{R}^m$  is a *polyhedron* if

- (i)  $P$  is a finite union of simplices  $A_1, \dots, A_q$ ,
- (ii) For each pair of indices  $i \neq j$ , the intersection  $A_i \cap A_j$  is a common face.

The simplices  $A_1, \dots, A_q$  are said to form a *simplicial decomposition* of  $P$ , and if  $\mathbf{K}$  is the collection of simplices given by the  $A_j$  and all their faces, then the ordered pair  $(P, \mathbf{K})$  is called a (finite) *simplicial complex*.

If  $X$  is an arbitrary topological space, then a (finite) *triangulation* of  $X$  consists of a simplicial complex  $(P, \mathbf{K})$  and a homeomorphism  $t : P \rightarrow X$ .

With these definitions, we can say that Green's Theorem holds for "decent" closed plane regions because Such regions have nice triangulations.

**SIMPLE EXAMPLE.** Consider the solid rectangle in the plane given by  $[a, b] \times [c, d]$ , where  $a < b$  and  $c < d$ . Everyday geometrical experience shows this can be split into two 2-simplices along a diagonal, and in fact it is the union of two 2-simplices, one with vertices  $(a, c)$ ,  $(a, d)$  and  $(b, d)$ , and the other with vertices  $(a, c)$ ,  $(b, c)$  and  $(b, d)$ . A point  $(x, y)$  which lies in the solid rectangle will be in the first simplex if and only if

$$(y - c)(b - a) \leq (d - c)(x - a)$$

and this point will be in the second simplex if and only if

$$(y - c)(b - a) \geq (d - c)(x - a)$$

Generalizations of this example will play an important role in the standard approach to algebraic topology.

If  $(P, \mathbf{K})$  is a simplicial complex, then a subset  $\mathbf{L} \subset \mathbf{K}$  is said to be a *subcomplex* if  $\sigma \in \mathbf{L}$  implies that every face of  $\sigma$  also lies in  $\mathbf{L}$ . The union of the simplices in  $\mathbf{L}$  is a closed subspace of  $P$  which is denoted by  $|\mathbf{L}|$ . With this notation we have  $P = |\mathbf{K}|$ .

LINEAR GRAPHS. The final chapter of Munkres studies 1-dimensional complexes (called *linear graphs* on p. 394) in considerable detail, and the **commentaries** file in the 205B directory contains some comments (see the discussion for Section 64 which begins at the bottom of page 29 and continues into page 30). One way of viewing this section and the next is to think of them as laying the foundations for effective study of similar objects in higher dimensions.

The study of 1-dimensional complexes is the subject called *graph theory*; it is significant for both its theory and applications, but all of this is well beyond the scope of this course. Here are some written and electronic references:

**J. A. Bondy and U. S. R. Murty.** *Graph Theory: An Advanced Course*. Springer-Verlag, New York-etc., 2008. ISBN: 1-846-28969-6.

**G. Chartrand.** *Introductory Graph Theory [UNABRIDGED]*. Dover Publications, New York, 1984. ISBN: 0-486-24775-9.

[http://en.wikipedia.org/wiki/Graph\\_theory](http://en.wikipedia.org/wiki/Graph_theory)

<http://www.utm.edu/departments/math/graph/>

<http://www.math.fau.edu/locke/GRAPHTHE.HTM>

[http://www.math.uni-hamburg.de/home/diestel/books/\[continue\]graph.theory/GraphTheoryIII.counted.pdf](http://www.math.uni-hamburg.de/home/diestel/books/[continue]graph.theory/GraphTheoryIII.counted.pdf)

SIMPLICIAL COMPLEXES AND  $\Delta$ -COMPLEXES. Our definition of simplicial complex is more restrictive than Hatcher's definition; this is explained on page 107 of Hatcher (see the third paragraph following Example 2.5). Each concept has its advantages and disadvantages. However, terms like  $\Delta$ -complex or  $\Delta$ -set are often also used for other constructions, and one should not assume that the meanings in other publications are "obviously" equivalent to the meaning in Hatcher.

#### *Decompositions of prisms*

The rectangular example has the following important generalization:

**PROPOSITION 3.** *Suppose that  $A \subset \mathbb{R}^m$  is a simplex with vertices  $\mathbf{v}_0, \dots, \mathbf{v}_n$ . Then  $A \times [0, 1] \subset \mathbb{R}^{m+1}$  has a simplicial decomposition with exactly  $n + 1$  simplices of dimension  $n + 1$ .*

**Proof.** For each  $i$  between 0 and  $n$  let  $\mathbf{x}_i = (\mathbf{v}_i, 0)$  and  $\mathbf{y}_i = (\mathbf{v}_i, 1)$ . We claim that the vectors

$$\mathbf{x}_0, \dots, \mathbf{x}_i, \mathbf{y}_i, \dots, \mathbf{y}_n$$

are affinely independent and the corresponding simplices

$$\mathbf{x}_0 \cdots \mathbf{x}_i \mathbf{y}_i \cdots \mathbf{y}_n$$

(where  $0 \leq i \leq n$ ) form a simplicial decomposition of  $A \times [0, 1]$ .

An illustration for the case  $n = 2$  is given in Figure I.2.11 of [algotop1figures01w09.pdf](#).

To prove affine independence, take a fixed value of  $i$  and suppose we have

$$\sum_{j < i} t_j \mathbf{x}_j + a \mathbf{x}_i + b \mathbf{y}_i + \sum_{j > i} t_j \mathbf{y}_j =$$

$$\sum_{j < i} t'_j \mathbf{x}_j + a' \mathbf{x}_i + b' \mathbf{y}_i + \sum_{j > i} t'_j \mathbf{y}_j$$

where the coefficients in each expression add up to 1; the summation will be taken to be zero if the limits reduce to  $j < 0$  or  $j > n$ . If we view  $\mathbb{R}^{m+1}$  as  $\mathbb{R}^m \times \mathbb{R}$  and project down to  $\mathbb{R}^m$  we obtain the equation

$$\sum_{j < i} t_j \mathbf{v}_j + (a+b) \mathbf{x}_i + \sum_{j > i} t_j \mathbf{v}_j = \sum_{j < i} t'_j \mathbf{v}_j + (a'+b') \mathbf{v}_i + \sum_{j > i} t'_j \mathbf{v}_j$$

and by the affine independence of the vectors  $\mathbf{v}_k$  it follows that  $t_j = t'_j$  if  $j \neq i$  and also that  $a+b = a'+b'$ . On the other hand, if we project down to the second coordinate (the copy of  $\mathbb{R}$ ), then we obtain

$$b + \sum_{j > i} t_j = b' + \sum_{j > i} t'_j$$

and since  $t_j = t'_j$  for all  $j$  it follows that  $b = b'$ . Finally, since the sum of all the coefficients is equal to 1, the preceding observations imply that  $1-a = 1-a'$ , and therefore we also have  $a = a'$ . Therefore the vectors

$$\mathbf{x}_0, \dots, \mathbf{x}_i, \mathbf{y}_i, \dots, \mathbf{y}_n$$

are affinely independent.

We shall next check that every point  $(\mathbf{z}, u) \in A \times [0, 1]$  lies in one of the simplices

$$\mathbf{x}_0 \cdots \mathbf{x}_i \mathbf{y}_i \cdots \mathbf{y}_n$$

listed above. Write  $\mathbf{z} = \sum_j t_j \mathbf{v}_j$  where  $t_j \geq 0$  for all  $j$  and  $\sum t_j = 1$ . It follows that  $u \leq 1 = \sum_{j \geq 0} t_j$ ; let  $i \leq n$  be the largest nonnegative integer such that  $u \leq \sum_{j \geq i} t_j$ . We claim that  $(\mathbf{z}, u)$  lies in the simplex  $\mathbf{x}_0 \cdots \mathbf{x}_i \mathbf{y}_i \cdots \mathbf{y}_n$ . Let  $b = \sum_{j \geq i} (t_j - u)$ , and let  $a = u - \left( \sum_{j > i} t_j \right) = t_i - b$ . Then we have  $a, b \geq 0$ , and

$$(\mathbf{z}, u) = \sum_{j < i} t_j \mathbf{x}_j + a \mathbf{x}_i + b \mathbf{y}_i + \sum_{j > i} t_j \mathbf{y}_j$$

where all the coefficients are nonnegative and add up to 1.

To conclude the proof, we need to show that the intersection of two simplices as above is a common face. Suppose that  $k < i$  and

$$(\mathbf{z}, u) \in (\mathbf{x}_0 \cdots \mathbf{x}_i \mathbf{y}_i \cdots \mathbf{y}_n) \cap (\mathbf{x}_0 \cdots \mathbf{x}_k \mathbf{y}_k \cdots \mathbf{y}_n).$$

Then we must have

$$\sum_{j \leq i} p_j \mathbf{x}_j + \sum_{j \geq i} q_j \mathbf{y}_j = \sum_{j \leq k} p'_j \mathbf{x}_j + \sum_{j \geq k} q'_j \mathbf{y}_j$$

where all the coefficients are nonnegative and the coefficients on each side of the equation add up to 1. If we project down to  $\mathbb{R}^m$  we obtain  $p_j + q_j = p'_j + q'_j$  for all  $j$  (by convention, we take a coefficient to be zero if it does not lie in the corresponding summation as above). It follows immediately that

$p_j = p'_j$  if  $j < k$ , while  $p_j = q'_j$  if  $k < j < i$  and  $q_j = q'_j$  if  $j > i$ . Furthermore, if we project down to the last coordinate we see that

$$u = \sum_{j \geq i} q_j = \sum_{j \geq k} q'_j.$$

Since  $q_j = q'_j$  if  $j > i$ , it follows that

$$q_i = \sum_{k \leq j \leq i} q'_j$$

and since all the coefficients are nonnegative, it follows that  $q_i \geq q'_i$ . On the other hand, we also have  $q'_i = p'_i + q'_i = p_i + q_i$ , and hence we conclude that  $q_i = q'_i$  and  $p_i = 0$ . Applying the first of these, we see that

$$0 = \sum_{k \leq j < i} q'_j$$

and hence the nonnegativity of the coefficients implies that  $q'_j = 0$  for all  $j$  such that  $k \leq j < i$ . We also know that  $p'_j = 0$  for  $j > k$ , and therefore it follows that  $p'_j + q'_j = 0$  when  $k < j < i$ . The equations  $p_j + q_j = p'_j + q'_j$  and the nonnegativity of all terms now imply that  $p_j = q_j = 0$  when  $k < j < i$ .

The conclusions of the preceding paragraph imply that the point  $(\mathbf{z}, u)$  actually lies on the simplex

$$\mathbf{x}_0 \cdots \mathbf{x}_k \mathbf{y}_i \cdots \mathbf{y}_n$$

and since the latter is a common face of  $\mathbf{x}_0 \cdots \mathbf{x}_i \mathbf{y}_i \cdots \mathbf{y}_n$  and  $\mathbf{x}_0 \cdots \mathbf{x}_k \mathbf{y}_k \cdots \mathbf{y}_n$  it follows that the  $(n+1)$ -simplices

$$\mathbf{x}_0 \cdots \mathbf{x}_i \mathbf{y}_i \cdots \mathbf{y}_n$$

(where  $0 \leq i \leq n$ ) form a simplicial decomposition of  $A \times [0, 1]$ . ■

**COROLLARY 4.** *If  $P \subset \mathbb{R}^m$  is a polyhedron, then  $A \times [0, 1] \subset \mathbb{R}^{m+1}$  is also a polyhedron.*

Before discussing the proof of this we note one important special case.

**COROLLARY 5.** *For each positive integer  $m$ , the hypercube  $[0, 1]^m \subset \mathbb{R}^m$  is a polyhedron.*

**Proof of Corollary 5 from Corollary 4.** If  $m = 1$  this follows because the unit interval is a 1-simplex; by Corollary 4, if the result is true for  $m = k$  then it is also true for  $m = k + 1$ . Therefore the result is true for all  $m$  by induction. ■

**Proof of Corollary 4.** Let  $\mathbf{K}$  be a simplicial decomposition for  $P$ , and let  $\mathbf{K}^*$  be obtained from  $\mathbf{K}$  by including all the faces of simplices in  $\mathbf{K}$ . Choose a linear ordering of the vertices in  $\mathbf{K}^*$  (note that there are only finitely many). For each vertex  $\mathbf{v}$  of  $\mathbf{K}^*$ , as before let  $\mathbf{x} = (\mathbf{v}, 0)$  and  $\mathbf{y} = (\mathbf{v}, 1)$ . Then  $P \times [0, 1]$  is the union of all simplices of the form

$$\mathbf{x}_0 \cdots \mathbf{x}_i \mathbf{y}_i \cdots \mathbf{y}_n$$

where  $\mathbf{v}_i < \mathbf{v}_{i+1}$  with respect to the given linear ordering of the vertices in  $\mathbf{K}^*$ , and furthermore the vertices  $\mathbf{v}_i$  are the vertices of a simplex in  $\mathbf{K}^*$ . The set  $P \times [0, 1]$  is the union of these simplices by Proposition 3 and the fact that  $P$  is the union of the simplices  $\mathbf{v}_0 \cdots \mathbf{v}_n$ . The fact that these simplices form a simplicial decomposition will follow from the construction and the next result.

**LEMMA 6.** *Suppose that we have two polyhedra  $P_1$  and  $P_2$  in  $\mathbb{R}^q$ , and there exist simplicial decompositions  $\mathbf{K}_1$  and  $\mathbf{K}_2$  such that the following hold:*

(i) Both  $\mathbf{K}_1$  and  $\mathbf{K}_2$  are closed under taking faces of simplices.

(ii) The set  $\mathbf{L}_1$  of all simplices in  $\mathbf{K}_1$  contained in  $P_1 \cap P_2$  equals the set  $\mathbf{L}_2$  of all simplices in  $\mathbf{K}_2$ , and this collection determines a simplicial decomposition of  $P_1 \cap P_2$ .

Then  $\mathbf{K}_1 \cup \mathbf{K}_2$  determines a simplicial decomposition of  $P_1 \cup P_2$ .

The hypothesis clearly applies to the construction in Proposition 3, so Corollary 4 indeed follows once we prove Lemma 6. ■

**Proof of Lemma 6.** It follows immediately that  $P_1 \cup P_2$  is the union of the points of the simplices in  $\mathbf{K}_1 \cup \mathbf{K}_2$ . Suppose now that we are given an intersection of two simplices in the latter. This intersection will be a common face if both simplices lie in either  $\mathbf{K}_1$  or  $\mathbf{K}_2$ , so the only remaining cases are those where one simplex  $\alpha$  lies in  $\mathbf{K}_1$  and the other simplex  $\beta$  lies in  $\mathbf{K}_2$ .

We know that  $\alpha \cap \beta$  is convex. Furthermore, by the hypotheses we know that  $\alpha \cap \beta$  must be a union of simplices that are faces of both  $\alpha$  and  $\beta$ . Therefore it follows that every point in  $\alpha \cap \beta$  is a convex combination of the vertices which lie in  $\alpha \cap \beta$ , and consequently  $\alpha \cap \beta$  is the common face determined by all vertices which lie in  $\alpha \cap \beta$ . ■

**GENERALIZATIONS — CONVEX LINEAR CELLS.** [Also known as CONVEX POLYTOPES] These are closed bounded subsets of some  $\mathbb{R}^n$  defined by a finite number of linear equations or inequalities. Note that sets defined by finite systems of this type are automatically convex. Prisms, simplices and cubes are obvious examples, but of course there are also many others. For every such object, there is a finite set  $E$  of *extreme points* such that the cell is the set of all convex combinations of the extreme points; in other words, for each  $\mathbf{x}$  in the cell and each extreme point  $\mathbf{e}$  there are scalars  $t_{\mathbf{e}}$  such that  $t_{\mathbf{e}} \geq 0$ ,  $\sum_{\mathbf{e}} t_{\mathbf{e}} = 1$ , and  $x = \sum_{\mathbf{e}} t_{\mathbf{e}} \mathbf{e}$ . A basic theorem states that every convex linear cell has a simplicial decomposition for which  $E$  is the set of vertices. Proofs of this statement appear in [MunkresEDT] and the book by Hudson; we shall discuss some additional facts about such objects later in these notes.

Some easily stated but challenging problems on convex polytopes in  $\mathbb{R}^3$  are contained in the file `swGeometrytest.pdf`, and solutions to these exercises using vector geometry are given in the file `swvectorproofs.pdf`.

**DEFAULT HYPOTHESIS.** Unless specifically indicated otherwise, we shall assume that the set of simplices in a simplicial decomposition  $\mathbf{K}$  is closed under taking faces. In order to justify this, we need to know that if  $\mathbf{K}^*$  is obtained from  $\mathbf{K}$  by adding all the faces of simplices in the latter, then the intersection of two simplices in  $\mathbf{K}^*$  is a (possibly empty) common face. — To see this, suppose that  $\alpha$  and  $\beta$  are simplices in  $\mathbf{K}^*$ , where  $\alpha$  and  $\beta$  are faces of the simplices  $\alpha'$  and  $\beta'$  in  $\mathbf{K}$ . If  $\mathbf{x} \in \alpha \cap \beta$ , then  $\mathbf{x}$  is a convex combination of vertices in  $\alpha' \cap \beta'$ , and in fact these vertices must lie in both  $\alpha$  and  $\beta$ . Since  $\alpha \cap \beta$  is convex, it follows that  $\alpha \cap \beta$  must be the simplex whose vertices lie in  $\alpha$  and in  $\beta$ . ■

### I.3 : Subdivisions

(Hatcher, § 2.1)

For many purposes it is convenient or necessary to replace a simplicial decomposition  $\mathbf{K}$  of a polyhedron  $P$  by another decomposition  $\mathbf{L}$  with smaller simplices. More precisely, we would like the smaller simplices in  $\mathbf{L}$  to determine simplicial decompositions for each of the simplices in  $\mathbf{K}$ .

### Simple examples

1. If  $P$  is a 1-simplex with vertices  $\mathbf{x}$  and  $\mathbf{y}$ , and  $\mathbf{K}$  is the standard decomposition given by  $P$  and the endpoints, then there is a subdivision  $\mathbf{L}$  given by trisecting  $P$ ; specifically, the vertices are given by  $\mathbf{x}$ ,  $\mathbf{y}$ ,  $\mathbf{z} = \frac{2}{3}\mathbf{x} + \frac{1}{3}\mathbf{y}$ , and  $\mathbf{w} = \frac{1}{3}\mathbf{x} + \frac{2}{3}\mathbf{y}$ , and the 1-simplices are  $\mathbf{xw}$ ,  $\mathbf{wz}$  and  $\mathbf{zy}$ . This is illustrated as Figure I.3.1 in the file `algotop1figures01w09.pdf`.
2. Similarly, if  $[a, b]$  is a closed interval in the real line and we are given a finite sequence  $a = t_0 < \cdots < t_m = b$ , then these points and the intervals  $[t_{j-1}, t_j]$ , where  $1 \leq j \leq m$ , form a subdivision of the standard decomposition of  $[a, b]$ .
3. If  $P$  is the 2-simplex with vertices  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z}$ , and  $\mathbf{K}$  is the standard decomposition given by  $P$  and its faces, then there is an obvious decomposition  $\mathbf{L}$  which splits  $P$  into two simplices  $\mathbf{xyz}$  and  $\mathbf{xyw}$ , where  $\mathbf{w} = \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}$  is the midpoint of the 1-simplex  $\mathbf{yz}$ . Similar examples exist if we take  $\mathbf{z} = a\mathbf{y} + (1 - a)\mathbf{z}$ , where  $a$  is an arbitrary number such that  $0 < a < 1$  (see Figure I.3.2 in the file `algotop1figures01w09.pdf`).

### Definition of subdivisions

Each of the preceding examples is consistent with the following general concept.

**Definition.** Let  $(P, \mathbf{K})$  be a simplicial complex, and let  $\mathbf{L}$  be a simplicial decomposition of  $P$ . Then  $\mathbf{L}$  is called a (linear) *subdivision* of  $\mathbf{K}$  if every simplex of  $\mathbf{L}$  is contained in a simplex of  $\mathbf{K}$ .

The following observation is very elementary, but we shall need it in the discussion below.

**PROPOSITION 0.** Suppose  $P$  is a polyhedron with simplicial decompositions  $\mathbf{K}$ ,  $\mathbf{L}$  and  $\mathbf{M}$  such that  $\mathbf{L}$  is a subdivision of  $\mathbf{K}$  and  $\mathbf{M}$  is a subdivision of  $\mathbf{L}$ . Then  $\mathbf{M}$  is also a subdivision of  $\mathbf{K}$ . ■

Figure I.3.3 in `algotop1figures01w09.pdf` depicts two subdivisions of a 2-simplex that are different from the one in Example 3 above. As indicated by Figure I.3.4 in the same document, in general if we have two simplicial decompositions of a polyhedron then neither is a subdivision of the other. However, it is possible to prove the following:

*If  $\mathbf{K}$  and  $\mathbf{L}$  are simplicial decompositions of the same polyhedron  $P$ , then there is a third decomposition which is a subdivision of both  $\mathbf{K}$  and  $\mathbf{L}$ .*

Proving this requires more machinery than we need for other purposes, and since we shall not need the existence of such subdivisions in this course we shall simply note that one can prove this result using methods from the second part of [MunkresEDT]:

**SUBDIVISION AND SUBCOMPLEXES.** These two concepts are related by the following elementary results.

**PROPOSITION 1.** Suppose that  $(P, \mathbf{K})$  is a simplicial complex and that  $(P_1, \mathbf{K}_1)$  is a subcomplex of  $(P, \mathbf{K})$ . If  $\mathbf{L}$  is a subdivision of  $\mathbf{K}$  and  $\mathbf{L}_1$  is the set of all simplices in  $\mathbf{L}$  which are contained in  $P_1$ , then  $(P_1, \mathbf{L}_1)$  is a subcomplex of  $(P, \mathbf{L})$ . ■

Recall our Default Hypothesis (at the end of Section I.2) that all simplicial decompositions should be closed under taking faces unless specifically stated otherwise.

**COROLLARY 2.** Let  $P$ ,  $\mathbf{K}$  and  $\mathbf{L}$  be as above, and let  $A \subset P$  be a simplex of  $\mathbf{K}$ . Then  $\mathbf{L}$  determines a simplicial decomposition of  $A$ . ■

### Barycentric subdivisions

We are particularly interested in describing a systematic construction for subdivisions that works for all simplicial complexes and allows one to form decompositions for which the diameters of all the simplices are very small. This will generalize a standard method for partitioning an interval  $[a, b]$  into small intervals by first splitting the interval in half at the midpoint, then splitting the two subintervals in half similarly, and so on. If this is done  $n$  times, the length of each interval in the subdivision is equal to  $(b - a)/2^n$ , and if  $\varepsilon > 0$  is arbitrary then for sufficiently large values of  $n$  the lengths of the subintervals will all be less than  $\varepsilon$ .

The generalization of this to higher dimensions is called the **barycentric subdivision**.

**Definition.** Given an  $n$ -simplex  $A \subset \mathbb{R}^m$  with vertices  $\mathbf{v}_0, \dots, \mathbf{v}_n$ , the *barycenter*  $\mathbf{b}_A$  of  $A$  is given by

$$\mathbf{b}_A = \frac{1}{n+1} \sum_{i=0}^n \mathbf{v}_i .$$

If  $n \leq m \leq 3$ , this corresponds to the physical center of mass for  $A$ , assuming the density in  $A$  is uniform.

**Definition.** If  $P \subset \mathbb{R}^m$  is a polyhedron and  $(P, \mathbf{K})$  is a simplicial complex, then the *barycentric subdivision*  $\mathbf{B}(\mathbf{K})$  consists of all simplices having the form  $\mathbf{b}_0 \cdots \mathbf{b}_k$ , where (i) each  $\mathbf{b}_j$  is the barycenter of a simplex  $A_j \in \mathbf{K}$ , (ii) for each  $j > 0$  the simplex  $A_{j-1}$  is a face of  $A_j$ .

In order to justify this definition, we need to prove the following result:

**PROPOSITION 3.** *Let  $A$  be an  $n$ -simplex, suppose that we are given simplices  $A_j \subset A$  such that  $A_{j-1}$  is a face of  $A_j$  for each  $j$ , and let  $\mathbf{b}_j$  be the barycenter of  $A_j$ . Then the set of vertices  $\{\mathbf{b}_0, \dots, \mathbf{b}_q\}$  is affinely independent.*

**Proof.** We can extend the sequence of simplices  $\{A_j\}$  to obtain a new sequence  $C_0 \subset \dots \subset C_n = A$  such that each  $C_k$  is obtained from the preceding one  $C_{k-1}$  by adding a single vertex, and it suffices to prove the result for the corresponding sequence of barycenters. Therefore we shall assume henceforth in this proof that each  $A_j$  is obtained from its predecessor by adding a single vertex and that  $A$  is the last simplex in the list.

It suffices to show that the vectors  $\mathbf{b}_j - \mathbf{b}_0$  are linearly independent. For each  $j$  let  $\mathbf{v}_{j_i}$  be the vertex in  $A_j$  that is not in its predecessor. Then for each  $j > 0$  we have

$$\mathbf{b}_j - \mathbf{b}_0 = \left( \frac{1}{j+1} \sum_{k \leq j} \mathbf{v}_{i_k} \right) - \mathbf{v}_0 = \frac{1}{j+1} \sum_{k \leq j} (\mathbf{v}_{i_k} - \mathbf{v}_{i_0}) .$$

which is a linear combination of the linearly independent vectors  $\mathbf{v}_{i_1} - \mathbf{v}_{i_0}, \dots, \mathbf{v}_{i_j} - \mathbf{v}_{i_0}$  such that the coefficient of the last vector in the set is nonzero.

If we let  $\mathbf{u}_k = \mathbf{v}_{i_k} - \mathbf{v}_{i_0}$ , then it follows that for all  $k > 0$  we have  $\mathbf{b}_k - \mathbf{b}_0 = a_k \mathbf{u}_k + \mathbf{y}_k$ , where  $\mathbf{y}_k$  is a linear combination of  $\mathbf{u}_1, \dots, \mathbf{u}_{k-1}$  and  $a_k \neq 0$ . Since the vectors  $\mathbf{u}_j$  are linearly independent, it follows that the vectors  $\mathbf{b}_k - \mathbf{b}_0$  (where  $0 < k \leq n$ ) are linearly independent and hence the vectors  $\mathbf{b}_0, \dots, \mathbf{b}_n$  are affinely independent. ■

The simplest nontrivial examples of barycentric subdivisions are given by 2-simplices, and Figure I.3.6 in `algotopfigures` gives a typical example. We shall enumerate the simplices in such a

barycentric subdivision using the definition. For the sake of definiteness, we shall call the simplex  $P$  and the vertices  $\mathbf{v}_0, \mathbf{v}_1$  and  $\mathbf{v}_2$ .

- (i) The 0-simplices are merely the barycenters  $\mathbf{b}_A$ , where  $A$  runs through all the nonempty faces of  $P$  and  $P$  itself. There are 7 such simplices and hence 7 vertices in  $\mathbf{B}(\mathbf{K})$ .
- (ii) The 1-simplices have the form  $\mathbf{b}_A\mathbf{b}_C$ , where  $A$  is a face of  $C$ . There are three possible choices for the ordered pair  $(\dim A, \dim C)$ ; namely,  $(0, 1)$ ,  $(0, 2)$  and  $(1, 2)$ . The number of pairs  $\{A, C\}$  for the case  $(0, 1)$  is equal to 6, the number for the case  $(0, 2)$  is equal to 3, and the number for the case  $(1, 2)$  is also equal to 3, so there are 12 different 1-simplices in  $\mathbf{B}(\mathbf{K})$ .
- (iii) The 2-simplices have the form  $\mathbf{b}_A\mathbf{b}_C\mathbf{b}_E$ , where  $A$  is a face of  $C$  and  $C$  is a face of  $E$ . There are 6 possible choices for  $\{A, C, E\}$ .

Obviously one could carry out a similar analysis for a 3-simplex but the details would be more complicated.

Of course, it is absolutely essential to verify that the barycentric subdivision construction actually defines simplicial decompositions.

**THEOREM 4.** *If  $(P, \mathbf{K})$  is a simplicial complex and  $\mathbf{B}(\mathbf{K})$  is the barycentric subdivision of  $\mathbf{K}$ , then  $(P, \mathbf{B}(\mathbf{K}))$  is also a simplicial complex (in other words, the collection  $\mathbf{B}(\mathbf{K})$  determines a simplicial decomposition of  $P$ ).*

**Proof.** We shall concentrate on the special case where  $P$  is a simplex. The general case can be recovered from the special case and Lemma I.2.6.

Suppose now that  $P$  is a simplex with vertices  $\mathbf{v}_0, \dots, \mathbf{v}_n$ . We first show that  $P$  is the union of the simplices in  $\mathbf{B}(\mathbf{K})$ . Given  $\mathbf{x} \in P$ , write  $\mathbf{x}$  as a convex combination  $\sum_j t_j \mathbf{v}_j$ , and rearrange the scalars into a sequence

$$t_{k_0} \geq t_{k_1} \geq \dots \geq t_{k_n}$$

(this is not necessarily unique, and in particular it is not so if  $t_u = t_v$  for  $u \neq v$ ). For each  $i$  between 0 and  $n$ , let  $A_i$  be the simplex whose vertices are  $\mathbf{v}_{k_0}, \dots, \mathbf{v}_{k_i}$ . We CLAIM that  $\mathbf{x} \in \mathbf{b}_0 \dots \mathbf{b}_n$ , where  $\mathbf{b}_i$  is the barycenter of  $A_i$ .

Let  $s_i = t_{k_i} - t_{k_{i+1}}$  for  $0 \leq i \leq n-1$  and set  $s_n = t_{k_n}$ . Then  $s_i \geq 0$  for all  $i$ , and it is elementary to verify that

$$\mathbf{x} = \sum_{i=0}^n (i+1) s_i \mathbf{b}_i, \quad \text{where} \quad \sum_{i=0}^n (i+1) s_i = \sum_{i=0}^n t_{k_i} = 1.$$

Therefore  $\mathbf{x} \in \mathbf{b}_0 \dots \mathbf{b}_n$ , so that every point in  $P$  lies on one of the simplices in the barycentric subdivision.

To conclude the proof, we must show that the intersection of two simplices in  $\mathbf{B}(\mathbf{K})$  is a common face. First of all, it suffices to show this for a pair of  $n$ -dimensional simplices; this follows from the argument following the Default Hypothesis at the end of Section I.2.

Suppose now that  $\alpha$  and  $\gamma$  are  $n$ -simplices in  $\mathbf{B}(\mathbf{K})$ . Then the vertices of  $\alpha$  are barycenters of simplices  $A_0, \dots, A_n$  where  $A_j$  has one more vertex than  $A_{j-1}$  for each  $j$ , and the vertices of  $\gamma$  are barycenters of simplices  $C_0, \dots, C_n$  where  $C_j$  has one more vertex than  $C_{j-1}$  for each  $j$ . Label the vertices of the original simplex as  $\mathbf{v}_{i_0}, \dots, \mathbf{v}_{i_n}$  where  $A_j = \mathbf{v}_{i_0} \dots \mathbf{v}_{i_j}$  and also as  $\mathbf{v}_{k_0}, \dots, \mathbf{v}_{k_n}$



where  $C_j = \mathbf{v}_{k_0} \cdots \mathbf{v}_{k_j}$ . The key point is to determine how  $(i_0, \dots, i_n)$  and  $(k_0, \dots, k_n)$  are related.

If  $\mathbf{x}$  lies on the original simplex and  $\mathbf{x}$  is written as a convex combination  $\sum_j t_j \mathbf{v}_j$ , then we have shown that  $\mathbf{x} \in A$  if  $t_{i_0} \leq \dots \leq t_{i_n}$ . In fact, we can reverse the steps in that argument to show that if  $\mathbf{x} \in A$  then conversely we have  $t_{i_0} \leq \dots \leq t_{i_n}$ . Similarly, if  $\mathbf{x} \in C$  then  $t_{k_0} \leq \dots \leq t_{k_n}$ . Therefore if  $\mathbf{x} \in A \cap C$  then  $t_{i_j} = t_{k_j}$  for all  $j$ . Choose  $m_0, \dots, m_q \in \{0, \dots, n\}$  such that  $t_{m_j} > t_{m_{j+1}}$ , with the convention that  $t_{n+1} = 0$ , and split  $\{0, \dots, n\}$  into equivalence classes  $\mathcal{M}_0, \dots, \mathcal{M}_q$  such that  $\mathcal{M}_j$  is the set of all  $u$  such that  $t_u = t_{m_j}$ . It follows that  $\mathbf{x}$  lies on the simplex  $\mathbf{z}_0 \cdots \mathbf{z}_q$ , where  $\mathbf{z}_j$  is the barycenter of the simplex whose vertices are  $\mathcal{M}_0 \cup \dots \cup \mathcal{M}_j$ . The vertices of this simplex are vertices of both  $A$  and  $C$ . Since  $A \cap C$  is convex, this implies that it is the simplex whose vertices are those which lie in  $A \cap C$ , and thus  $A \cap C$  is a face of both  $A$  and  $C$ . ■

**Terminology.** Frequently the complex  $(P, \mathbf{B}(\mathbf{K}))$  is called the *derived complex* of  $(P, \mathbf{K})$ . The barycentric subdivision construction can be iterated, and thus one obtains a sequence of decompositions  $\mathbf{B}^r(\mathbf{K})$ . The latter is often called the  $r^{\text{th}}$  barycentric subdivision of  $\mathbf{K}$  and  $(P, \mathbf{B}^r(\mathbf{K}))$  is often called the  $r^{\text{th}}$  derived complex of  $(P, \mathbf{K})$ .

### *Diameters of barycentric subdivisions*

Given a metric space  $(X, \mathbf{d})$ , its *diameter* is the least upper bound of the distances  $\mathbf{d}(y, z)$ , where  $y, z \in X$ ; if the set of distances is unbounded, we shall follow standard usage and say that the diameter is infinite or equal to  $\infty$ .

**PROPOSITION 5.** *Let  $A \subset \mathbb{R}^n$  be an  $n$ -simplex with vertices  $\mathbf{v}_0, \dots, \mathbf{v}_n$ . Then the diameter of  $A$  is the maximum of the distances  $|\mathbf{v}_i - \mathbf{v}_j|$ , where  $0 \leq i, j \leq n$ .*

**Proof.** Let  $\mathbf{x}, \mathbf{y} \in A$ , and write these as convex combinations  $\mathbf{x} = \sum_j t_j \mathbf{v}_j$  and  $\mathbf{y} = \sum_j s_j \mathbf{v}_j$ . Then

$$\mathbf{x} - \mathbf{y} = \left( \sum_i s_i \right) \mathbf{x} - \left( \sum_j t_j \right) \mathbf{y} = \sum_{i,j} s_i t_j \mathbf{v}_j - \sum_{i,j} s_i t_j \mathbf{v}_i.$$

Since  $0 \leq s_i, t_j \leq 1$  for all  $i$  and  $j$ , we have  $0 \leq s_i t_j \leq 1$  for all  $i$  and  $j$ , so that

$$\begin{aligned} \mathbf{d}(\mathbf{x}, \mathbf{y}) + |\mathbf{x} - \mathbf{y}| &\leq \left| \sum_{i,j} s_i t_j (\mathbf{x}_j - \mathbf{x}_i) \right| \leq \\ &\sum_{i,j} s_i t_j |\mathbf{v}_i - \mathbf{v}_j| \leq \sum_{i,j} s_i t_j \max |\mathbf{v}_k - \mathbf{v}_\ell| = \max |\mathbf{v}_k - \mathbf{v}_\ell| \end{aligned}$$

as required. ■

**Definition.** If  $\mathbf{K}$  is a simplicial decomposition of a polyhedron  $P$ , then the *mesh* of  $\mathbf{K}$ , written  $\mu(\mathbf{K})$ , is the maximum diameter of the simplices in  $\mathbf{K}$ .

**PROPOSITION 6.** *In the preceding notation, the mesh of  $\mathbf{K}$  is the maximum distance  $|\mathbf{v} - \mathbf{w}|$ , where  $\mathbf{v}$  and  $\mathbf{w}$  are vertices of some simplex in  $\mathbf{K}$ . ■*

The main result in this discussion is a comparison of the mesh of  $\mathbf{K}$  with the mesh of  $\mathbf{B}(\mathbf{K})$ .

**PROPOSITION 7.** *Suppose that  $(P, \mathbf{K})$  be a simplicial complex and that all simplices of  $\mathbf{K}$  have dimension  $\leq n$ . Then*

$$\mu(\mathbf{B}(\mathbf{K})) \leq \frac{n}{n+1} \cdot \mu(\mathbf{K}).$$

Before proving this result, we shall derive some of its consequences.

**COROLLARY 8.** *In the preceding notation, if  $r \geq 1$  then*

$$\mu(\mathbf{B}^r(\mathbf{K})) \leq \left(\frac{n}{n+1}\right)^r \cdot \mu(\mathbf{K}) \blacksquare$$

**COROLLARY 9.** *In the preceding notation, if  $\varepsilon > 0$  then there exists an  $r_0$  such that  $r \geq r_0$  implies  $\mu(\mathbf{B}^r(\mathbf{K})) < \varepsilon$ .*

Corollary 9 follows from Corollary 8 and the fact that

$$\lim_{r \rightarrow \infty} \left(\frac{n}{n+1}\right)^r = 0 \blacksquare$$

**Proof of Proposition 7.** By Proposition 5 and the definition of barycentric subdivision we know that  $\mu(\mathbf{B}(\mathbf{K}))$  is the maximum of all distances  $|\mathbf{b}_A - \mathbf{b}_C|$ , where  $\mathbf{b}_A$  and  $\mathbf{b}_C$  are barycenters of simplices  $A, C \in \mathbf{K}$  such that  $A \subset C$ . Suppose that  $A$  is an  $a$ -simplex and  $C$  is a  $c$ -simplex, so that  $0 \leq a < c \leq n$ . We then have

$$|\mathbf{b}_A - \mathbf{b}_C| = \left| \frac{1}{a+1} \sum_{\mathbf{v} \in A} \mathbf{v} - \frac{1}{c+1} \sum_{\mathbf{w} \in C} \mathbf{w} \right|$$

and as in the proof of Proposition 5 we have

$$\frac{1}{a+1} \sum_{\mathbf{v} \in A} \mathbf{v} - \frac{1}{c+1} \sum_{\mathbf{w} \in C} \mathbf{w} = \frac{1}{(a+1)(c+1)} \sum_{\mathbf{v}, \mathbf{w}} (\mathbf{v} - \mathbf{w}) .$$

There are  $(a+1)$  terms in this summation which vanish (namely, those for which  $\mathbf{w} = \mathbf{v}$ ), and therefore we have

$$\begin{aligned} |\mathbf{b}_A - \mathbf{b}_C| &= \left| \frac{1}{(a+1)(c+1)} \sum_{\mathbf{v} \neq \mathbf{w}} (\mathbf{v} - \mathbf{w}) \right| \leq \frac{1}{(a+1)(c+1)} \sum_{\mathbf{v} \neq \mathbf{w}} |\mathbf{v} - \mathbf{w}| \leq \\ &\frac{1}{(a+1)(c+1)} \cdot \left( \max_{\mathbf{v}, \mathbf{w}} |\mathbf{v} - \mathbf{w}| \right) \cdot \left[ (a+1)(c+1) - (a+1) \right] = \\ &\left( \max_{\mathbf{v}, \mathbf{w}} |\mathbf{v} - \mathbf{w}| \right) \cdot \left( 1 - \frac{1}{c+1} \right) \leq \left( 1 - \frac{1}{n+1} \right) . \end{aligned}$$

At the last step we use  $c \leq n$  and the fact that the function  $1 - (x/n)$  is an increasing function of  $x$  if  $x > 0$ . The inequality in the corollary follows directly from the preceding chain of inequalities.  $\blacksquare$

One further consequence of Proposition 7 will be important for our purposes.

**COROLLARY 10.** *Let  $(P, \mathbf{K})$  be a simplicial complex, and let  $\mathcal{W}$  be an open covering of  $P$ . Then there is a positive integer  $r_0$  such that  $r \geq r_0$  implies that every simplex of  $\mu(\mathbf{B}^r(\mathbf{K}))$  is contained in an element of  $\mathcal{W}$ .*

**Proof.** By construction,  $P$  is a compact subset of the metric space  $\mathbb{R}^m$ . Therefore the Lebesgue Covering Lemma implies the existence of a real number  $\eta > 0$  such that every subset of diameter

$< \eta$  is contained in an element of  $\mathcal{W}$ . If we choose  $r_0 > 0$  such that  $r \geq r_0$  implies  $\mu(\mathbf{B}^r(\mathbf{K})) < \eta$ , then  $\mathbf{B}^r(\mathbf{K})$  will have the required properties. ■

## I.4 : Cones and suspensions

(Hatcher, 0)

These two basic constructions are described on pages 8–9 of Hatcher. We shall say a little more about them and apply them to construct a homeomorphism from the standard  $n$ -disk and  $(n - 1)$ -sphere to the standard  $n$ -simplex and its boundary.

*The constructions and their properties*

**Definition.** Let  $X$  be a topological space. The *cone on  $X$* , usually written  $\mathbf{C}(X)$ , is the quotient of  $X \times [0, 1]$  modulo the equivalence relation whose equivalence classes are all one point subsets of the form  $\{ (x, t) \}$ , where  $t \neq 0$ , and the subset  $X \times \{0\}$ .

The first result explains the motivation for the name.

**PROPOSITION 1.** *If  $X$  is a compact subset of  $\mathbb{R}^n$ , then  $\mathbf{C}(X)$  is homeomorphic to a subset of  $\mathbb{R}^{n+1}$  so that the image of  $X \times \{1\}$  in  $\mathbf{C}(X)$  corresponds to  $X \times \{0\}$  and every point of the image is on a closed line segment joining a point of the latter to the last unit vector  $(0, \dots, 0, 1)$ .*

**Proof.** Define a continuous map  $g$  from  $X \times [0, 1]$  to  $\mathbb{R}^{n+1}$  sending  $(x, t)$  to  $(tx, 1 - t)$ . This passes to a continuous 1–1 mapping  $f$  from  $\mathbf{C}(X)$  to  $\mathbb{R}^{n+1}$  whose image is the set described in the statement of the result, and since  $\mathbf{C}(X)$  is a (continuous image of a) compact space it follows that  $f$  maps the cone homeomorphically onto its image. ■

**Examples.** The cone on  $S^n$  is canonically homeomorphic to  $D^{n+1}$ ; specifically, the map  $S^n \times [0, 1] \rightarrow D^{n+1}$  which sends  $(x, t)$  to  $(1 - t)x$  passes to a map of quotients  $\mathbf{C}(S^n) \rightarrow D^{n+1}$  which is a homeomorphism. Also, the cone on  $D^n$  is canonically homeomorphic to  $D^{n+1}$ . Perhaps the quickest way to see this is the following: The preceding argument shows that the cone on the upper hemisphere  $D_+^n$  of  $S^n$  (where the last coordinate is nonnegative) is the set of points in  $D^{n+1}$  whose last coordinate is nonnegative (its “upper half”), so we have to show that the latter is homeomorphic to  $D^{n+1}$ . If we let  $|x|_2$  and  $|x|_\infty$  denote the appropriate norms on  $\mathbb{R}^{n+1}$  (see the 205A notes), then the homeomorphism  $h$  of  $\mathbb{R}^{n+1}$  to itself defined by

$$h(x) = \frac{|x|_\infty}{|x|_2} \cdot x \quad \text{if } x \neq 0$$

and  $h(0) = 0$  (continuity here must be checked, but this is not difficult) will send the upper half of  $D^{n+1}$  to the subspace  $[-1, 1]^n \times [0, 1] \subset \mathbb{R}^{n+1}$ . Since this product of closed intervals is homeomorphic to  $[-1, 1]^n$  and the latter is homeomorphic to  $D^{n+1}$  by the inverse of the map  $h$ , the assertion about  $\mathbf{C}(D^n)$  and  $D^{n+1}$  follows. ■

The cone construction extends to a covariant functor as follows: If  $f : X \rightarrow Y$  is continuous, then the map  $f \times \text{id}_{[0,1]} : X \times [0, 1] \rightarrow Y \times [0, 1]$  is also continuous, and if  $q_W : W \times [0, 1] \rightarrow \mathbf{C}(W)$

is the quotient projection for  $W = X$  or  $Y$ , then passage to quotients defines a unique continuous mapping  $\mathbf{C}(f) : \mathbf{C}(X) \rightarrow \mathbf{C}(Y)$  such that

$$\mathbf{C}(f) \circ q_X = q_Y \circ (f \times \text{id}_{[0,1]}) .$$

It is a routine exercise to verify that this construction satisfies the covariant functor identities  $\mathbf{C}(\text{id}_{[0,1]}) = \text{id}_{\mathbf{C}(X)}$  and  $\mathbf{C}(g \circ f) = \mathbf{C}(g) \circ \mathbf{C}(f)$ .

**Definition.** Let  $X$  be a topological space. The (*unreduced*) *suspension on  $X$* , usually written  $\mathbf{S}(X)$ , is the quotient of  $X \times [-1, 1]$  modulo the equivalence relation whose equivalence classes are all one point subsets of the form  $\{ (x, t) \}$ , where  $|t| < 1$ , and the subsets  $X \times \{\pm 1\}$ .

The suspension of a circle is illustrated in the **figures** file. The name arises because the original space is effectively “suspended” between the north and south poles (the classes of  $X \times \{\pm 1\}$  in the quotient), being held in place by the “cables”  $\{x\} \times [-1, 1]$ .

We have the following analog of Propositions 1 for cones.

**PROPOSITION 3.** *If  $X$  is a compact subset of  $\mathbb{R}^n$ , then  $\mathbf{S}(X)$  is homeomorphic to a subset of  $\mathbb{R}^{n+1}$  so that the images of  $X \times \{\pm 1\}$  in  $\mathbf{S}(X)$  correspond to the point  $(0, \dots, 0, \pm 1)$  and the homeomorphism is the inclusion on  $X \times \{0\}$ .*

**Proof.** This is very similar to the proof for cones. Define a continuous map  $g$  from  $X \times [-1, 1]$  to  $\mathbb{R}^{n+1}$  sending  $(x, t)$  to  $((1-|t|x), t)$ . This passes to a continuous 1–1 mapping  $f$  from  $\mathbf{S}(X)$  to  $\mathbb{R}^{n+1}$  whose image is the set described in the statement of the result, and since  $\mathbf{C}(X)$  is a (continuous image of a) compact space it follows that  $f$  maps the suspension homeomorphically onto its image.■

**Examples.** The suspension on  $S^n$  is canonically homeomorphic to  $S^{n+1}$  by the map sending the class of  $(x, t) \in S^n \times [0, 1]$  to  $(\sqrt{1-t^2} \cdot x, t) \in \mathbb{R}^{n+1}$ . Similarly, the suspension of  $D^n$  is canonically homeomorphic to  $D^{n+1}$ , and this can be shown by adapting the previous argument which proved that the cone on  $D^n$  is homeomorphic to the upper half of  $D^{n+1}$  (the cone is just the upper half of the suspension; use symmetry considerations to define the homeomorphism on the lower halves of everything).■

The suspension construction extends to a covariant functor as follows: If  $f : X \rightarrow Y$  is continuous, then the map  $f \times \text{id}_{[-1,1]} : X \times [0, 1] \rightarrow Y \times [-1, 1]$  is also continuous, and if  $q_W : W \times [-1, 1] \rightarrow \mathbf{S}(W)$  is the quotient projection for  $W = X$  or  $Y$ , then passage to quotients defines a unique continuous mapping  $\mathbf{S}(f) : \mathbf{S}(X) \rightarrow \mathbf{S}(Y)$  such that

$$\mathbf{S}(f) \circ q_X = q_Y \circ (f \times \text{id}_{[0,1]}) .$$

It is a routine exercise to verify that this construction satisfies the covariant functor identities  $\mathbf{S}(\text{id}_{[0,1]}) = \text{id}_{\mathbf{S}(X)}$  and  $\mathbf{S}(g \circ f) = \mathbf{S}(g) \circ \mathbf{S}(f)$ .

Observe that projection onto the second coordinate from  $X \times [-1, 1]$  to  $[-1, 1]$  passes to a continuous map from  $\mathbf{S}(X)$ , and we shall say that the value of this map on a point is the latter’s *second coordinate* or *latitude* (the second term is suggested by the drawing in the **figures** file).

**Definition.** If  $X$  is a topological space, then the *upper and lower cones*  $\mathbf{C}_{\pm}(X)$  are the subspaces of  $\mathbf{S}(X)$  consisting of all classes of all point whose second coordinates are nonnegative and nonpositive respectively.

By construction, both the upper and lower cones on  $X$  are canonically homeomorphic to the cone on  $X$ ; in fact, these concepts extend to subfunctors  $\mathbf{C}_{\pm}$  of the suspension functor (in other words, the inclusions of the upper and lower cones are natural transformations).

*A homeomorphism problem*

Given a simplex  $A \subset \mathbb{R}^k$  with vertex set  $V = \{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ , its boundary  $\partial A$  is the union of the faces with vertex sets  $V_i = V - \{\mathbf{x}_i\}$ , where  $i = 0, \dots, n$ . For each  $i$ , the simplex  $A_i$  with vertex set  $V_i$  is called the  $i^{\text{th}}$  face of  $A$ .

We shall use the concepts of cones and suspensions to prove the following result, which will be needed in subsequent units.

**THEOREM 4.** *For each  $n \geq 0$  there is a homeomorphism from the  $n$ -simplex  $\Delta_n$  to the  $n$ -disk  $D^n$  which maps  $\partial\Delta_n$  onto  $S^{n-1}$  and sends the barycenter of  $\Delta_n$  to the center  $\mathbf{0}$  of  $D^n$ .*

This result is obvious if  $n = 0$  because  $\Delta_0$  and  $D^0$  each contain only one point. Let  $\mathbf{A}_n$  be the statement of the theorem for a fixed nonnegative integer  $n$ , and let  $\mathbf{B}_n$  be the following statement:

*There is a homeomorphism from  $\partial\Delta_n$  to  $S^{n-1}$  such that the lower half corresponds to the  $0^{\text{th}}$  face and the upper half corresponds to the union of the other faces.*

We shall prove that  $\mathbf{A}_n$  implies  $\mathbf{B}_{n+1}$  for all  $n \geq 0$  and  $\mathbf{B}_n$  implies  $\mathbf{A}_n$  for all  $n \geq 1$ . If we combine this with the validity of  $\mathbf{A}_0$ , we obtain Theorem 4. ■

**Proof that  $\mathbf{B}_n$  implies  $\mathbf{A}_n$  for all  $n \geq 1$ .** Let  $h : \partial\Delta_n \rightarrow S^{n-1}$  be the homeomorphism which exists by  $\mathbf{B}_n$ . Consider the maps  $f_0 : S^{n-1} \times [0, 1] \rightarrow D^n$  and  $g_0 : \partial\Delta_n \times [0, 1] \rightarrow \Delta_n$  defined by  $f_0(\mathbf{x}, t) = t\mathbf{x}$  and  $g_0(\mathbf{x}, t) = t\mathbf{x} + (1-t)\mathbf{b}$ , where  $\mathbf{b}$  is the barycenter of  $\Delta_n$ . Since each of these maps is constant on the set of all points where  $t = 0$ , it follows that they pass to continuous maps on the cones of the domains, and we shall denote these maps by  $f : \mathbf{C}(S^{n-1}) \rightarrow D^n$  and  $g : \mathbf{C}(\partial\Delta_n) \rightarrow \Delta_n$ . Elementary considerations from linear algebra imply that  $f$  is bijective, and the basic results on barycentric subdivisions imply that  $g$  is also bijective; since all relevant spaces are compact Hausdorff, it follows that these maps are homeomorphisms and that the composite  $f \circ \mathbf{C}(h) \circ g^{-1}$  defines a homeomorphism from  $\Delta_n$  to  $D^n$ . By construction the maps  $f$  and  $g$  send the bases of the cones to  $S^{n-1}$  and  $\partial\Delta_n$  respectively, and since the cone homeomorphism  $\mathbf{C}(h)$  sends the base of one cone to the base of the other it follows that the composite homeomorphism sends the boundary of the simplex to the unit sphere. ■

We shall prove that  $\mathbf{A}_n$  implies  $\mathbf{B}_{n+1}$  for all  $n \geq 0$  and  $\mathbf{B}_n$  implies  $\mathbf{A}_n$  for all  $n \geq 1$ . If we combine this with the validity of  $\mathbf{A}_0$ , we obtain Theorem 4. ■

**Proof that  $\mathbf{A}_n$  implies  $\mathbf{B}_{n+1}$  for all  $n \geq 0$ .** The idea in this case is similar, but we shall use suspensions instead of cones.

As usual, let  $\partial_{n+1}\Delta_{n+1}$  denote the face opposite the last vertex  $\mathbf{e}_{n+1}$  (so the vertices of this face are  $\mathbf{e}_i$  for  $0 \leq i \leq n$ ). Then  $\mathbf{A}_n$  implies the existence of a homeomorphism from  $\Delta_n = \partial_{n+1}\Delta_n$  to  $D^n \cong \mathbf{C}_-(S^{n-1})$ . Let  $E$  denote the union of all the remaining faces of  $\Delta_{n+1}$ , and let  $\varphi$  be the homeomorphism from  $\partial\Delta_n$  to  $S^{n-1}$  which is given by  $\mathbf{A}_n$  as above. Define a map  $k_0$  from  $\partial\Delta_n \times [0, 1]$  to  $E$  which sends  $\mathbf{x}$  to  $(1-t)\mathbf{x} + t\mathbf{e}_{n+1}$ . Since every point on  $E$  lies on a line segment joining  $\mathbf{e}_{n+1}$  to a point on  $\partial\Delta_n$ , one can proceed as before to conclude that  $k$  passes to a homeomorphism  $k$  from  $\mathbf{C}_+(\partial\Delta_n)$  to  $E$ , and its restriction to  $\partial\Delta_n$  is the identity. If we piece together these two homeomorphisms, we obtain a homeomorphism from  $\mathbf{S}(\partial\Delta_n)$  to  $\partial\Delta_{n+1}$ .

Since the suspension construction is functorial, we also know that the suspension of the mapping  $\varphi$  defines a homeomorphism from the suspension of  $\partial\Delta_n$  to the suspension of  $S^{n-1}$ . To

complete the proof, we need to construct a homeomorphism from  $\mathbf{S}(S^{n-1})$  to  $S^n$  which sends the upper and lower cones to the upper and lower hemispheres respectively. The ideal behind this is given by the drawing in the `figures` file. Formally, the homeomorphism  $\psi$  is given by taking the continuous map  $\psi_0 : S^{n-1} \times [-1, 1] \rightarrow S^n$  sending  $(\mathbf{x}, t)$  to  $(\sqrt{1-t^2}\mathbf{x}, t)$  and verifying that it passes to a continuous bijective map  $\psi$  defined on  $\mathbf{S}(S^{n-1})$ .■

**NOTE.** Another proof of this result is given in an earlier version of these notes. The latter does not require the use of cones and suspensions, but the argument is considerably longer.

## II. Homotopy and cell complexes

The notion of homotopy is introduced in Mathematics 205B, and it is central to both algebraic and geometric topology as well as many of the applications of topology to algebra and analysis. Part of the material is a review of topics from the second part of Munkres' book; some of the review topics and most of the new material are also covered in Chapters 0 and 1 of Hatcher.

The new material covers two related topics. The first (in Section 3) is to describe generalizations of simplicial complexes called **cell complexes** that are more convenient for many purposes of algebraic topology, and the second (in Section 4) provides a fundamental illustration of the usefulness of such objects. One objective is an important result on the following central problem:

**EXTENSION QUESTION.** *Suppose that  $X$  and  $Y$  are topological spaces, that  $A$  is a subspace of  $X$ , and  $g : A \rightarrow Y$  is continuous. Is there an extension of  $g$  to a continuous mapping  $f : X \rightarrow Y$  (in other words, a continuous mapping  $f$  such that the restriction  $f|_A$  is equal to  $g$ )?*

One of the main results in Section 4 provides an extremely useful answer to this question in terms of the main concepts of this unit: *If  $X$  is a cell complex and  $A$  is a subcomplex, then  $g$  has a continuous extension to  $X$  if and only if some mapping homotopic to  $g$  has such an extension.*

This and subsequent units of the notes will be less self-contained than Unit I, and there will be numerous references to Munkres or Hatcher for details.

### II.1 : Homotopic mappings

(Hatcher, Ch. 0; Munkres, §§ 51, 58)

The general notion of homotopy for (continuous) mappings is defined on page 323 of Munkres and page 3 of Hatcher. Following standard practice we shall write  $f \simeq g$  to indicate that  $f$  is homotopic to  $g$ . We shall state some basic properties of homotopic mappings that are particularly important for our purposes.

**PROPOSITION 1.** (Munkres, Lemma 51.1, p. 324.) *The binary relation  $\simeq$  of homotopy on the set of continuous mappings from one topological space  $X$  to a second topological space  $Y$  is an equivalence relation.■*

In the proposition above, we allow the possibility that  $X = Y$ . The set of homotopy classes of continuous mappings from  $X$  to  $Y$  is generally denoted by  $[X, Y]$ .

**PROPOSITION 2.** (Munkres, Exercise 1, p. 330.) *If we are given continuous maps  $f_0 \simeq f_1 : X \rightarrow Y$  and  $g_0 \simeq g_1 : Y \rightarrow Z$ , then  $g_0 \circ f_0 \simeq g_1 \circ f_1$ .■*

**COROLLARY 3.** *There is a category **HTOP** (the homotopy category) whose objects are topological spaces and whose morphisms are given by  $[X, Y]$  such that if  $u \in [X, Y]$  is represented by  $f$  and  $v \in [X, Y]$  is represented by  $g$ , then  $v \circ u = [g \circ f]$ .■*

Not surprisingly, the identity morphism in  $[X, X]$  is the homotopy class of the identity on  $X$ .

Given a continuous mapping  $f : X \rightarrow Y$ , then  $f$  represents an isomorphism in **HTOP** if and only if there is a mapping  $g : Y \rightarrow X$  such that  $g \circ f \simeq 1_X$  and  $f \circ g \simeq 1_Y$ . A mapping  $f$  which

satisfies these properties is said to be a *homotopy equivalence*. — Since every map is homotopic to itself, it follows immediately that every *homeomorphism is a homotopy equivalence*.

**Definition.** Two topological spaces  $X$  and  $Y$  are *homotopy equivalent* if there is a homotopy equivalence from  $X$  to  $Y$  (in which case there is also a homotopy equivalence from  $Y$  to  $X$ ). Note that the relation “ $X$  is homotopy equivalent to  $Y$ ” is reflexive, symmetric and transitive. Frequently one also says that  $X$  and  $Y$  *have the same homotopy type*.

*Special types of homotopy equivalences*

We shall begin with a homotopy between to basic types of continuous mappings.

**Definition.** A *contracting homotopy* of a topological space  $X$  is a mapping  $H : X \times [0, 1] \rightarrow X$  such that  $H(x, 0) = x$  for all  $x \in X$  and  $H|X \times \{1\}$  is a constant mapping.

We shall say that a topological space is **contractible** if it admits a contracting homotopy.

An arbitrary topological space  $X$  is not necessarily contractible, and in some sense most spaces are not. For example, if  $X$  is the circle  $S^1$  this is not the case because in  $[S^1, S^1] \cong \pi_1(S^1, 1)$  the identity map and the constant map determine different homotopy classes. In fact, one can manufacture many similar examples using the following lemma.

**PROPOSITION 4.** *If  $A, B$  and  $C$  are topological spaces, then there is an isomorphism*

$$\theta : [A, B \times C] \cong [A, B] \times [A, C]$$

*sending a homotopy class  $[f]$  to the ordered pair  $([p_B \circ f], [p_C \circ f])$ , where  $p_B : B \times C \rightarrow B$  and  $p_C : B \times C \rightarrow C$  are the coordinate projections.*

**Sketch of proof.** The mapping  $\theta$  is well-defined by the preceding two results. It is onto, for if we are given an ordered pair of homotopy classes  $([g], [h])$ , then this class is  $\theta([f])$ , where  $f : A \rightarrow B \times C$  is the unique continuous mapping such that  $p_B \circ f = g$  and  $p_C \circ f = h$ . To see it is also 1–1, suppose  $\theta([f]) = \theta([f'])$ . Then there are homotopies  $K : p_B \circ f \simeq p_B \circ f'$  and  $L : p_C \circ f \simeq p_C \circ f'$ , and if we take the map  $H$  whose projections onto  $B$  and  $C$  are  $K$  and  $L$  respectively, then  $H$  defines a homotopy from  $f$  to  $f'$ . ■

**COROLLARY 5.** *If  $X$  is a nonempty topological space, then  $X \times S^1$  is NOT contractible.*

The proof of this result is relatively simple and formal, but it is important to understand it because the argument reflects the viewpoint underlying much of algebraic topology.

**Proof.** It will suffice to show that the identity map on  $X \times S^1$  is not homotopic to a constant map. Let  $q : X \times S^1 \rightarrow S^1$  be projection onto the second coordinate, let  $j : S^1 \rightarrow X \times S^1$  project to the constant map on the first factor and to the identity on the second, and let  $k$  be a constant map from  $X \times S^1$  to itself. If the identity on  $X \times S^1$  is homotopic to a constant map, then we have

$$\begin{aligned} [\text{id}(S^1)] &= [q \circ j] = [q] \circ [j] = [q] \circ [\text{id}(X \times S^1)] \circ [j] = \\ &[q] \circ [k] \circ [j] = [q \circ k \circ j] = [\text{constant}] \end{aligned}$$

which contradicts the fact that the identity on  $S^1$  is not homotopic to a constant. Therefore the identity on  $X \times S^1$  cannot be homotopic to a constant map. ■

One can clearly “leverage” this result to construct further examples; in particular, if  $T^k$  is the product of  $k$  copies of  $S^1$ , then an inductive argument combined with the preceding corollary shows that  $X \times T^k$  is not contractible. ■



**Example.** If  $K$  is a convex subset of  $\mathbb{R}^n$ , then  $K$  is contractible by a so-called *straight line homotopy*: Take an arbitrary point  $\mathbf{y} \in K$  and set

$$H(\mathbf{x}, t) = (1 - t)\mathbf{x} + t\mathbf{y}$$

so that  $H$  shrinks  $K$  down to  $\{\mathbf{y}\}$  along the straight lines joining points  $\mathbf{x} \in K$  to the chosen point  $\mathbf{y}$ . ■

In the preceding example, the inclusion of  $\{\mathbf{y}\}$  in  $K$  is a special case of the following general concept.

**Definition.** Let  $X$  be a topological space, and let  $A \subset X$  with inclusion mapping  $i_A$ . Then  $A$  is said to be a *deformation retract* of  $X$  if there is a map  $r : X \rightarrow A$  such that  $r|_A$  is the identity and  $i_A \circ r_A$  is homotopic to the identity on  $X$ . — If there is a homotopy  $H : i_A \circ r_A \simeq 1_X$  such that  $H(a, t) = a$  for all  $(a, t) \in A \times [0, 1]$  (i.e., the homotopy is fixed on  $A$ ), we say that  $A$  is a *strong deformation retract* of  $X$ .

More generally, in a category  $\mathbf{C}$ , a morphism  $f : X \rightarrow Y$  is said to be a *retract* if there is a morphism  $g : Y \rightarrow X$  such that  $g \circ f = 1_X$ , and a morphism  $h : A \rightarrow B$  is said to be a *retraction* if there is a morphism  $k : B \rightarrow A$  such that  $k \circ h = 1_B$ . — If  $A$  is a deformation retract of  $X$ , then the inclusion  $i_A$  is a retract and the mapping  $r$  is a retraction.

**Example.** The sphere  $S^n$  is a strong deformation retract of  $\mathbb{R}^{n+1} - \{0\}$ . The standard choice of  $r$  in this case is given by  $r(\mathbf{x}) = |\mathbf{x}|^{-1} \cdot \mathbf{x}$  and  $i \circ r$  is homotopic to the identity by the straight line homotopy sending  $(\mathbf{x}, t)$  to  $t\mathbf{x} + (1 - t)r(\mathbf{x})$ .

### *Counting homotopy classes*

We shall conclude this section by proving a result mentioned earlier.

**THEOREM 6.** *If  $K$  is a compact subset of  $\mathbb{R}^n$  for some  $n$  and  $U$  is an open subset of  $\mathbb{R}^m$  for some  $m$ , then  $[K, U]$  is countable.*

One major step in the proof is the following result of independent interest:

**LEMMA 7.** *Let  $X$  and  $U$  be as above, and let  $f : K \rightarrow U$  be continuous. Then there is some  $\delta > 0$  such that if  $g : K \rightarrow U$  is another continuous function satisfying  $\mathbf{d}(f(\mathbf{x}), g(\mathbf{x})) < \delta$  for all  $\mathbf{x}$ , then  $g$  is homotopic to  $f$  as mappings from  $X$  to  $U$ .*

**Sketch of proof of Lemma 7.** We can define a continuous function  $h : K \rightarrow \mathbb{R}$  by  $h(\mathbf{x}) = \mathbf{d}(f(\mathbf{x}), \mathbb{R}^m - U)$ . In fact, this function is positive valued because  $f$  maps  $K$  into  $U$ , and by the compactness of  $K$  it takes a minimum value  $\delta$ . Therefore, if  $\mathbf{x}$  is an arbitrary point in  $K$  and  $\mathbf{d}(f(\mathbf{x}), \mathbf{v}) < \delta$ , then the closed line segment joining  $f(\mathbf{x})$  to  $\mathbf{v}$  lies entirely in  $U$ . Consequently, if  $g$  satisfies the condition in the lemma for this choice of  $\delta$ , the image of the straight line homotopy from  $f$  to  $g$  lies entirely in  $U$ . ■

**NOTE AND EXAMPLE.** The preceding lemma reflects one reason for including the codomain as an extra piece of data in our definition of a function. Given any two functions  $f$  and  $g$  as above, they are always homotopic as maps into  $\mathbb{R}^m$  by a straight line homotopy. The crucial point in the lemma is that the image of the homotopy is contained in  $U$ . — Without the constraint involving a positive constant  $\delta$ , the result is false. To see this, let  $K = S^1$  and  $U = \mathbb{R}^2 - \{\mathbf{0}\}$ , and take  $f$  to be the usual inclusion. Then  $f$  is not homotopic to a constant map, for if  $r : U \rightarrow K$  is the retraction

described above, then  $r \circ f$  is not homotopic to a constant, but if  $f$  were homotopic to a constant map  $k$ , then we would have

$$\text{id}(S^1) \simeq r \circ f \simeq r \circ k = \text{constant}$$

and we know this is not the case.

The observations of the previous paragraph have the following positive implication: *If  $H : S^1 \times [0, 1] \rightarrow \mathbb{R}^2$  is a homotopy from the inclusion map to the constant map, then there is some  $(\mathbf{x}_0, t_0) \in S^1 \times [0, 1]$  such that  $H(\mathbf{x}_0, t_0) = \mathbf{0}$ .*■

A major objective of the course is to develop tools that will yield generalizations of the preceding observation to mappings from  $S^n \times [0, 1] \rightarrow \mathbb{R}^{n+1}$ .

**Sketch of proof of Theorem 6.** Suppose that  $f : K \rightarrow U$  as above is continuous, and let  $\delta > 0$  be given as in Lemma 7. Denote the coordinate projections of  $f$  by  $f_i$ , where  $1 \leq i \leq m$ .

By the Stone-Weierstrass Approximation Theorem, there are polynomial functions  $p_i$  on  $K \subset \mathbb{R}^n$  such that

$$|(p_i|_K) - f_i| < \frac{\delta}{2\sqrt{n}}$$

for each  $i$ , and in fact we can also find polynomials  $g_i$  with rational coefficients such that

$$|(p_i|_K) - (g_i|_K)| < \frac{\delta}{2\sqrt{n}}.$$

If we let  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the function whose coordinates are given by the polynomials  $g_i$ , it follows that  $|f - (g|_K)| < \delta$ .

Standard set-theoretic computations show that there are only countably many polynomials in  $n$  variables with rational coefficients, and it follows that there are only countably many choices for  $g$ .

Combining the preceding two paragraphs with Lemma 7, we conclude that  $f$  is homotopic to one of the countable family of continuous functions whose coordinates are given by polynomials in  $n$  variables with rational coefficients, and therefore the set  $[K, U]$  is countable.■

Using the fact that the inclusion of  $S^1$  in  $\mathbb{R}^2 - \{\mathbf{0}\}$  is a homotopy equivalence, one can show that

$$\mathbb{Z} \cong [S^1, S^1] \cong [S^1, \mathbb{R}^2 - \{\mathbf{0}\}]$$

(see the exercises for this section) and therefore the cardinality bound of  $\aleph_0$  on  $[K, U]$  is the best possible general result.■

#### *Important standard notation*

Unless stated otherwise, in the remainder of these notes the symbol  $I$  will denote the closed unit interval  $[0, 1]$ .

## II.2 : The fundamental group

(Hatcher, §§ 1.1 – 1.3, 1.A – 1.B; Munkres, §§ 52, 54)

This subject was treated in Mathematics 205B, and it might be useful to review this material before proceeding.

Section 1.B of Hatcher is devoted to proving a fundamental result in topology which has numerous uses in geometry and complex variables:

**THEOREM 1.** *Let  $G$  be an arbitrary group. Then there is an arcwise connected, locally arcwise connected, and locally simply connected Hausdorff space  $BG$  such that  $\pi_1(BG, \text{pt.})$  is isomorphic to  $G$  and the universal covering space of  $G$  is contractible. Furthermore, if  $X$  and  $Y$  are spaces which have these properties, then  $X$  is homotopy equivalent to  $Y$ .*

The existence argument is contained in Example 1.B.7 of Hatcher, while the uniqueness up to homotopy type is stated as Theorem 1.B.8 and established by the argument in Proposition 1.B.9.

**Definition.** A topological space  $X$  is (strongly) **aspherical** if it is arcwise connected and it has a contractible covering space.

As noted in Hatcher, the torus  $T^k$  is aspherical because its universal covering space is  $\mathbb{R}^k$ , and the covering space projection is given by  $p(x_1, \dots, x_k) = (\exp(2\pi i x_1), \dots, \exp(2\pi i x_k))$ . Also, as noted in Hatcher, all compact connected surfaces except  $S^2$  and  $\mathbb{R}P^2$  are aspherical.

**Generalization.** (For students who have taken Mathematics 205C or are familiar with the notion of sectional curvature in a riemannian manifold.) There is an important generalization of all these facts due to J. Hadamard (1865–1963): *If  $M$  is a compact smooth  $n$ -manifold which has a riemannian metric whose sectional curvature is everywhere nonpositive, then the universal covering of  $M$  is diffeomorphic to  $\mathbb{R}^n$ .* — We shall not use this result at any future point in the course.

## II.3 : Abstract cell complexes

(Hatcher, Ch. 0)

One possible way to view a polyhedron is to think of it as an object that is constructible in a finite number of steps as follows:

- (0) Start with the finite set  $P_0$  of vertices,
- ( $n$ ) If  $P_{n-1}$  is the partial polyhedron constructed at Step ( $n-1$ ), at Step ( $n$ ) one adds finitely many simplices  $S_j$ , identifying each face of each simplex  $S_j$  with a simplex in  $P_{n-1}$ .

In fact, one can do this in order of increasing dimension, attaching all 1-simplices to the vertices at Step 1, then attaching 2-simplices along the boundary faces at Step 2, and so on. It is often useful in topology to consider objects that are generalizations of this procedure that are more flexible in certain key respects. The objects used these days in algebraic topology are known as **cell complexes**.

One immediate difference between cell complexes and simplicial complexes is that the former use the closed unit disk  $D^n \subset \mathbb{R}^n$  and its boundary  $S^{n-1}$  in place of an  $n$ -simplex  $\Delta$  and its

boundary  $\partial\Delta_n$ . Since the results of Section I.4 imply that  $D^n$  is homeomorphic to  $\Delta_n$  such that  $S^{n-1}$  corresponds to  $\partial\Delta_n$ , it follows that one can view simplicial complexes as special cases of cell complexes.

### *Adjoining cells to a space*

We shall now give the basic step in the construction of cell complexes. The discussion below relies heavily on the material in Unit V of the online Mathematics 205A notes that were previously cited.

**Definition.** Let  $X$  be a compact Hausdorff space and let  $A$  be a closed subset of  $X$ . If  $k$  is a nonnegative integer, we shall say that *the space  $X$  is obtained from  $A$  by adjoining finitely many  $k$ -cells* if there are continuous mappings  $f_i : S^{k-1} \rightarrow A$  for  $i = 1, \dots, n$  such that  $X$  is homeomorphic to the quotient space of the topological disjoint union

$$A \amalg (\{1, \dots, n\} \times D^k)$$

modulo the equivalence relation generated by identifying  $(j, \mathbf{x}) \in \{j\} \times S^{k-1}$  with  $f_j(\mathbf{x}) \in A$ , where the homeomorphism maps  $A \subset X$  to the image of  $A$  in the quotient by the canonical mapping.

By construction, there is a 1–1 correspondence of sets between  $X$  and

$$A \amalg (\{1, \dots, n\} \times \mathbf{open}(D^k))$$

where  $\mathbf{open}(D^k) \subset D^k$  is the complement of the boundary sphere. The set  $E_j \subset X$  corresponding to the image of  $\{j\} \times D^k$  in the quotient is called a *(closed)  $k$ -cell*, and the subset  $E_j^{\mathbf{O}}$  corresponding to the image of  $\{j\} \times \mathbf{open}(D^k)$  in the quotient is called an *open  $k$ -cell*. One can then restate the observation in the first sentence of the paragraph to say that  $X$  is a union of  $A$  and the open  $k$ -cells, and these subsets are pairwise disjoint.

Before discussing some topological properties of a space obtained by adjoining  $k$ -cells, we shall consider some special cases.

**Example 1.** Let  $(P, \mathbf{K})$  be a simplicial complex, let  $P_k$  be the union of all  $k$ -simplices in  $\mathbf{K}$ , and let  $P_{k-1}$  be defined similarly. Then the whole point of stating and proving Theorem 1 was to justify an assertion that  $P_k$  is obtained from  $P_{k-1}$  by attaching  $k$ -cells, one for each  $k$ -simplex in  $\mathbf{K}$ . Specifically, for each  $k$ -simplex  $A$  the map  $f_A$  is given by the composite of the homeomorphism  $S^{k-1} \rightarrow \partial A$  with the inclusion  $\partial A \subset P_{k-1}$ . The homeomorphism from the quotient of the disjoint union to  $P_k$  is given by starting with the composite

$$P_{k-1} \amalg (\amalg \{1, \dots, n\} \times D^k) \longrightarrow P_{k-1} \amalg_A A \longrightarrow P_k$$

where  $\amalg_A$  runs over all the  $k$ -simplices of  $\mathbf{K}$ , the first map is a disjoint union of homeomorphisms on the pieces where the maps of Theorem 1 are used to define the homeomorphisms  $\{j\} \times D^k \cong A$ , and the second map is inclusion on each disjoint summand. This composite passes to a map of the quotient of the space on the left modulo the equivalence relation described above, and it is straightforward to show this map is 1–1 onto and hence a homeomorphism (all relevant spaces are compact Hausdorff).

**Example 2.** (GRAPHS) As in Section 64 of Munkres, one may define a finite (vertex-edge) graph to be a space obtained from a finite discrete space by adjoining 1-cells. Frequently there is an added condition that the attaching maps for the boundaries should be 1–1 (so that each

1-cell has two endpoints), and the weaker notion introduced here (and in Hatcher) is then called a *pseudograph*. The graph corresponds to a simplicial decomposition of a simplicial complex if and only if different 1-cells have different endpoints. The simplest example of a graph structure that is not a pseudograph and does not come from a simplicial complex is given by taking  $X = S^1$  and  $A = S^0$  with two 1-cells corresponding to the upper and lower semicircles  $E_{\pm}^1$  in the complex plane. The attaching maps are defined to map the endpoints of  $D^1 = [-1, 1]$  bijectively to  $-1, 1$ . — Another example that is historically noteworthy is the Königsberg Bridge Graph, in which the vertices correspond to four land masses in the city of Königsberg (now Kaliningrad, Russia) and the 1-cells (or *edges*) correspond to the bridges which joined pairs of land masses in the 18<sup>th</sup> century (see the **figures** file for drawing). This is another example of a graph that does not come from a simplicial complex but is not a pseudograph; if there are two bridges joining the same pairs of land masses, then the graph has two edges with the same boundary points. — In the next unit we shall see how Euler’s analysis of this graph may be stated in terms of algebraic topology.

We shall encounter further examples after we define the main concept of this section. For the time being, we mention a few simple properties of spaces obtained by attaching  $k$ -cells for some  $k$

**PROPOSITION 2.** *If  $X$  is obtained from  $A$  by attaching 0-cells, then  $X$  is homeomorphic to the disjoint union of  $A$  with a finite discrete space.*

This is true because the 0-disk  $D^0$  has an empty unit sphere, so there are no attaching maps and the equivalence relation on the space  $A \amalg \{1, \dots, N\}$  is the equality relation.■

**PROPOSITION 3.** *If  $X$  is obtained from  $A$  by attaching  $k$ -cells, then each open cell  $E_j^{\mathcal{O}}$  is an open subset of  $X$ , and each such open cell is homeomorphic to  $\mathbf{open}(D^k)$ .*

**Proof.** Each closed cell is compact because it is a continuous image of  $D^k$ , and hence each such subset is closed in  $X$ . By the set-theoretic description given above, the open cell  $E_j^{\mathcal{O}}$  is just the complement of the closed set

$$A \cup \bigcup_{i \neq j} E_i$$

and hence it is open in  $X$ . Since the quotient space map from the disjoint union to  $X$  defines a 1–1 onto continuous mapping from  $\mathbf{open}(D^k)$  to  $E_j^{\mathcal{O}}$ , it suffices to show that an open subset of  $\mathbf{open}(D^k)$  is sent to an open subset of  $E_j^{\mathcal{O}}$ . Let

$$\varphi : A \amalg (\{1, \dots, N\} \times D^k) \longrightarrow X$$

be the continuous onto quotient map corresponding to the cell attachments, and suppose that  $U$  is open in  $\{j\} \times \mathbf{open}(D^k)$ . By construction we then have

$$U = \varphi^{-1}[\varphi[U]]$$

and thus  $\varphi[U]$  is open in  $X$  by the definition of the quotient topology.■

The last result in this subsection implies that the inclusion of  $A$  in  $X$  is homotopically well-behaved if  $X$  is obtained from  $A$  by adjoining  $k$ -cells.

**PROPOSITION 4.** *If  $X$  is obtained from  $A$  by attaching  $k$ -cells and  $U$  is an open subset of  $X$  containing  $A$ , then there is an open subset  $V$  such that*

$$A \subset V \subset \overline{V} \subset U$$

and  $A$  is a strong deformation retract of both  $V$  and  $\overline{V}$ .

The **figures** file contains an drawing for the case  $N = 1$ .

**Proof.** As in the preceding argument, take

$$\varphi : A \amalg \{1, \dots, N\} \times D^k \longrightarrow X$$

to be the continuous onto map corresponding to the  $k$ -cell attachments.

Let  $F = X - U$ , and let  $F_0 = \varphi^{-1}[F]$ , so that  $F_0$  corresponds to a disjoint union  $\amalg_j F_j$ , where each  $F_j$  is a compact subset of  $\mathbf{open}(D^k)$ ; compactness follows because the image of each  $F_j$  in  $X$  is a closed subset of the compact  $k$ -cell  $E_j$ . Therefore we can find constants  $c_j$  such that  $0 < c_j < 1$  and  $F_j$  is contained in the open disk of radius  $c_j$  about the origin in  $\{j\} \times D^k$ ; let  $c$  be the maximum of the numbers  $c_j$ , and let  $V \subset X$  be the image under  $\varphi$  of the set

$$W = A \amalg \left( \bigcup_j \{j\} \times \{ \mathbf{x} \in D^k \mid c < |\mathbf{x}| \leq 1 \} \right).$$

Then  $V$  is open because it is the complement of a compact set, and it follows that  $\overline{V}$  is the image of

$$Y = A \left( \amalg \bigcup_j \{j\} \times \{ \mathbf{x} \in D^k \mid c \leq |\mathbf{x}| \leq 1 \} \right).$$

Each of the sets  $W$  and  $Y$  is a strong deformation retract of

$$B = A \amalg \left( \bigcup_j \{j\} \times S^{k-1} \right).$$

Specifically, the homotopies deforming  $W$  and  $Y$  into  $B$  are the identity on  $A$  and map each of the sets  $\{ c < |\mathbf{x}| \leq 1 \}$ ,  $\{ c \leq |\mathbf{x}| \leq 1 \}$  to  $S^{k-1}$  by sending a (necessarily nonzero) vector  $\mathbf{y}$  to  $|\mathbf{y}|^{-1}\mathbf{y}$  and taking a straight line homotopy to join these two points. A direct check of the equivalence relation defining  $\varphi$  shows that the associated maps and homotopies  $W \rightarrow B \rightarrow W$  and  $Y \rightarrow B \rightarrow Y$  pass to the quotients  $V \rightarrow A \rightarrow V$  and  $\overline{V} \rightarrow A \rightarrow \overline{V}$ , and these quotient maps display  $A$  as a strong deformation retract of both  $V$  and  $\overline{V}$ . ■

### *Cell complex structures*

By the preceding discussion, a simplicial complex  $(P, \mathbf{K})$  has a finite, linearly ordered chain of closed subspaces

$$\emptyset = P_{-1} \subset P_0 \subset \dots \subset P_m = P$$

such that for each  $k$  satisfying  $0 \leq k \leq m$ , the subspace  $P_k$  is obtained from  $P_{k-1}$  by attaching finitely many  $k$ -cells. We shall generalize this property into a definition for arbitrary cell complex structures.

**Definition.** Let  $X$  be a topological space. A *finite cell complex structure* (or finite *CW* structure) on  $X$  is a chain  $\mathcal{E}$  of closed subspaces

$$\emptyset = X_{-1} \subset X_0 \subset \dots \subset X_m = X$$

such that for each  $k$  satisfying  $0 \leq k \leq m$ , the subspace  $X_k$  is obtained from  $X_{k-1}$  by attaching finitely many  $k$ -cells. The subspace  $X_k$  is called the  $k$ -skeleton of  $X$ , or more correctly the  $k$ -skeleton of  $(X, \mathcal{E})$ .

At this level of abstraction, the notion of cell complex structure is due to J. H. C. Whitehead (1904–1960); his definition extended to infinite cell complex structures and the letters  $CW$  were described as abbreviations for two properties of the infinite complexes that are explained in the Appendix of Hatcher’s book, but one should also note that the letters also represent Whitehead’s last two initials.

It follows immediately that simplicial complexes are examples of cell complexes. Numerous further examples appear on pages 5–8 of Hatcher. Furthermore, the  $\Delta$ -complexes discussed on pages 102–104 are also examples of cell complexes. In analogy with (edge-vertex) graphs, the main difference between  $\Delta$ -complexes and simplicial complexes is that two  $k$ -simplices in a  $\Delta$ -complex may have the same faces, but two  $k$ -simplices in a simplicial complex have at most a single  $(k - 1)$ -face in common.

Because of the following result, one often describes a cell complex structure as a *cellular decomposition* of  $X$ .

**PROPOSITION 5.** *If  $X$  is a space and  $\mathcal{E}$  is a cell decomposition of  $X$ , then every point of  $X$  lies on exactly one open cell of  $X$ .*

**Proof.** Since  $X = \cup_k (X_k - X_{k-1})$ , it follows that every point  $y \in X$  lies in a exactly subset of the form  $X_k - X_{k-1}$ . Therefore there is at most one value of  $k$  such that  $x$  can lie on an open  $k$ -cell. Furthermore, since  $X_k - X_{k-1}$  is a union of the opne  $k$ -cells and the latter are pairwise disjoint, it follows that  $x$  lies on exactly one of these open  $k$ -cells.■

**NOTE.** If a cell complex has an  $n$ -cell for some  $n > 0$  and  $0 < m < n$ , the cell complex might not have any  $m$ -cells (in contrast to the situation for, say, simplicial complexes); see Example 0.3 on page 6 of Hatcher.

Finally, we shall give a slightly different definition of subcomplex than the one in Hatcher.

**Definition.** If  $(X, \mathcal{E})$  is a cell complex, we say that a closed subspace  $A \subset X$  determines a *cell subcomplex* if for each  $k \geq 0$  the set  $A_k = X_k \cap A$  is obtained from  $A_{k-1}$  by attaching  $k$ -cells such that the every  $k$ -cell for  $A$  is also a  $k$ -cell for  $X$ .

There is an simple relationship between this notion of cell subcomplex and the previous definition of subcomplex for a simplicial complex; the proof is straightforward.

**PROPOSITION 6.** *If  $(P, \mathbf{K})$  is a simplicial complex and  $(P_1, \mathbf{K}_1)$  is a simplicial subcomplex, then  $P_1$  also determines a cell subcomplex.■*

Finally, here are two further observations regarding subcomplexes. Again, the proofs are straightforward.

**PROPOSITION 7.** *If  $X$  is a cell complex such that  $A \subset X$  determines a subcomplex of  $X$  and  $B \subset A$  determines a subcomplex of  $A$ , then  $B$  also determines a subcomplex of  $A$ . Likewise, if  $B$  determines a subcomplex of  $X$  then  $B$  determines a subcomplex of  $A$ .■*

**PROPOSITION 8.** *If  $X$  is a cell complex such that  $A \subset X$  determines a subcomplex of  $X$ , then for each  $k \geq 0$  the set  $X_k \cup A$  determines a subcomplex of  $X$ .■*

## II.4 : The Homotopy Extension Property

(Hatcher, Ch. 0, § 2.1)

In this section we shall bring together several concepts from the preceding sections. The basis is the following central Extension Question stated at the beginning of this unit, and our first result describes a condition under which this question always has an affirmative answer.

**PROPOSITION 1.** *Suppose that  $X$  and  $Y$  are topological spaces, that  $A \subset X$  is a retract, and that  $g : A \rightarrow Y$  is continuous. Then there is an extension of  $g$  to a continuous mapping  $f : X \rightarrow Y$ .*

**Proof.** Let  $r : X \rightarrow A$  be a continuous function such that  $r|_A$  is the identity, and define  $f = g \circ r$ . Then if  $a \in A$  we have  $f(a) = g \circ r(a) = g(r(a))$ , and the latter is equal to  $g(a)$  because  $r|_A$  is the identity. ■

The hypothesis of the proposition is fairly rigid, but the result itself is a key step in proving a general result on the Extension Question.

**THEOREM 2.** (HOMOTOPY EXTENSION PROPERTY) *Let  $(X, \mathcal{E})$  be a cell complex, and suppose that  $A$  determines a subcomplex. Suppose that  $Y$  is a topological space, that  $g : A \rightarrow Y$  is a continuous map, and  $f : X \rightarrow Y$  is a continuous map such that  $f|_A$  is homotopic to  $g$ . Then there is a continuous map  $G : X \rightarrow Y$  such that  $G|_A = g$ .*

**COROLLARY 3.** *Suppose that  $X$  and  $A$  are as above and that  $g : A \rightarrow Y$  is homotopic to a constant map. Then  $g$  extends to a continuous function from  $X$  to  $Y$ .*

**COROLLARY 4.** *Suppose that  $X$  and  $A$  are as above and that  $g : A \rightarrow X$  is homotopic to the inclusion map. Then  $g$  extends to a continuous function from  $X$  to itself.*

Corollary 3 follows because every constant map from  $A$  to  $Y$  extends to the analogous constant map from  $X$  to  $Y$ , and Corollary 4 follows because the inclusion of  $A$  in  $X$  extends continuously to the identity map from  $X$  to itself. ■

One important step in the proof of the Homotopy Extension Property relies upon the following result:

**PROPOSITION 5.** *For all  $k > 0$  the set  $D^k \times \{0\} \cup S^{k-1} \times [0, 1]$  is a strong deformation retract of  $D^k \times [0, 1]$ .*

**Proof.** This argument is outlined in Proposition 0.16 on page 15 of Hatcher, and there is a drawing to illustrate the proof in the `figures` document.

The retraction  $r : D^k \times [0, 1] \rightarrow D^k \times \{0\} \cup S^{k-1} \times [0, 1]$  is defined by a radial projection with center  $(\mathbf{0}, 2) \in D^k \times \mathbb{R}$ . As indicated by the drawing, the formula for  $r$  depends upon whether  $2|\mathbf{x}| + t \geq 2$  or  $2|\mathbf{x}| + t \leq 2$ . Specifically, if  $2|\mathbf{x}| + t \geq 2$  then

$$r(\mathbf{x}, t) = \frac{1}{|\mathbf{x}|} (\mathbf{x}, 2|\mathbf{x}| + t - 2)$$

while if  $2|\mathbf{x}| + t \leq 2$  then we have

$$r(\mathbf{x}, t) = \frac{1}{2} ((2 - t)\mathbf{x}, 0)$$



and these are consistent when  $2|\mathbf{x}| + t = 2$  then both formulas yield the value  $|\mathbf{x}|^{-1}(\mathbf{x}, 0)$ . Elementary but slightly tedious calculation then implies that  $r(\mathbf{x}, t)$  always lies in  $D^k \times [0, 1]$ , and likewise that  $r$  is the identity on  $D^k \times \{0\} \cup S^{k-1} \times [0, 1]$ . The homotopy from inclusion  $\circ r$  to the identity is then the straight line homotopy

$$H(\mathbf{x}, t; s) = (1 - s) \cdot r(\mathbf{x}, t) + s \cdot (\mathbf{x}, t)$$

and this completes the proof of the proposition. ■

**Proof of Theorem 2.** In the course of the proof we shall need the following basic fact: *If  $A$  and  $B$  are compact Hausdorff spaces and  $\varphi : A \rightarrow B$  is a quotient map in the sense of Munkres' book, then for each compact Hausdorff space  $C$  the product map  $\varphi \times 1_C : A \times C \rightarrow B \times C$  is also a quotient map.* — This follows because  $\varphi \times 1_C$  is closed, continuous and surjective.

Since the homotopy relation on continuous functions is transitive, a standard recursive argument reduces the proof of the theorem to the special cases subcomplex inclusions

$$X_{k-1} \cup A \subset X_k \cup A.$$

In other words, *it will suffice to prove the theorem when  $X$  is obtained from  $A$  by attaching  $k$ -cells.*

We now assume the condition in the preceding sentence. Let  $h : A \times [0, 1] \rightarrow Y$  be the homotopy from  $f$  (when  $t = 0$ ) to  $g$  (when  $t = 1$ ). If we can show that the inclusion

$$A \times [0, 1] \cup X \times \{0\} \subset X \times [0, 1]$$

is a retract, then we can use Proposition 1 to find an extension of the map

$$\theta = "h \cup f" : A \times [0, 1] \cup X \times \{0\} \longrightarrow Y$$

to  $X \times [0, 1]$ , and the restriction of this extension to  $X \times \{1\}$  will be a continuous extension of  $g$ . — In fact, we shall show that *the space  $A \times [0, 1] \cup X \times \{0\}$  is a strong deformation retract of  $X \times [0, 1]$ .*

As in earlier discussions let

$$\varphi : A \amalg (\{1, \dots, N\} \times D^k) \longrightarrow X$$

be the topological quotient map which exists by the definition of attaching  $k$ -cells. By Proposition 5 we know that the space

$$A \times [0, 1] \amalg (\{1, \dots, N\}) \times (S^{k-1} \times [0, 1] \cup D^k \times \{0\})$$

is a strong deformation retract of

$$(A \amalg \{1, \dots, N\} \times D^k) \times [0, 1]$$

because we can the mappings piecewise using the identity on  $A \times [0, 1]$  and the functions from Proposition 5 on each of the pieces  $\{j\} \times D^k \times [0, 1]$ . Let

$$\begin{aligned} r' : (A \amalg (\{1, \dots, N\} \times D^k)) \times [0, 1] &\longrightarrow \\ A \times [0, 1] \amalg (\{1, \dots, N\} \times (S^{k-1} \times [0, 1] \cup D^k \times \{0\})) & \end{aligned}$$

be the retraction obtained in this fashion, and let

$$H' : \left( (A \amalg \{1, \dots, N\} \times D^k) \times [0, 1] \right) \times [0, 1] \longrightarrow (A \amalg \{1, \dots, N\} \times D^k) \times [0, 1]$$

be defined similarly. It will suffice to show that these pass to continuous mappings of quotient spaces; in other words, we want to show there are (continuous) mappings  $r$  and  $H$  such that the following diagrams are commutative, in which  $\psi$  is the mapping whose values are given by  $\varphi$ :

$$\begin{array}{ccc} (A \amalg \dots) \times [0, 1] & \xrightarrow{r'} & A \times [0, 1] \amalg (\{1, \dots, N\} \times [\dots]) \\ \downarrow \varphi \times 1 & & \downarrow \psi \\ X \times [0, 1] & \xrightarrow{r} & A \times [0, 1] \cup X \times \{0\} \end{array}$$

$$\begin{array}{ccc} \left( (A \amalg \dots) \times [0, 1] \right) \times [0, 1] & \xrightarrow{H'} & (A \amalg \dots) \times [0, 1] \\ \downarrow \varphi \times 1 \times 1 & & \downarrow \phi \times 1 \\ (X \times [0, 1]) \times [0, 1] & \xrightarrow{H} & X \times [0, 1] \end{array}$$

Standard results on factoring maps through quotient spaces imply that such commutative diagrams exist if and only if (i) if two points map to the same point under  $\psi \circ r'$ , then they map to the same point under  $\varphi \times 1$ , (ii) if two points map to the same point under  $\phi \times 1 \circ H'$ , then they map to the same point under  $\varphi \times 1 \times 1$ . It is a routine exercise to check both of these statements are true. ■

**COROLLARY 6.** *Suppose that  $X$  and  $Y$  are as in the theorem and  $Y$  is contractible. Then every continuous mapping  $f : X \rightarrow Y$  has a continuous extension to  $X$ .*

**Proof.** It will suffice to prove that an arbitrary continuous mapping  $f : A \rightarrow Y$  is homotopic to a constant. We know that  $1_Y$  is homotopic to a constant map  $k$ , and therefore  $f = 1_Y \circ f$  is homotopic to the constant map  $k \circ f$ . ■

### III. Simplicial homology

The goal of this unit is to define a sequence of abelian groups associated to a simplicial complex  $(P, \mathbf{K})$  which are called **homology groups** and denoted by  $H_n(P, \mathbf{K})$ , where  $n$  runs through all the integers but the groups are all zero if  $n$  is negative. These groups may be interpreted as furnishing an “algebraic picture” of the underlying topological space  $P$ . In order to develop the important properties of these groups it will be necessary to introduce some basic concepts and results from *homological algebra*, but efforts will be made to keep this to a minimum.

We have stated that the groups provide information about the underlying space  $P$  rather than the simplicial complex  $(P, \mathbf{K})$  because *these groups turn out to depend only upon  $P$  itself*. This fact will drop out of the more general constructions in the next unit, where homology groups are defined for an arbitrary topological space and shown to agree with the groups of this unit if the space  $P$  has a simplicial decomposition.

#### *Some motivation from vector analysis*

Suppose that  $U$  is an open subset of  $\mathbb{R}^3$  and  $\Sigma$  is some sort of compact oriented surface in  $U$  (for our purposes, it suffices to think of  $\Sigma$  as having a continuously defined unit normal vector at every point). Then the boundary of  $\Sigma$  is some union of closed curves  $\Gamma_i$ , where the sense of  $\Gamma_i$  is chosen such that for each point of such a curve the ordered triple of vectors given by

the chosen unit normal vector to the surface at the point,

the unit tangent vector to the curve at the point,

the unit vector which is tangent to the surface at the point, but perpendicular to the curve’s tangent vector and directed **into** the surface

will form a right handed triad (see the illustration in the **figures** document); we shall not try to make everything rigorous here because the goal is to provide some intuition. In such a situation one sometimes says that the formal sum  $\sum_i \Gamma_i$  of the sensed curves  $\Gamma_i$  is homologous to zero in  $U$ , and by Stokes’ Theorem we have the following:

*If  $\sum_i \Gamma_i$  is homologous to zero in  $U$  and  $\mathbf{F}$  is a smooth vector field defined on  $U$  such that  $\nabla \times \mathbf{F} = \mathbf{0}$ , then*

$$\sum_i \int_{\Gamma_i} \mathbf{F} \cdot d\mathbf{x} = 0 \blacksquare$$

It is important to note that if  $V$  is an open subset of  $U$  and  $\sum_i \Gamma_i$  is homologous to zero in  $U$ , then  $\sum_i \Gamma_i$  is *not necessarily homologous to zero in  $V$* . The standard example for this involves the ordinary unit circle  $\Gamma$  in  $\mathbb{R}^2 \subset \mathbb{R}^3$  whose center is the origin and whose radius is 1. This curve is homologous to zero in  $\mathbb{R}^3$  because it bounds the closed unit disk. To see it is not homologous to zero in  $V = (\mathbb{R}^2 - \{\mathbf{0}\})$ , consider the vector field given by

$$\mathbf{F}(u, v) = \left( \frac{v}{u^2 + v^2}, \frac{-u}{u^2 + v^2}, 0 \right)$$

and note that  $\nabla \times \mathbf{F} = \mathbf{0}$  and the standard computation

$$\int_{\Gamma} \mathbf{F} \cdot d\mathbf{x} = 2\pi$$

imply that  $\Gamma$  cannot be homologous to zero in  $V$ .

Suppose now that we have a union of pairwise disjoint closed oriented surface  $\Sigma_j$  in our open set  $U$ ; the term “closed” means that the surfaces have no boundary curves, just like the unit sphere defined by  $u^2 + v^2 + w^2 = 1$ . We shall say that the formal sum  $\Sigma_1 + \cdots + \Sigma_j$  is homologous to zero in  $U$  if  $\cup_j \Sigma_j$  bounds a region  $W \subset U$  such that the closure of  $W$  is equal to the union of  $W$  and  $\cup_j \Sigma_j$  and the normal directions to  $\Sigma$  are all outward pointing. — For example, the unit sphere is homologous to zero in  $\mathbb{R}^3$  because it bounds a unit disk, and if  $\Sigma_r$  denotes the sphere of radius  $r$  in  $\mathbb{R}^3$ , then  $\Sigma_1 \cup \Sigma_2$  is homologous to zero if we orient the pieces so that the normal vectors on  $\Sigma_2$  point outward (away from the origin) and the normal vectors on  $\Sigma_1$  point inward (towards the origin). The Divergence Theorem from vector analysis then has the following implication:

If  $\Sigma_1 + \cdots + \Sigma_n$  is homologous to zero in  $U$  and  $\mathbf{F}$  is a smooth vector field defined on  $U$  such that  $\nabla \cdot \mathbf{F} = 0$ , then

$$\sum_i \iint_{\Sigma_i} \mathbf{F} \cdot d\mathbf{\Sigma} = 0 \blacksquare$$

We can now show that  $\Sigma_1$  is not homologous to zero in  $\mathbb{R}^3 - \{\mathbf{0}\}$  by an argument similar to the preceding one. Let  $\mathbf{F}$  be the vector field on  $\mathbb{R}^3 - \{\mathbf{0}\}$  defined by  $\mathbf{F}(\mathbf{x}) = |\mathbf{x}|^{-1}\mathbf{x}$ ; then it is a routine exercise to prove that  $\nabla \cdot \mathbf{F} = 0$  but direct computation shows that

$$\iint_{\Sigma_1} \mathbf{F} \cdot d\mathbf{\Sigma} = 4\pi .$$

Homology theory provides an organized algebraic framework for studying such phenomena.

### III.1 : Exact sequences and chain complexes

(Hatcher, § 2.1)

This section is basically algebraic, and at first the need for formally introducing the concepts may be unclear. However, the notions described here arise repeatedly in algebraic topology and other subjects.

**Definition.** Suppose we are given a diagram of the form

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in which the objects are abelian groups (possibly with some additional structure) and the morphisms are abelian group homomorphisms (possibly preserving the extra structure). We shall say that the diagram is *exact* at  $B$  if the kernel of  $g$  is equal to the image of  $f$ .

More generally, if we are given a linear diagram such as

$$\cdots \longrightarrow Z \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow \cdots$$

we shall say that it is an *exact sequence* if it is exact at every object which is the domain of one morphism and the codomain of another.

## Examples

There are many standard exact sequences in elementary algebra.

1. A *short exact sequence* is one having the form  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ . Exactness at  $A$  means that the kernel of  $A \rightarrow B$  is the image of  $0 \rightarrow A$ , which is equivalent to saying that the map is injective. Similarly, exactness at  $C$  means that the kernel of  $C \rightarrow 0$  is the image of  $B \rightarrow C$ , which is equivalent to saying that the map is surjective. The short exact sequence property is then equivalent to saying that  $A \rightarrow B$  is injective, and  $C$  is isomorphic to the quotient of  $B$  by the image of  $A$ .
2. The *cokernel* of a homomorphism  $f : A \rightarrow B$  is defined to be the quotient group  $B/f[A]$ . Given an arbitrary homomorphism  $f : A \rightarrow B$ , one then has the following *kernel – cokernel exact sequence*:

$$0 \longrightarrow \text{Ker}(f) \longrightarrow A \longrightarrow B \longrightarrow \text{Coker}(f) \longrightarrow 0$$

3. Let  $U$  be a connected open subset of  $\mathbb{R}^2$ , let  $\mathbf{C}^\infty(U)$  denote the infinitely differentiable real valued functions on  $U$ , and let  $\mathbf{VF}(U)$  denote the infinitely differentiable (2-dimensional) vector fields on  $U$  in the sense of vector analysis. If we let  $\mathbb{R} \rightarrow \mathbf{C}^\infty(U)$  denote the inclusion of the constant functions and take the gradient map from  $\mathbf{C}^\infty(U)$  to  $\mathbf{VF}(U)$ , then it follows that the sequence  $\mathbb{R} \rightarrow \mathbf{C}^\infty(U) \rightarrow \mathbf{VF}(U)$  is exact. Furthermore, if we take the map  $\mathbf{VF}(U) \rightarrow \mathbf{C}^\infty(U)$  which sends a vector field  $\mathbf{F} = (P, Q)$  to its “scalar curl”  $Q_1 - P_2$ , then the sequence  $\mathbf{C}^\infty(U) \rightarrow \mathbf{VF}(U) \rightarrow \mathbf{C}^\infty(U)$  will be exact **provided**  $U$  is convex (or more generally star-shaped). — On the other hand, the second sequence is not exact if  $U = \mathbb{R}^2 - \{\mathbf{0}\}$ , for the previously described vector field on  $U$  with coordinate functions  $v/r$  and  $-u/r$  has zero scalar curl but is not the gradient of any smooth function on  $U$ ; this follows from Green’s Theorem and the previous line integral calculation.

We can extend the preceding if  $U$  is a connected open set in  $\mathbb{R}^3$  by considering the following sequence:

$$\mathbb{R} \xrightarrow{\text{constants}} \mathbf{C}^\infty(U) \xrightarrow{\text{grad}} \mathbf{VF}(U) \xrightarrow{\text{curl}} \mathbf{VF}(U) \xrightarrow{\text{div}} \mathbf{C}^\infty(U)$$

This is again exact at the left hand object  $\mathbf{C}^\infty(U)$ , and standard results in vector analysis imply that the kernel of the curl is contained in the image of the gradient, while the kernel of the divergence is contained in the image of the curl. If  $U$  is convex, then one can also show that the sequence is exact, but in general this is not true. Our previous examples give a vector field on  $\mathbb{R}^2 - \{\mathbf{0}\} \times \mathbb{R}$  whose curl is zero but cannot be expressed as a gradient over  $U$ , and a vector field on  $\mathbb{R}^3 - \{\mathbf{0}\}$  whose divergence is zero but cannot be expressed as the curl of another vector field over  $U$ .

## Graded objects

The next concept is simple but indispensable.

**Definition.** Let  $A$  be a set, and let  $\mathbf{C}$  be a category. A *graded object* over  $\mathbf{C}$  with grading set  $A$  is a function  $X$  from  $A$  to the objects of  $\mathbf{C}$ . The object corresponding to  $a$  is generally denoted by  $X_a$ .

For example, one can construct a graded vector space over the reals with grading set the integers  $\mathbb{Z}$  by taking  $V_n = \mathbb{R}^n$  for  $n \geq 0$  and setting  $V_n$  equal to the zero space if  $n < 0$ .

Another example is obtainable from an algebra of polynomials  $\mathbb{R}[x_1, \dots, x_n]$  in finitely many indeterminates. Here we can take  $V_n$  to be the set of all homogeneous polynomials of degree  $n$  together with the zero polynomial.

In this course we shall mainly be interested in *nonnegatively graded objects*, where the indexing set is  $\mathbb{Z}$  and the object  $X_n$  is a suitable zero object if  $n < 0$ . For the categories of abelian groups or modules over some associative ring with unit, the meaning of zero object is obvious, and these categories are the only ones to be considered here.

**Definition.** If  $X$  and  $Y$  are nonnegatively graded objects over a category  $\mathbf{C}$ , then a *graded morphism of degree zero* or *grade preserving morphism* is a function  $f$  which assigns to each  $n \in \mathbb{Z}$  a morphism  $f_n : X_n \rightarrow Y_n$  in the category  $\mathbf{C}$ .

In the polynomial example, one can define a grade preserving homomorphism by sending the homogeneous polynomial  $p(x_1, x_2, \dots, x_n)$  to the homogeneous polynomial  $q(x_1, x_2, \dots, x_n) = p(x_1, x_1 + x_2, \dots, x_n)$ . Obviously there are many other maps of this type.

The following observation is immediate:

**PROPOSITION 1.** *Given a category  $\mathbf{C}$ , the  $\mathbb{Z}$ -graded objects over  $\mathbf{C}$  and graded morphisms of degree zero form a category. ■*

In fact, this category has many structural properties that are direct analogs of properties that hold for  $\mathbf{C}$  (for example, subobjects, quotient objects, direct products, and so on).

### *Chain complexes*

The following concept is absolutely fundamental.

**Definition.** Let  $\mathbf{C}$  be the category of abelian groups and homomorphisms or a category of unital modules over an associative ring with unit  $R$ . A **chain complex** over  $\mathbf{C}$  is a pair  $(A, d)$  consisting of a graded object  $A$  over  $\mathbf{C}$  indexed by the integers together with morphisms  $d_j : A_j \rightarrow A_{j-1}$  such that  $d_{j-1} \circ d_j = 0$  for all  $j$ .

Here are a few simple examples.

1. Given an arbitrary graded module  $A$ , one can make it into a chain complex by taking  $d_j = 0$  for all  $j$ . More generally, given a sequence of homomorphisms  $f_{2j} : A_{2j} \rightarrow A_{2j-1}$ , one can define a chain complex whose graded module is  $A$  with  $d_{2j} = f_{2j}$  and  $d_{2j-1} = 0$ .
2. Suppose we are given three modules  $B, H$ , and  $B'$ . Then we can define a chain complex with  $A_2 = B$ ,  $A_1 = B \oplus H \oplus B'$ , and  $A_0 = B'$  and  $A_j = 0$  otherwise such that  $d_2$  is injection into the first summand,  $d_1$  is projection onto the third summand, and all other maps  $d_j$  must be zero (since either their domain or codomain is zero).
3. If  $U$  is open in  $\mathbb{R}^2$ , then one can obtain a chain complex from the previous sequence involving  $\mathbf{C}^\infty(U)$  and  $\mathbf{VF}(U)$ , if one takes  $A_3$  to be the reals,  $A_2$  and  $A_0$  to be the smooth functions,  $A_1$  to be the vector fields, with morphisms given by inclusion of constants from  $A_3$  to  $A_2$ , gradient from  $A_2$  to  $A_1$ , scalar curl from  $A_1$  to  $A_0$ , and with all other real vector spaces and morphisms equal to zero. Similarly, if  $U$  is open in  $\mathbb{R}^3$  one has a system with  $A_4$  equal to the reals,  $A_3$  and  $A_0$  equal to the smooth functions,  $A_2$  and  $A_1$  equal to the vector fields, with morphisms given by inclusion of constants from  $A_4$  to  $A_3$ , gradient from  $A_3$  to  $A_2$ , curl from  $A_2$  to  $A_1$ , divergence from  $A_1$  to  $A_0$ , and with all other real vector spaces and morphisms equal to zero.

The mapping  $d$  is often called a *differential*; the motivation is related to the preceding examples where the maps are given by some form of differentiation.

**Definition.** Given two chain complexes  $(A, d)$  and  $(B, e)$  a **chain map**  $f : A \rightarrow B$  is a graded morphism such that for all integers  $j$  we have  $e_j \circ f_j = f_{j-1} \circ d_j$ . In other words, the following diagram is commutative:

$$\begin{array}{ccc} A_j & \xrightarrow{f_j} & B_j \\ \downarrow d_j & & \downarrow e_j \\ A_{j-1} & \xrightarrow{f_{j-1}} & B_{j-1} \end{array}$$

If the differential in a chain complex  $(A, d)$  is unambiguous from the context we shall frequently write  $A$  instead of  $(A, d)$ .

The following consequences of the definitions are elementary but important.

**PROPOSITION 2.** *Given a category  $\mathbf{C}$ , the chain complexes over  $\mathbf{C}$  and chain complex morphisms form a category.*

**PROPOSITION 3.** *If  $(A, d)$  and  $(B, e)$  are chain complexes over  $\mathbf{C}$  and  $f : (A, d) \rightarrow (B, e)$  is a morphism of chain complex such that the mappings  $f_j$  are all isomorphisms, then the map  $f^{-1}$  of graded modules defined by  $(f^{-1})_j = f_j^{-1}$  is also a chain map.*

**Proof.** To simplify the formulas let  $g_j = f_j^{-1}$ . The conclusion of the proposition is equivalent to the identities  $d_j \circ g_j = g_{j-1} \circ e_j$  as maps from  $B_j$  to  $A_{j-1}$ .

Let  $b \in B_j$  be arbitrary. Since  $f_{j-1}$  is injective, it follows that  $d_j \circ g_j(b) = g_{j-1} \circ e_j(b)$  if and only if  $f_{j-1} \circ d_j \circ g_j(b) = f_{j-1} \circ g_{j-1} \circ e_j(b)$ . The left hand side is equal to

$$f_{j-1} \circ d_j \circ g_j(b) = e_j \circ f_j \circ g_j(b) = e_j(b)$$

by the defining identity for chain maps and the fact that  $g$  is inverse to  $f$ , and the latter fact also implies that the right hand side is equal to  $e_j(b)$ . Therefore it follows that the maps  $g_j$  satisfies the defining conditions for a chain map. ■

As before, the category of chain complexes over  $\mathbf{C}$  has many structural properties that are direct analogs of properties that hold for  $\mathbf{C}$  and the category of graded objects over  $\mathbf{C}$  (such as subobjects, quotient objects, direct products).

A few additional remarks about subcomplexes and quotient complexes seem worthwhile. If  $(A, d')$  is a chain subcomplex of  $(B, d)$ , then it follows that  $A_j \subset B_j$  for all  $j$  and that  $d_j$  maps  $A_j$  to  $A_{j-1}$  via  $d'_j$ . The quotient complex has a differential  $d''$  such that  $d''_j[x] = [d_j x]$ , where “[ $\dots$ ]” denotes the equivalence class in the appropriate quotient module. There is a well-defined map of this sort because  $d_j$  maps  $A_j$  into  $A_{j-1}$ .

**ONE MORE EXAMPLE.** Let  $\Delta$  be a 2-simplex with vertices  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z}$ , let  $C_0$  be the free abelian group generated by these vertices, let  $C_1$  be the free abelian group generated by the three edges  $\mathbf{yz}$ ,  $\mathbf{xz}$  and  $\mathbf{xy}$ , let  $C_2$  be the free abelian group generated by the element  $\Delta$ , and define maps  $d_2 : C_2 \rightarrow C_1$  and  $d_1 : C_1 \rightarrow C_0$  by

$$\begin{aligned} d_2(\Delta) &= \mathbf{yz} - \mathbf{xy} + \mathbf{xz} \\ d_1(\mathbf{xy}) &= \mathbf{y} - \mathbf{x}, \quad d_1(\mathbf{yz}) = \mathbf{z} - \mathbf{y} \quad \text{and} \quad d_1(\mathbf{xz}) = \mathbf{z} - \mathbf{x}. \end{aligned}$$

We set all other groups  $C_j$  equal to zero, and it follows that all remaining homomorphisms must also be zero. Direct examination shows that the kernel of  $d_1$  is the set of all multiples of  $d_2(\Delta)$ .

Geometrically,  $d_2(\Delta)$  represents the boundary of the simplex  $A$  with the edges oriented so that they correspond to a simple closed curve. More generally, if  $(A, d)$  is a chain complex then elements in the kernel of  $d_j$  are frequently called *cycles*, while elements in the image of  $d_{j+1}$  are frequently called *boundaries*, and the homomorphisms  $d_j$  are frequently called *boundary homomorphisms*.

### III.2: Homology groups

(Hatcher, § 2.1)

If  $(A, d)$  is a chain complex, then the condition  $d_j \circ d_{j+1} = 0$  implies that the kernel of  $d_j$  (the submodule of cycles) contains the image of  $d_{j+1}$  (the submodule of boundaries). The sequence determined by the chain complex is exact at  $A_j$  if and only if these two submodules are equal. One can view homology groups as measuring the extent to which a chain complex is not an exact sequence.

**Definition.** Let  $(A, d)$  be a chain complex. The  $j^{\text{th}}$  homology group  $H_j(A) = H_j(A, d)$  is equal to the quotient module

$$(\text{Kernel } d_j) / (\text{Image } d_{j+1}).$$

By the definitions, the sequence of morphisms determined by a chain complex  $(A, d)$  is exact at  $A_j$  if and only if  $H_j(A) = 0$ .

Computation of the homology groups for the examples in Section III.1 is fairly straightforward.

1. If we take an arbitrary graded module  $A$  and make it into a chain complex by taking  $d_j = 0$  for all  $j$ , then  $H_j(A, 0) = A_j$ . If we are given a sequence of homomorphisms  $f_{2j} : A_{2j} \rightarrow A_{2j-1}$  and define a chain complex whose graded module is  $A$  with  $d_{2j} = f_{2j}$  and  $d_{2j-1} = 0$ , then  $H_{2j}(A) = \text{Kernel } d_{2j}$  and  $H_{2j-1}(A) = A_{2j-1} / \text{Image } d_{2j}$ .
2. In Example 2 from the previous section, the homology is zero if  $U$  is a convex open subset of  $\mathbb{R}^2$  or  $\mathbb{R}^3$ .
3. In “ONE MORE EXAMPLE” from the previous section, we have  $H_j(C) = 0$  if  $j \neq 0$ , while  $H_0(C)$  is infinite cyclic, with the generator represented by the class of  $\mathbf{x}$  (and the same generator also turns out to be represented by  $\mathbf{y}$  and  $\mathbf{z}$ ).

The next result is fairly simple to prove but absolutely fundamental.

**THEOREM 1.** *If  $f : (A, d^A) \rightarrow (B, d^B)$  is a map of chain complexes, then there are unique homomorphisms  $f_* : H_k(A) \rightarrow H_k(B)$  such that if  $u \in H_k(A)$  is represented by  $z \in A_q$ , then  $f_*(u)$  is represented by  $f_q(z)$ . Furthermore, if  $f$  is a chain identity map then  $f_*$  is also the identity, and if  $g : (B, d^B) \rightarrow (C, d^C)$  is another chain map, then  $(g \circ f)_* = g_* \circ f_*$ .*

The second sentence of the theorem implies that the construction sending  $f$  to  $f_*$  defines a covariant functor from chain complexes to graded modules. Thus the following is immediate.

**COROLLARY 2.** *In the setting above, if  $f$  is an isomorphism then so is  $f_*$ . ■*

**Proof of Theorem 1.** The condition in the first sentence of the theorem implies uniqueness, and the formula for  $f_*$  immediately yields the functoriality properties in the second sentence. Thus everything reduces to showing that there is indeed a homomorphism  $f_*$  with the asserted property.



First of all, we must check that  $f_q(z)$  is a cycle if  $z$  is a cycle. To see this note that

$$d_q^B \circ f_q(z) = f_{q-1} \circ d_q^A(z) = f_{q-1}(0) = 0$$

so there is no problem here. Next, we need to check that if  $z$  and  $w$  represent the same class in  $A_q$ , then  $f_q(z)$  and  $f_q(w)$  represent the same class in  $B_q$ . However, if  $z$  and  $w$  represent the same class, then  $z - w = d_{q+1}^A(y)$ , and hence we have

$$f_q(z) - f_q(w) = f_q(z - w) = f_q \circ d_{q+1}^A(y) = d_q^B \circ f_{q+1}(y)$$

so that the images of  $z$  and  $w$  represent the same class in  $H_q(B)$ . The identities  $f_*(u_1 + u_2) = f_*(u_1) + f_*(u_2)$  and  $f_*(r \cdot u) = r \cdot f_*(u)$  now follow immediately from the definition of  $f_*$  and the standard choices of representatives for  $u_1 + u_2$  and  $r \cdot u$ . ■

### III.3 : Homology and simplicial complexes

(Hatcher, § 2.1)

In this section we shall take the first step towards defining homology groups for topological spaces. At this stage we can only handle special classes of spaces with additional geometrical structures, and our definitions will also depend upon the extra structure. In fact, we shall give three different definitions of homology here, and a major objective of the rest of this unit will be to show that they are naturally equivalent. The following citation from a set of online lecture notes by J. W. Morgan summarize the situation quite well.

The main trouble with algebraic topology is that there are many different approaches to defining the basic ... homology ... groups. Each approach brings with it a fair amount of required technical baggage ... one must pay a fairly high price ... as one slogs through the basic constructions and proves the basic results. Furthermore, possibly the most striking feature of the subject, the interrelatedness (and often equality) of the theories ... requires even more machinery.

A more precise citation is pages 4 and 5 of the following document:

<http://www.math.columbia.edu/~jm/algtop.ps>

[This link no longer seems to be functioning.]

Some earlier passages from the same notes describe some reasons why such a “large and complicated array of tools” has proven to be worth knowing:

The subject has turned out to have a vast ... range of applicability ... The power of algebraic topology is the generality of its application. The tools apply in situations so disparate as seemingly to have nothing to do with each other, yet the common thread linking them is algebraic topology. One of the most impressive arguments by analogy of twentieth century mathematics is the work of the French school of algebraic geometry, mainly Weil, Serre, Grothendieck and Deligne, to apply the machinery of algebraic topology to projective varieties defined over finite fields in order to prove the Weil Conjectures. On the face of it these conjectures, which dealt with counting the number of solutions over finite fields of polynomial equations, have nothing to do with usual topological spaces and algebraic topology. The powerful insight ... was to recognize that in fact there was a relationship and then to establish the vast array of technical results in algebraic geometry over finite fields necessary to implement this relationship. ... A quote from [the celebrated mathematician Solomon] Lefschetz [1884 – 1972] seems appropriate to capture the spirit of the subject; after a long and complicated study ... he said, ‘we have succeeded in planting the harpoon of algebraic topology in the whale of algebraic geometry.’

Further examples of the uses of ideas from algebraic topology are noted on page 4 of Morgan’s notes.

*Three definitions of simplicial homology groups*

One central feature of algebraic topology is that there are usually several different chain complexes which yield the same homology groups, each of which has its own advantages and disadvantages. We shall start with a definition involving a relatively small chain complex.

**First Definition.** Suppose that  $(P, \mathbf{K})$  is a simplicial complex, and choose a linear ordering  $L$  for the vertices of  $\mathbf{K}$ ; we shall use the usual notation  $\mathbf{v} < \mathbf{w}$  to indicate that one vertex precedes another. For each integer  $k$ , the  $k$ -dimensional ordered simplicial chain group of  $(P, \mathbf{K})$ , written  $C_k^{\text{ordered}}(P, \mathbf{K})$  is a free abelian group on all objects  $\mathbf{v}_0 \cdots \mathbf{v}_k$ , where  $\mathbf{v}_0 < \cdots < \mathbf{v}_k$ . By construction, it follows that  $C_k^{\text{ordered}}(P, \mathbf{K}) = 0$  if  $k < 0$  or  $k > \dim \mathbf{K}$ . The boundary homomorphism

$$d_k : C_k^{\text{ordered}}(P, \mathbf{K}) \longrightarrow C_{k-1}^{\text{ordered}}(P, \mathbf{K})$$

is defined on free generators by the formula

$$d_k(\mathbf{v}_0 \cdots \mathbf{v}_k) = \sum_{j=0}^k (-1)^j \mathbf{v}_0 \cdots \widehat{\mathbf{v}}_j \cdots \mathbf{v}_k$$

where  $\widehat{\mathbf{v}}_i$  means that  $\mathbf{v}_i$  is omitted; by the definition of free generators, it follows that there is a unique extension to the group  $C_k^{\text{ordered}}(P, \mathbf{K})$ .

Since our purpose is to define homology groups, presumably we want to verify that the preceding data define a chain complex. For this purpose it will be helpful to introduce some additional definitions.

If  $k > 0$  and  $\mathbf{v}_0 \cdots \mathbf{v}_k$  is as above, then the  $i^{\text{th}}$  **face operator**  $\partial_i^{[k]}(\mathbf{v}_0 \cdots \mathbf{v}_k)$  is given by

$$\mathbf{v}_0 \cdots \widehat{\mathbf{v}}_i \cdots \mathbf{v}_k .$$

Frequently we shall suppress the superscript  $[k]$  to simplify notation. The following identity for iterated faces is elementary but fundamentally important:

**LEMMA 1.** *If  $k - 1 \geq j \geq i$ , then  $\partial_j^{[k-1]} \circ \partial_i^{[k]} = \partial_i^{[k-1]} \circ \partial_{j+1}^{[k]}$ .*

The identity is true because the result of applying both composites to  $\mathbf{v}_0 \cdots \mathbf{v}_k$  is given by deleting  $\mathbf{v}_i$  and  $\mathbf{v}_{j+1}$ . ■

With Lemma 1, it is fairly easy to prove that the boundary maps  $d_k$  define a chain complex.

**THEOREM 2.** *In the setting above we have  $d_{k-1} \circ d_k = 0$ .*

The proof of this result is given in Lemma 2.1 on pages 105–106 of Hatcher. ■

We now define the  $k$ -dimensional simplicial homology group of  $(P, \mathbf{K})$  for ordered simplicial chains, also called the  $k$ -dimensional ordered simplicial homology group and denoted by

$$H_k^{\text{ordered}}(P, \mathbf{K})$$

to be the  $k$ -dimensional homology of the chain complex  $C_*^{\text{ordered}}(P, \mathbf{K})$ , where the differential or boundary is given as above.

The preceding definition depends not only upon the choice of a simplicial decomposition but also upon choosing a linear ordering of the vertices. Ultimately we want to show the homology groups depend only upon the underlying space  $P$ , and as a first step we would like to prove the groups do not depend upon the choice of linear ordering. Our approach to doing will involve finding other definitions of homology that do not depend upon the choice of a vertex orderings and showing that the new definitions yield the same homology groups.

**Second Definition.** Given  $(P, \mathbf{K})$  as above, the *unordered simplicial chain group*  $C_k(P, \mathbf{K})$  is the free abelian group on all symbols  $\mathbf{u}_0 \cdots \mathbf{u}_k$ , where the  $\mathbf{u}_j$  are all vertices of some simplex in  $\mathbf{K}$  and repetitions of vertices are allowed. A family of differential or boundary homomorphisms  $d_k$  is defined as before, and the  $k$ -dimensional simplicial homology  $H_k(P, \mathbf{K})$  is defined to be the  $k$ -dimensional homology of this chain complex.

The unordered simplicial chain complex  $C_*(P, \mathbf{K})$  contains the ordered simplicial chain complex  $C_*^{\text{ordered}}(P, \mathbf{K})$  as a chain subcomplex, and we shall let  $i$  denote the resulting inclusion map of chain complexes. If we can show that the associated homology maps  $i_*$  are isomorphisms, then it will follow that the homology groups for the ordered simplicial chain complex agree with the corresponding groups for the unordered simplicial chain complex.

One major difference between the unordered and ordered simplicial chain groups is that the latter are nontrivial in every positive dimension. In particular, if  $\mathbf{v}$  is a vertex of  $\mathbf{K}$ , then the free generator  $\mathbf{v} \cdots \mathbf{v} = \mathbf{u}_0 \cdots \mathbf{u}_k$ , with  $\mathbf{u}_j = \mathbf{v}$  for all  $j$ , represents a nonzero element of  $C_k(P, \mathbf{K})$ . On the other hand, the ordered simplicial chain groups are nonzero for only finitely many values of  $k$ .

In order to analyze the mappings  $i_*$ , we shall introduce yet another definition of homology groups.

**Third Definition.** In the setting above, define the subgroup  $C'_k(P, \mathbf{K})$  of *degenerate* simplicial  $k$ -chains to be the subgroup generated by

- (a) all elements  $\mathbf{v}_0 \cdots \mathbf{v}_k$  such that  $\mathbf{v}_i = \mathbf{v}_{i+1}$  for some (at least one)  $i$ ,
- (b) all sums  $\mathbf{v}_0 \cdots \mathbf{v}_i \mathbf{v}_{i+1} \cdots \mathbf{v}_k + \mathbf{v}_0 \cdots \mathbf{v}_{i+1} \mathbf{v}_i \cdots \mathbf{v}_k$ , where  $0 \leq i < k$ .

We claim these subgroups define a chain subcomplex, and to show this we need to verify the following.

**LEMMA 3.** *The boundary homomorphism  $d_k$  sends elements of  $C'_k(P, \mathbf{K})$  to  $C'_{k-1}(P, \mathbf{K})$ .*

It suffices to prove that the boundary map sends the previously described generators into degenerate chains, and checking this is essentially a routine calculation. ■

We now define the complex of *alternating simplicial chains*  $C_*^{\text{alt}}(P, \mathbf{K})$  to be the quotient complex  $C_*(P, \mathbf{K})/C'_*(P, \mathbf{K})$  with the associated differential or boundary map.

**PROPOSITION 4.** *The composite  $\varphi : C_*^{\text{ordered}}(P, \mathbf{K}) \rightarrow C_*(P, \mathbf{K}) \rightarrow C_*^{\text{alt}}(P, \mathbf{K})$  is an isomorphism of chain complexes.*

**COROLLARY 5.** *The morphism  $i_* : H_*^{\text{ordered}}(P, \mathbf{K}) \rightarrow H_*(P, \mathbf{K})$  is injection onto a direct summand.*

**Proof that Proposition 4 implies Corollary 5.** Let  $q$  be the projection map from unordered to alternating chains, so that  $\varphi_* = q_* \circ i_*$ . General considerations imply that  $\varphi_*$  is an isomorphism.

Suppose now that  $i_*(a) = i_*(b)$ . Applying  $q_*$  to each side we obtain

$$\varphi_*(a) = q_* \circ i_*(a) = q_* \circ i_*(b) = \varphi_*(b)$$

and since  $\varphi_*$  is bijective it follows that  $a = b$ .

Now let  $B_*$  be the kernel of  $q_*$ . We shall prove that every element of  $H_*(P, \mathbf{K})$  has a unique expression as  $i_*(a) + c$ , where  $c \in B_*$ . Given  $u \in H_*(P, \mathbf{K})$ , direct computation implies that

$$u - i_*(\varphi_*)^{-1}q_*(u) \in B_*$$

and thus yields existence. Suppose now that  $u = i_*(a) + c$ , where  $c \in B_*$ . It then follows from the definitions that

$$i_*(a) = i_*(\varphi_*)^{-1}q_*(u)$$

and hence we also have

$$c = u - i_*(a) = u - i_*(\varphi_*)^{-1}q_*(u)$$

which proves uniqueness. ■

**Proof of Proposition 4.** Analogs of standard arguments for determinants yield the following observations:

- (1) *The generator  $\mathbf{v}_0 \cdots \mathbf{v}_k \in C_k(P, \mathbf{K})$  lies in the subgroup of degenerate chains if two vertices are equal.*
- (2) *If  $\sigma$  is a permutation of  $\{0, \dots, k\}$ , then  $\mathbf{v}_0 \cdots \mathbf{v}_k - (-1)^{\text{sgn}(\sigma)} \mathbf{v}_{\sigma(0)} \cdots \mathbf{v}_{\sigma(k)}$  is a degenerate chain.*

Define a map of graded abelian groups  $\Psi$  from  $C_*(P, \mathbf{K})$  to  $C_*^{\text{ordered}}(P, \mathbf{K})$  which sends  $\mathbf{v}_0 \cdots \mathbf{v}_k$  to zero if there are repeated vertices and sends  $\mathbf{v}_0 \cdots \mathbf{v}_k$  to  $(-1)^{\text{sgn}(\sigma)} \mathbf{v}_{\sigma(0)} \cdots \mathbf{v}_{\sigma(k)}$  if the vertices are distinct and  $\sigma$  is the unique permutation which puts the vertices in the proper order:

$$\mathbf{v}_{\sigma(0)} < \cdots < \mathbf{v}_{\sigma(k)}$$

It follows that  $\Psi$  passes to a map  $\psi$  of quotients from  $C_*^{\text{alt}}(P, \mathbf{K})$  to  $C_*^{\text{ordered}}(P, \mathbf{K})$  such that  $\psi \circ \varphi$  is the identity. In particular, it follows that  $\varphi$  is injective. To prove it is surjective, note that (1) and (2) imply that  $C_k^{\text{alt}}(P, \mathbf{K})$  is generated by the image of  $\varphi$  and hence *varphi* is also surjective. It follows that  $\varphi$  determines an isomorphism of chain complexes as required. ■

### *Acyclic complexes*

**Definition.** An *augmented chain complex* over a ring  $R$  consists of a chain complex  $(C_*, d)$  and a homomorphism  $\varepsilon : C_0 \rightarrow R$  (the augmentation map) such that  $\varepsilon$  is onto and  $\varepsilon \circ d_1 = 0$ .

All of the simplicial chain complexes defined above have canonical augmentations given by sending expressions of the form  $\sum n_{\mathbf{v}} \mathbf{v}$  to the corresponding integers  $\sum n_{\mathbf{v}}$ .

**Definition.** A simplicial complex is said to be *acyclic* (“has no nontrivial cycles”) if  $H_j(P, \mathbf{K}) = 0$  for  $j \neq 0$  and  $H_0 \cong \mathbb{Z}$ , with the generator in homology represented by an arbitrary free generator of  $C_0(P, \mathbf{K})$ .

There is a simple geometric criterion for a simplicial chain complex to be acyclic.

**Definition.** A simplicial complex  $(P, \mathbf{K})$  is said to be *star shaped* with respect to some vertex  $\mathbf{v}$  in  $\mathbf{K}$  if for each simplex  $A$  in  $\mathbf{K}$  either  $\mathbf{v}$  is a vertex of  $A$  or else there is a simplex  $\mathbf{B}$  in  $\mathbf{K}$  such that  $A$  is a face of  $\mathbf{B}$  and  $\mathbf{v}$  is a vertex of  $\mathbf{B}$ .

Some examples are described in the **figures** document. One particularly important example for the time being is the standard simplex  $\Delta_n$  with its standard decomposition.

**PROPOSITION 6.** *If the simplicial complex  $(P, \mathbf{K})$  is star shaped with respect to some vertex, then it is acyclic, and the map  $i_* : H_*^{\text{ordered}}(P, \mathbf{K}) \rightarrow H_*(P, \mathbf{K})$  is an isomorphism.*

**Proof.** Define a map of graded abelian groups  $\eta : C_*(P, \mathbf{K}) \rightarrow C_*(P, \mathbf{K})$  such that  $\eta_q : C_q(P, \mathbf{K}) \rightarrow C_q(P, \mathbf{K})$  is zero if  $q \neq 0$  and  $\eta_0$  sends a chain  $y$  to  $\varepsilon(y) \mathbf{v}$ . Then  $\eta$  is a chain map because  $\varepsilon \circ d_1 = 0$ .

We next define homomorphisms  $D_q : C_q(P, \mathbf{K}) \rightarrow C_{q+1}(P, \mathbf{K})$  such that

$$d_{q+1} \circ D_q = \text{identity} - d_q \circ D_{q-1}$$

if  $q$  is positive and

$$d_1 \circ D_0 = \text{identity} - \eta_0$$

on  $C_0$ . We do this by setting  $D_q(\mathbf{x}_0 \cdots \mathbf{x}_q) = \mathbf{v} \mathbf{x}_0 \cdots \mathbf{x}_q$  and taking the unique extension which exists since the classes  $\mathbf{x}_0 \cdots \mathbf{x}_q$  are free generators for  $C_q$ . Elementary calculations show that the mappings  $D_q$  satisfy the conditions given above.

To see that  $H_q(P, \mathbf{K}) = 0$  if  $q > 0$ , suppose that  $d_q(z) = 0$ . Then the first formula implies that  $z = d_{q+1} \circ D_q(z)$ . Therefore  $H_q = 0$  if  $q > 0$ . On the other hand, if  $z \in C_0$ , then the second formula implies that  $d_1 \circ D_0(z) = z - \varepsilon(z) \mathbf{v}$ . Furthermore, since  $\varepsilon \circ d_1 = 0$  and  $d_0 = 0$ , it follows that

- (i) the map  $\varepsilon$  passes to a homomorphism from  $H_0$  to  $\mathbb{Z}$ ,
- (ii) since  $\varepsilon(\mathbf{v}) = 1$  this homomorphism is onto,
- (iii) the multiples of the class  $[\mathbf{v}]$  give all the classes in  $H_0$ .

Taken together, these imply that  $H_0(P, \mathbf{K}) \cong \mathbb{Z}$ , and it is generated by  $[\mathbf{v}]$ . This completes the computation of  $H_*(P, \mathbf{K})$ .

By Corollary 5 we know that  $H_q^{\text{ordered}}(P, \mathbf{K})$  is isomorphic to a direct summand of  $H_q(P, \mathbf{K})$  and since the latter is zero if  $q > 0$  it follows that the former is also zero if  $q > 0$ . Similarly, we know that  $H_0^{\text{ordered}}(P, \mathbf{K})$  is isomorphic to a direct summand of  $H_0(P, \mathbf{K}) \cong \mathbb{Z}$ . By construction we know that the generating class  $[\mathbf{v}]$  for the latter lies in the image of  $i_*$ , and therefore it follows that the map from  $H_0^{\text{ordered}}(P, \mathbf{K})$  to  $H_0(P, \mathbf{K})$  must also be an isomorphism. ■

**COROLLARY 6.** *If  $\Delta$  is a simplex with the standard simplicial decomposition, then*

$$H_q^{\text{ordered}}(P, \mathbf{K}) \cong H_q(P, \mathbf{K})$$

*is trivial if  $q \neq 0$  and infinite cyclic if  $q = 0$ . ■*

Clearly we would like to “leverage” this result into a proof for an arbitrary simplicial complex  $(P, \mathbf{K})$ . This will require some additional algebraic tools, and it will be done in the next section. We shall conclude this section by using simplicial chains to solve the problem which is often viewed as the beginning of algebraic topology.

### *The Königsberg Bridge Problem*

In this problem one has four masses of land joined by various bridges. This can be modeled by a 1-dimensional cell complex with vertices  $\mathbf{w}$ ,  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z}$  representing the land masses and edges representing one bridge each from  $\mathbf{w}$  to  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z}$  along with two bridges joining  $\mathbf{y}$  to each of  $\mathbf{x}$  and  $\mathbf{z}$ . This configuration is homotopic to a simplicial complex if we add extra vertices  $\mathbf{u}_1$  and  $\mathbf{u}_2$  on each of the bridges joining  $\mathbf{y}$  to  $\mathbf{x}$  and  $\mathbf{v}_1$  and  $\mathbf{v}_2$  on each of the bridges joining  $\mathbf{y}$  to  $\mathbf{z}$ . This will be our simplicial complex  $(P, \mathbf{K})$ , and we shall let  $C_*$  denote the ordered chain complex associated to some ordering of the vertices.

The problem is to determine whether there is a path on this complex in which each bridge is crossed exactly once, and the first step is to formulate this in terms of the chain complex  $C_*$ . What we want is a 1-chain  $\sum_{\mathbf{E}} \theta(\mathbf{E}) \mathbf{E}$ , where the sum runs over all free generators of  $C_1$  and  $\theta_{\mathbf{E}} \in \{\pm 1\}$  for all  $\mathbf{E}$ , such that the boundary of this 1-chain has the form  $\mathbf{p} - \mathbf{q}$  for two vertices in  $C_0$  (the case  $\mathbf{p} = \mathbf{q}$  is allowed). The problem is then to determine if such a 1-chain exists.

Euler’s crucial insight into the problem can be stated as follows:

**PROPOSITION 7.** *Let  $(P, \mathbf{K})$  be a 1-dimensional simplicial complex, let  $\gamma \in C_1^{\text{ordered}}(P, \mathbf{K})$  be a 1-chain  $\sum_{\mathbf{E}} \theta(\mathbf{E}) \mathbf{E}$ , where the sum runs over all free generators of  $C_1$  and  $\theta_{\mathbf{E}} \in \{\pm 1\}$  for all ordered edges  $\mathbf{E}$ , and write  $d(\gamma) = \sum_{\mathbf{v}} n(\mathbf{v}) \mathbf{v}$  for suitable integers  $n(\mathbf{v})$ , where the sum runs over all vertices of  $(P, \mathbf{K})$ . Then  $n(\mathbf{v})$  is congruent modulo 2 to the number  $m(\mathbf{v})$  of 1-simplices  $\mathbf{E}$  that have  $\mathbf{v}$  as one of their endpoints.*

**Proof.** The integer  $n_{\mathbf{v}}$  is equal to  $\sum_{\mathbf{F}} e(\mathbf{F})$ , where the sum runs over all edges containing  $\mathbf{v}$  as a vertex and  $e(\mathbf{F}) \in \{\pm 1\}$ . Since  $e(\mathbf{F}) - 1$  is either equal to 0 or  $\pm 2$  for each  $\mathbf{F}$ , it follows that the sum of these differences, which is merely  $n(\mathbf{v}) - m(\mathbf{v})$ , must be a multiple of 2. ■

**COROLLARY 8.** *In the preceding setting, if there is a 1-chain  $\gamma$  such that  $d(\gamma) = \mathbf{p} - \mathbf{q}$ , then  $m_{\mathbf{v}}$  must be even if  $\mathbf{v} \neq \mathbf{p}, \mathbf{q}$ . ■*

The impossibility of finding a suitable 1-chain for our Königsberg bridge network now follows by observing that  $m = 3$  for  $\mathbf{w}$ ,  $\mathbf{x}$  and  $\mathbf{z}$ , while  $m = 5$  for  $\mathbf{y}$ . In particular, if  $\gamma$  is a chain as in the statement of the theorem, then in  $d(\gamma)$  the coefficients of all four of these vertices must be nonzero. ■

It is left as an exercise for the reader to show that the homology groups of this simplicial complex are given by  $H_1 \cong \mathbb{Z}^4$  and  $H_0 \cong \mathbb{Z}$ . This is essentially an exercise in linear algebra (however, the scalars here are integers rather than elements of some field).

### III.4 : Comparison principles

(Hatcher, §§ 2.1 – 2.2)

We have already stated the goal of proving that the mappings  $i_*$  define isomorphisms from  $H_*^{\text{ordered}}(P, \mathbf{K})$  to  $H_*(P, \mathbf{K})$  for every finite simplicial complex  $(P, \mathbf{K})$ . The proof of this requires some purely algebraic theorems involving large commutative diagrams, and the results involve a technique known as *diagram chasing*. We shall begin with a simple observation and a related question.

**PROPOSITION 1.** (Effaceability Property) *If  $(A, d)$  is a chain complex and  $u \in H_k(A)$  for some  $k$ , then there is a chain complex  $(B, d')$  containing  $(A, d)$  as a chain subcomplex and the inclusion map  $i : A \rightarrow B$  satisfies  $i_*(u) = 0$ .*

**Proof.** Define  $B_q = A_q$  if  $q \neq k + 1$ , set  $B_{k+1} = A_{k+1} \oplus R$ , where  $R$  is the ring for the underlying category of modules, and define  $d'_q = d_q$  if  $q \neq k + 1$  with  $d_{k+1}(a, r) = d_{k+1}(a) + rz$ , where  $z$  is a cycle representing  $u$ . There is an obvious inclusion of chain complexes which is the identity in degrees  $\neq k + 1$  and is given in the remaining case by  $i_{k+1}(a) = (a, 0)$ . It is then straightforward to verify that the conclusion of the proposition is true. ■

The preceding result leads naturally to the following question:

*If  $i : A \rightarrow B$  defines an inclusion of chain complexes, how can we analyze the kernel and cokernel of  $i_*$  in a relatively effective manner?*

As in many other instances, the answer to this question involves some additional constructions. Let  $A \subset B$  be a chain complex inclusion, and consider the quotient complex  $B/A$ ; let  $i : A \rightarrow B$  denote the inclusion map, and let  $j : B \rightarrow B/A$  denote the projection. We then have the following result:

**PROPOSITION 2.** *Let  $i : A \rightarrow B$  and  $j : B \rightarrow B/A$  be injection and projection maps of chain complexes as above. Then for each  $k$  there is a homomorphism  $\partial : H_k(B/A) \rightarrow H_{k-1}(A)$  defined as follows: If  $u \in H_k(B/A)$  and  $x \in B_k$  is such that  $j(x)$  represents  $u$ , then  $\partial(u)$  is represented by  $y \in A_{k-1}$  such that  $i(y) = d(x)$ . Furthermore, if we are given a second pair  $i' : A' \rightarrow B'$  and  $j' : B' \rightarrow B'/A'$  as above and a chain map  $f : B \rightarrow B'$  such that  $f$  maps  $A$  to  $A'$  by a chain map  $g$  and  $h : B/A \rightarrow B'/A'$  is the map given by passage to quotients, then the corresponding homomorphisms  $\partial$  and  $\partial'$  satisfy  $g_* \circ \partial = \partial' \circ h_*$ .*

**Proof.** First of all, we should check that the definition makes sense. The first step in doing so is to verify that if we are given  $x$  there is always a suitable choice of  $y$ . In general the class  $x$  need not be a cycle, but we know that  $j(x)$  is a cycle representing  $u$ , and therefore  $0 = d \circ j(x) = j \circ d(x)$ , which means that  $d(x) = i(a)$  for some  $a$ . This element is a cycle; we know that  $d(a) = 0$  if and only if  $i \circ d(a) = 0$ , and since  $i \circ d(a) = d \circ i(a) = d \circ d(x) = 0$ , it does follow that  $d(a) = 0$  as required.

Next, we need to check that the construction is well defined when one passes to homology. Suppose that  $j(x)$  and  $j(x')$  represent the same class in  $H_k(B/A)$ . It then follows that  $j(x - x')$  is a boundary, which means there is some  $w \in B_{k+1}$  such that  $d(w) - (x - x')$  lies in  $A$ , which is the image of  $i$ . Express the difference element as  $i(z)$ ; then we have

$$i(dz) = d(iz) = d(d(w) - (x - x')) = d(x') - d(x)$$

so that  $d(x) = i(a)$  and  $d(x') = i(a')$  imply that  $a' - a = d(z)$ .

Next, we need to check that  $\partial$  is a module homomorphism. Given classes  $u$  and  $u'$  represented by  $x$  and  $x'$ , it follows that  $x + x'$  represents  $u + u'$ , while  $d(x) = i(a)$  and  $d(x') = i(a')$  imply  $d(x + x') = i(a + a')$ . Thus  $a + a'$  represents  $u + u'$ , showing that  $\partial$  is additive. If  $r \in R$ , then similar considerations show that  $\partial(r \cdot u)$  is represented by  $r \cdot a$ , and therefore  $\partial$  is compatible with scalar multiplication.

Finally, suppose we have chain maps as described in the proposition, let  $u \in H_k(B/A)$ , and let  $x \in B_k$  be such that  $j(x)$  represents  $u$ . Then a representative for  $g_*\partial(u)$  is given by  $g(a)$ , where  $ia = dx$ , while a representative for  $\partial'h_*(u)$  is given by  $z$  such that  $i'(z) = d'f(x)$ . The right hand side equals  $f \circ d(x) = f \circ i(a) = i' \circ g(a)$ , and therefore we see that  $z = g(a)$ , which means that  $g_*\partial(u) = \partial'h_*(u)$  as desired. ■

We may now state and prove the following basic result:

**THEOREM 3.** (Long Exact Homology Sequence Theorem — Algebraic Version). *Let  $i : A \rightarrow B$  and  $j : A \rightarrow A/B$  be injection and projection maps of chain complexes as above. Then there is a long exact sequence of homology groups as follows:*

$$\cdots \quad H_{k+1}(B/A) \xrightarrow{\partial} H_k(A) \xrightarrow{i_*} H_k(B) \xrightarrow{j_*} H_k(B/A) \xrightarrow{\partial} H_{k-1}(A) \quad \cdots$$

*This sequence extends indefinitely to the left and right. Furthermore, if we are given chain maps  $f, g$  and  $h$  as in Proposition 2, then we have the following commutative diagram in which the two rows are exact:*

$$\begin{array}{cccccccccccc} \cdots & H_{k+1}(B/A) & \xrightarrow{\partial} & H_k(A) & \xrightarrow{i_*} & H_k(B) & \xrightarrow{j_*} & H_k(B/A) & \xrightarrow{\partial} & H_{k-1}(A) & \cdots \\ \cdots & \downarrow h_* & & \downarrow g_* & & \downarrow f_* & & \downarrow h_* & & \downarrow g_* & \\ \cdots & H_{k+1}(B'/A') & \xrightarrow{\partial'} & H_k(A') & \xrightarrow{i'_*} & H_k(B') & \xrightarrow{j'_*} & H_k(B'/A') & \xrightarrow{\partial'} & H_{k-1}(A') & \cdots \end{array}$$

A proof of this theorem appears on page 117 of Hatcher. ■

### *Application to simplicial complexes*

In order to apply the preceding algebraic results, we need to define *relative homology groups* associated to a *simplicial complex pair*

$$( (P, \mathbf{K}), (Q, \mathbf{L}) )$$

consisting of a simplicial complex  $(P, \mathbf{K})$  and a subcomplex  $(Q, \mathbf{L})$ . To simplify notation, we shall usually denote such a pair by  $(\mathbf{K}, \mathbf{L})$ .

**Definition.** In the setting above the *relative simplicial chain groups*, denoted by  $C_*^{\text{ordered}}(\mathbf{K}, \mathbf{L})$  and  $C_*(\mathbf{K}, \mathbf{L})$ , are respectively given by the corresponding quotient complexes

$$C_*^{\text{ordered}}(\mathbf{K})/C_*^{\text{ordered}}(\mathbf{L}) \quad \text{and} \quad C_*(\mathbf{K})/C_*(\mathbf{L}) .$$

Since the chain complex mappings from ordered to unordered chains send ordered chains on  $\mathbf{L}$  to unordered chains on  $\mathbf{L}$ , it follows that there are canonical homomorphisms

$$\varphi : C_*^{\text{ordered}}(\mathbf{K})/C_*^{\text{ordered}}(\mathbf{L}) \longrightarrow C_*(\mathbf{K})/C_*(\mathbf{L})$$



defined by passage to quotients. The *relative simplicial homology groups*, denoted by  $H_*^{\text{ordered}}(\mathbf{K}, \mathbf{L})$  and  $H_*(\mathbf{K}, \mathbf{L})$  respectively, are the homology groups of the associated chain complexes; by the preceding sentence, we have canonical homomorphisms from the relative homology groups for ordered chains to the relative homology groups for unordered chains. We should also note that the previously defined absolute chain groups may be viewed as special cases of this definition where  $\mathbf{L} = \emptyset$ .

By Theorem 3 above, we have the following result:

**THEOREM 4.** (Long Exact Homology Sequence Theorem — Simplicial Version). *Let  $i : \mathbf{L} \rightarrow \mathbf{K}$  denote a simplicial subcomplex inclusion. Then there are long exact sequences of homology groups, and they fit into the following commutative diagram, in which the rows are exact and the horizontal arrows represent the canonical maps from ordered to unordered chains:*

$$\begin{array}{cccccccccccc}
 \cdots & H_{k+1}(\mathbf{K}, \mathbf{L}) & \xrightarrow{\partial} & H_k(\mathbf{L}) & \xrightarrow{i_*} & H_k(\mathbf{K}) & \xrightarrow{j_*} & H_k(\mathbf{K}, \mathbf{L}) & \xrightarrow{\partial} & H_{k-1}(\mathbf{L}) & \cdots \\
 \cdots & \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \varphi_* & \\
 \cdots & H_{k+1}(\mathbf{K}, \mathbf{L}) & \xrightarrow{\partial} & H_k(\mathbf{L}) & \xrightarrow{i_*} & H_k(\mathbf{K}) & \xrightarrow{j_*} & H_k(\mathbf{K}, \mathbf{L}) & \xrightarrow{\partial} & H_{k-1}(\mathbf{L}) & \cdots
 \end{array}$$

This follows immediately from the definitions and Theorem 3. ■

### *The Five Lemma*

Theorem 4 provides one fundamental piece of algebraic input which is needed to show that ordered simplicial chains and unordered simplicial chains define isomorphic homology groups. Another is given by the following result:

**PROPOSITION 5.** *Suppose we are given a commutative diagram of modules as below in which the rows are exact and the horizontal maps  $a, b, d$  and  $e$  are isomorphisms. Then the mapping  $c$  is also an isomorphism:*

$$\begin{array}{ccccccccc}
 \cdots & A & \xrightarrow{f_*} & B & \xrightarrow{g_*} & C & \xrightarrow{j_*} & D & \xrightarrow{k_*} & E \\
 & \downarrow a_* & & \downarrow b_* & & \downarrow c_* & & \downarrow d_* & & \downarrow e_* \\
 & A' & \xrightarrow{f'_*} & B' & \xrightarrow{g'_*} & C' & \xrightarrow{h'_*} & D' & \xrightarrow{k'_*} & E'
 \end{array}$$

A proof of this theorem appears on page 129 of Hatcher. ■

### *The isomorphism theorem*

Here is the result that has been our main objective:

**THEOREM 6.** *If  $(\mathbf{K}, \mathbf{L})$  is a simplicial complex pair, then the canonical map*

$$\varphi_* : H_*^{\text{ordered}}(\mathbf{K}, \mathbf{L}) \rightarrow H_*(\mathbf{K}, \mathbf{L})$$

*is an isomorphism.*

**Proof.** Consider the following statements:

( $\mathbf{X}_n$ ) *The map  $\varphi$  above is an isomorphism for all simplicial complex pairs  $(\mathbf{K}, \mathbf{L})$  such that  $\dim \mathbf{K} \leq n$ .*

( $\mathbf{Y}_{n+1}$ ) The map  $\varphi$  above is an isomorphism for all  $(\mathbf{K}, \mathbf{L})$  such that  $\dim \mathbf{K} \leq n$  and also for  $(\Delta_{n+1}, \partial\Delta_{n+1})$ .

( $\mathbf{W}_{n+1,m}$ ) The map  $\varphi$  above is an isomorphism for all  $(\mathbf{K}, \mathbf{L})$  such that  $\dim \mathbf{K} \leq n$  and also for all  $(\mathbf{K}, \mathbf{L})$  such that  $\dim \mathbf{K} \leq n + 1$  and  $\mathbf{K}$  has at most  $m$  simplices of dimension equal to  $n + 1$ .

The theorem is then established by the following double inductive argument:

[F] The statement  $(\mathbf{X}_0)$  and the equivalent statement  $(\mathbf{W}_{1,0})$  are true.

[G] For all nonnegative integers  $n$ , the statement  $(\mathbf{X}_n)$  implies  $(\mathbf{Y}_{n+1})$ .

[K] For all nonnegative integers  $n$  and  $m$ , the statements  $(\mathbf{W}_{n+1,m})$  and  $(\mathbf{Y}_{n+1})$  imply  $(\mathbf{W}_{n+1,m+1})$ .

Since statement  $(\mathbf{X}_n)$  is true if and only if  $(\mathbf{W}_{n,m})$  is true for all  $m$ , and the latter are all true if and only if  $(\mathbf{W}_{n+1,0})$  is true, we also have the following:

[L] For all  $n$  the statements  $(\mathbf{X}_n) \iff (\mathbf{W}_{n+1,0})$  and  $(\mathbf{Y}_{n+1})$  imply  $(\mathbf{W}_{n+1,m})$  for all  $m$ , and hence  $(\mathbf{X}_n)$  implies  $(\mathbf{X}_{n+1})$ .

Therefore  $(\mathbf{X}_n)$  is true for all  $n$ , and this is the conclusion of the theorem.

**Proof of [F].** By the Five Lemma it suffices to prove the result when  $\mathbf{L}$  is empty. Since the 0-dimensional complex determined by  $\mathbf{K}$  is merely a finite set of vertices, write these vertices as  $\mathbf{w}_1, \dots, \mathbf{w}_m$ . We then have canonical chain complex isomorphisms

$$\bigoplus_{j=1}^m C_*^{\text{ordered}}(\{\mathbf{w}_j\}) \longrightarrow C_*^{\text{ordered}}(\mathbf{K}), \quad \bigoplus_{j=1}^m C_*(\{\mathbf{w}_j\}) \longrightarrow C_*(\mathbf{K})$$

and these pass to homology isomorphisms

$$\bigoplus_{j=1}^m H_*^{\text{ordered}}(\{\mathbf{w}_j\}) \longrightarrow H_*^{\text{ordered}}(\mathbf{K}), \quad \bigoplus_{j=1}^m H_*(\{\mathbf{w}_j\}) \longrightarrow H_*(\mathbf{K}).$$

These maps commute with the homomorphisms  $\varphi_*$  sending ordered to unordered chains. and since the maps  $\varphi_*$  are isomorphisms for one point complexes (= 0-simplices), it follows that  $\varphi$  defines an isomorphism from  $H_*^{\text{ordered}}(\mathbf{K})$  to  $H_*(\mathbf{K})$ . This completes the proof of  $(\mathbf{X}_0)$ .

**Proof of [G].** By  $(\mathbf{X}_n)$  we know that  $\varphi_*$  is an isomorphism for the complex  $\partial\Delta_{n+1}$ . Since  $\varphi_*$  is also an isomorphism for  $\Delta_{n+1}$  by Corollary III.3.6. Therefore the Five Lemma implies that  $\varphi_*$  is an isomorphism for  $\Delta_{n+1}, \partial\Delta_{n+1}$ .

**Proof of [K].** This is the crucial step. Let  $\mathbf{K}$  be an  $(n + 1)$ -dimensional complex, and let  $\mathbf{M}$  be a subcomplex obtained by removing exactly one  $(n + 1)$ -simplex from  $\mathbf{K}$ , so that  $\varphi_*$  is an isomorphism for  $\mathbf{M}$  by the inductive hypothesis. If we can show that  $\varphi_*$  is an isomorphism for  $(\mathbf{K}, \mathbf{M})$ , then it will follow that  $\varphi_*$  is an isomorphism for  $\mathbf{K}$ , and the relative case will follow from the Five Lemma.

Let  $\mathbf{S}$  be the extra simplex of  $\mathbf{K}$  and let  $\partial\mathbf{S}$  be its boundary. Then there are canonical isomorphism from the chain groups of  $\Delta_{n+1}, \partial\Delta_{n+1}$  and  $(\Delta_n, \partial\Delta_{n+1})$  to the chain groups of  $\mathbf{S}, \partial\mathbf{S}$

and  $(\mathbf{S}, \partial\mathbf{S})$ . We then have the following commutative diagram, in which the morphisms  $\alpha$  and  $\beta$  are determined by subcomplex inclusions:

$$\begin{array}{ccc} C_*^{\text{ordered}}(\mathbf{S}, \partial\mathbf{S}) & \xrightarrow{\alpha} & C_*^{\text{ordered}}(\mathbf{S}, \partial\mathbf{S}) \\ \downarrow \varphi(\mathbf{S}, \partial\mathbf{S}) & & \downarrow \varphi(\mathbf{K}, \mathbf{M}) \\ C_*^{\text{ordered}}(\mathbf{S}, \partial\mathbf{S}) & \xrightarrow{\beta} & C_*^{\text{ordered}}(\mathbf{S}, \partial\mathbf{S}) \end{array}$$

We CLAIM that  $\alpha$  and  $\beta$  are isomorphisms of chain complexes. For the mapping  $\alpha$ , this follows because the relative ordered chain groups of a pair  $(\mathbf{T}, \mathbf{T}_0)$  are free abelian groups on the simplices in  $\mathbf{T} - \mathbf{T}_0$ , and each of the sets  $\mathbf{S} - \partial\mathbf{S}$  and  $\mathbf{K} - \mathbf{M}$  is given by the same  $(n + 1)$ -simplex. For the mapping  $\beta$ , this follows because the relative unordered chain groups of a pair  $(\mathbf{T}, \mathbf{T}_0)$  are free abelian groups on the generators  $\mathbf{v}_0 \cdots \mathbf{v}_k$ , where the  $\mathbf{v}_j$  are vertices of a simplex that is in  $\mathbf{T}$  but not in  $\mathbf{T}_0$  (with repetitions allowed as usual), and once again these free generators are identical for the pairs  $(\mathbf{S}, \partial\mathbf{S})$  and  $(\mathbf{K}, \mathbf{M})$  because  $\mathbf{S} - \partial\mathbf{S}$  and  $\mathbf{K} - \mathbf{M}$  are the same.

By  $(\mathbf{Y}_{n+1})$  we know that  $\varphi(\mathbf{S}, \partial\mathbf{S})$  defines an isomorphism in homology, and therefore it follows that the homology map

$$\varphi(\mathbf{K}, \mathbf{M})_* = \beta_* \circ \varphi(\mathbf{S}, \partial\mathbf{S})_* \circ \alpha_*^{-1}$$

also defines an isomorphism in homology. We can now use the Five Lemma and  $(\mathbf{W}_{n+1,m})$  to conclude that the map  $\varphi(\mathbf{K})$  defines an isomorphism in homology, and finally we can use the Five Lemma once more to see that the statement  $(\mathbf{W}_{n+1,m+1})$  is true. This completes the proof of  $[\mathbf{K}]$ , and as noted above it also yields  $[\mathbf{L}]$  and the theorem. ■

The preceding result can be reformulated in an abstract setting that will be needed later. We begin by defining a category **SCPairs** whose objects are pairs of simplicial complexes  $(\mathbf{K}, \mathbf{K}_0)$  and whose morphisms are given by subcomplex inclusions  $(\mathbf{L}, \mathbf{L}_0) \subset (\mathbf{K}, \mathbf{K}_0)$ ; in other words,  $\mathbf{L}_0$  is a subcomplex of both  $\mathbf{L}$  and  $\mathbf{K}_0$  while  $\mathbf{L}$  is also a subcomplex of  $\mathbf{K}$ . A *homology theory* on this category is a covariant functor  $h_*$  valued in some category of modules together with a natural transformation

$$\partial(\mathbf{K}, \mathbf{L}) : h_*(\mathbf{K}, \mathbf{L}) \longrightarrow h_{*-1}(\mathbf{L})$$

such that

- (a) one has long exact homology sequences,
- (b) if  $\mathbf{K}$  is a simplex and  $\mathbf{v}$  is a vertex of  $\mathbf{K}$  then  $h_*(\{\mathbf{v}\}) \rightarrow h_*(\mathbf{K})$  is an isomorphism,
- (c) if  $\mathbf{K}$  is 0-dimensional with vertices  $\mathbf{v}_j$  then the associated map from  $\bigoplus_j h_j(\{\mathbf{v}_j\})$  to  $h_*(\mathbf{K})$  is an isomorphism,
- (d) if  $\mathbf{K}$  is obtained from  $\mathbf{M}$  by adding a single simplex  $\mathbf{S}$ , then  $h_*(\mathbf{S}, \partial\mathbf{S}) \rightarrow h_*(\mathbf{M}, \mathbf{K})$  is an isomorphism,
- (d) if  $\mathbf{K}$  is complex consisting only of a single vertex then  $h_0(\mathbf{K})$  is the underlying ring  $R$  and  $h_j(\mathbf{K}) = 0$  if  $j \neq 0$ .

A *natural transformation* from one such theory  $(h_*, \partial)$  to another  $(h'_*, \partial')$  is a natural transformation of  $\theta$  of functors that is compatible with the mappings  $\partial$  and  $\partial'$ ; specifically, we want

$$\theta(\mathbf{L}) \circ \partial = \partial' \circ \theta(\mathbf{K}, \mathbf{L}) .$$

These conditions imply the existence of a commutative ladder diagram as in Theorem 4, where the rows are the long exact sequences determined by the two abstract homology theories. The

definition is set up so that the proof of the next result is formally parallel to the proof of Theorem 6:

**THEOREM 7.** *Suppose we are given a natural transformation of homology theories  $\theta$  as above such that  $\theta(\mathbf{K})$  is an isomorphism if  $\mathbf{K}$  consists of just a single vertex. Then  $\theta(\mathbf{K}, \mathbf{L})$  is an isomorphism for all pairs  $(\mathbf{K}, \mathbf{L})$ .■*

*Application to barycentric subdivisions*

We shall now use the preceding results to show that the homology groups of a barycentric subdivision  $B(\mathbf{K})$  are isomorphic to the homology groups of the original complex  $\mathbf{K}$ . In this case the homology theories will be  $H_*^{\text{ordered}}(\mathbf{K}, \mathbf{L})$  and  $H_*^{\text{ordered}}(B(\mathbf{K}), B(\mathbf{L}))$ , and the natural transformation will be associated to maps defined on the chain level. It will suffice to define these chain maps for a simplex and to extend to arbitrary complexes and pairs by putting things together in an obvious manner.

**PROPOSITION 8.** *Given a nonnegative integer  $n$ , let  $\partial_j : \Delta_{n-1} \rightarrow \Delta_n$  be the order preserving affine map sending  $\Delta_{n-1}$  to the face of  $\Delta_n$  opposite the  $j^{\text{th}}$  vertex, and let  $(\delta_j)_\#$  generically denote an associated chain map. Then there are classes  $\beta_n \in C_n^{\text{ordered}}(\Delta_n)$  such that  $\beta_0$  is just the standard generator and if  $n > 0$  then*

$$d_n(\beta_n) = \sum_{j=0}^n (-1)^j (\partial_j)_\#(\beta_{n-1}) .$$

**Proof.** Since  $\Delta_n$  is acyclic, it suffices to show that the right hand side lies in the kernel of  $d_{n-1}$  if  $n > 1$  and in the kernel of  $\varepsilon$  if  $n = 1$ . Both of these are routine (but tedious) calculations.■

Using the chains  $\beta_n$  one can piece together chain maps

$$C_*^{\text{ordered}}(\mathbf{K}, \mathbf{L}) \longrightarrow C_*^{\text{ordered}}(B(\mathbf{K}), B(\mathbf{L})) .$$

We claim these define a natural transformation of homology theories, but in order to do this we must first show that  $H_*^{\text{ordered}}(B(\mathbf{K}), B(\mathbf{L}))$  actually defines a homology theory. Properties (a), (c) and (e) follow directly from the construction. Property (b) follows because  $B(\Delta_n)$  is star shaped with respect to the vertex  $\mathbf{b}$  given by the barycenter of  $\Delta_n$ . Thus it only remains to verify property (d); in fact, direct inspection similar to an argument in the proof of Theorem 6 shows that the map on the chain level is an isomorphism.

By Theorem 7, it suffices to check that the natural transformation of homology theories is an isomorphism for a simplicial complex consisting of a single vertex; in fact, for such complexes the map is already an isomorphism on the chain level. Therefore the barycentric subdivision chain maps determine isomorphism of homology groups as asserted in the proposition.■

### III.5 : Chain homotopies

(Hatcher, § 2.1)

In this section we shall generalize a key step in the proof of Proposition III.3.6. Recall that the latter gives the homology  $H_*(\mathbf{K})$  if  $\mathbf{K}$  is star shaped with respect to some vertex  $\mathbf{v}$ , and it does

so by constructing an algebraic analog of the straight line contracting homotopy from the identity to the constant map whose value is  $\mathbf{v}$ .

**Definition.** Let  $(A, d)$  and  $(B, e)$  be chain complexes, and let  $f$  and  $g$  be chain maps from  $A$  to  $B$ . A *chain homotopy* from  $f$  to  $g$  is a sequence of mappings  $d_k : A_k \rightarrow B_{k+1}$  satisfying the following condition for all integers  $k$ :

$$d_{k+1}^B \circ D_k + D_{k-1} \circ d_k^A = g_k - f_k$$

Two chain mappings  $f, g$  from  $A$  to  $B$  are said to be *chain homotopic* if there is a chain homotopy from the first to the second, and this is often written  $f \simeq g$ .

The proof of the following result is an elementary exercise:

**PROPOSITION 1.** *The relation  $\simeq$  is an equivalence relation on chain maps from one chain complex  $(A, d)$  to another  $(B, e)$ . Furthermore, if  $f$  and  $g$  are chain homotopic chain maps from  $(A, d)$  to  $(B, e)$ , and  $h$  and  $k$  are chain homotopic chain maps from  $(B, e)$  to  $(C, \theta)$ , then the composites  $h \circ f$  and  $k \circ g$  are also chain homotopic. Finally, if  $f, g, h, k$  are chain maps from  $A$  to  $B$  and  $r \in R$ , then  $f \simeq g$  and  $h \simeq k$  imply  $f + h \simeq g + k$  and  $rf \simeq rg$ .*

**Proof.** For the first part of the proof let  $f, g$  and  $h$  be chain maps from  $(A, d)$  to  $(B, e)$ . The zero homomorphisms define a chain homotopy from  $f$  to itself. If  $D$  is a chain homotopy from  $f$  to  $g$  then  $-D$  is a chain homotopy from  $g$  to  $f$ . Finally, if  $D$  is a chain homotopy from  $f$  to  $g$  and  $E$  is a chain homotopy from  $g$  to  $h$ , then  $D + E$  is a chain homotopy from  $f$  to  $h$ .

To prove the assertion in the second sentence, let  $D$  be a chain homotopy from  $f$  to  $g$  and let  $E$  be a chain homotopy from  $g$  to  $h$ . Then one can check directly that

$$h \circ D + E \circ g$$

defines a chain homotopy from  $h \circ f$  to  $k \circ g$ .

The proof of the final assertion is also elementary and is left to the reader. ■

Chain homotopies are useful and important because of the following result:

**PROPOSITION 2.** *If  $f$  and  $g$  are chain homotopic chain maps from one chain complex  $(A, d)$  to another complex  $(B, e)$ , then the associated homology mappings  $f_*$  and  $g_*$  are equal.*

**Proof.** Suppose that  $u \in H_k(A)$  and  $x \in A_k$  is a cycle representing  $u$ , so that  $d_k(x) = 0$ . If  $D$  is a chain homotopy from  $f$  to  $gh$ , then by definition we have

$$d_{k+1}^B \circ D_k(x) + D_{k-1} \circ d_k^A(x) = g_k(x) - f_k(x)$$

and since  $d_k^A(x) = 0$  it follows that the expression above is a boundary. Therefore  $g_*(u) - f_*(u)$  must be the zero element of  $H_k(B)$ . ■

### *An important example*

The following basic construction gives an explicit connection between the topological notion of homotopy and the algebraic notion of chain homotopy. Let  $n \geq 0$ , and let  $\mathbf{P}_{n+1}$  denote the *standard  $(n+1)$ -dimensional prism*  $\Delta_n \times [0, 1]$  with the simplicial decomposition given in Unit II. As in that unit, label the vertices of this prism decomposition by  $\mathbf{x}_j = (\mathbf{e}_j, 0)$  and  $\mathbf{y}_j = (\mathbf{e}_j, 1)$ .

**PROPOSITION 3.** *The simplicial chain complexes  $C_*^{\text{ordered}}(\mathbf{P}_{n+1})$  and  $C_*(\mathbf{P}_{n+1})$  are acyclic.*

**Proof.** These follow from the isomorphism theorem and the fact that  $\mathbf{P}_{n+1}$  is star shaped with respect to  $\mathbf{y}_n$ . ■

For each integer  $j$  satisfying  $0 \leq j \leq n$ , let  $\partial_j : \Delta_{n-1} \rightarrow \Delta_n$  be the affine map which sends  $\Delta_{n-1}$  to the face opposite the vertex  $\mathbf{e}_j$  and is order preserving on the vertices, and let  $\partial_j \times \mathbf{I}$  denote the product of the map  $\partial_j$  with the identity on  $[0, 1]$ . It then follows immediately that we have associated injections of simplicial chain groups

$$(\partial_j)_\# : C_j(\Delta_{n-1}) \longrightarrow C_j(\Delta_n), \quad (\partial_j \times \mathbf{I})_\# : C_*(\mathbf{P}_{n-1}) \longrightarrow C_*(\mathbf{P}_n)$$

and these are chain maps. Furthermore, these chain maps send ordered chains to ordered chains.

Similarly, for  $t = 0, 1$  we also have injections of simplicial chain groups

$$(i_t)_\# : C_*(\Delta_n) \longrightarrow C_*(\mathbf{P}_n)$$

which send a free generator  $\mathbf{v}_0 \cdots \mathbf{v}_q$  to  $i_t(\mathbf{v}_0) \cdots i_t(\mathbf{v}_q)$ , where  $i_t(\mathbf{v}) = (\mathbf{v}, t)$ .

We then have the following result:

**THEOREM 4.** *For all  $n \geq 0$  there are chains  $P_{n+1} \in C_{n+1}^{\text{ordered}}(\mathbf{P}_n)$  such that*

$$d_{n+1}(P_{n+1}) = \mathbf{y}_0 \cdots \mathbf{y}_n - \mathbf{x}_0 \cdots \mathbf{x}_n - \sum_j (-1)^j (\partial_j \times \mathbf{I})_\#(P_{n-1}).$$

**Sketch of proof.** Not surprisingly, the construction is inductive, with  $P_0 = 0$ . Suppose we have constructed the chains  $P_j$  for  $j \leq n$ . There is a chain  $P_{n+1}$  with the required properties if and only if the expression on the right hand side of the equation is a cycle, so we need to show that the right hand side vanishes if we apply  $d_n$ . This is a straightforward but messy calculation like several previous ones. Some key details are worked out in the bottom half of page 112 of Hatcher. ■

The preceding result implies that the inclusion mappings  $i_t$ , which are topologically homotopic, determine algebraic chain maps that are chain homotopic. Specifically, if we are given a free generator  $\mathbf{v}_0 \cdots \mathbf{v}_q$  of  $C_q(\Delta_n)$  then we may form a chain

$$D_q(\mathbf{v}_0 \cdots \mathbf{v}_q) \in C_{q+1}(\Delta_n \times \mathbf{I})$$

by substituting  $i_0(\mathbf{v})$  for  $\mathbf{x}$  and  $i_1(\mathbf{v})$  for  $\mathbf{y}$ . In fact, one can carry out all of this for an arbitrary simplicial complex  $(P, \mathbf{K})$ , and one has the following conclusion.

**PROPOSITION 5.** *In the setting above the maps  $(i_0)_\#$  and  $(i_1)_\#$  from  $C_*(\mathbf{K})$  to  $C_*(\mathbf{K} \times \mathbf{I})$  are chain homotopic, and hence the associated homology maps*

$$(i_0)_*, (i_1)_* : H_*(\mathbf{K}) \longrightarrow H_*(\mathbf{K} \times \mathbf{I})$$

are equal. ■

... to be continued ...