# EXERCISES FOR MATHEMATICS 246A

## WINTER 2009

The references denote sections of the text for the course:

J. R. Munkres, *Topology* (Second Edition), Prentice-Hall, Saddle River NJ, 2000. ISBN: 0–13–181629–2.

Solutions to nearly all the exercises below are given in separate files called `solutions`$n$`.pdf` (in the course directory). Here is another web site with solutions to exercises in Munkres (including some not given in our files):

`http://www.math.ku.dk/~moller/e03/3gt/3gt.html`

# I. Foundational material

### I.1 : Categories and functors

**1. Definition.** A morphism $f : A \to B$ in a category is a *monomorphism* if for all $g, h : C \to A$ we have that $f \circ h = f \circ g$ only if $h = g$. Dually, a morphism $f : A \to B$ in a category is an *epimorphism* if for all $u, v : B \to D$ we have that $u \circ f = v \circ f$ only if $u = v$.

($a$) Prove that a monomorphism in the category **Set** is $1 - 1$ and an epimorphism in **Set** is onto. [*Hint:* Prove the contrapositives.]

($b$) Prove that in the category of Hausdorff topological spaces (and continuous maps) a morphism $f : A \to B$ is an epimorphism if $f(A)$ is dense in $B$.

($c$) Prove that the composite of two monomorphisms is a monomorphism and the composite of two epimorphisms is an epimorphism.

($d$) A morphism $r : X \to Y$ in a category is called a *retract* if there is a morphism $q : Y \to X$ such that $qr = \mathrm{id}_X$. For example, in the category of sets or topological spaces the diagonal map $d_X : X \to X \times X$ is a retract with $q =$ projection onto either factor. Prove that every retract is a monomorphism.

($e$) A morphism $p : A \to B$ in a category is called a *retraction* if there is a morphism $s : B \to A$ such that $q \circ r = \mathrm{id}_B$. For example, if $r$ and $q$ are as in (d) then $q$ is a retraction. Prove that every retract is a monomorphism and every retraction is an epimorphism.

**2.** Let **A** be a category, and let $f : A \to B$ be a morphism in **A** such that

$$\mathsf{Morph}\,(f, C) : \mathsf{Morph}\,(B, C) \to \mathsf{Morph}\,(A, C)$$

is an isomorphism for all objects $C$ in **A**. Prove that $f$ is an isomorphism. [*Hint:* Choose $C = B$ or $A$ and consider the preimages of the identity elements.] Also prove the (relatively straightforward) converse.

**3.** An object $\mathbf{0}$ is called an *initial object* in the category $\mathbf{A}$ if for each object $A$ in $\mathbf{A}$ there is a unique morphism $\mathbf{0} \to A$. An object $\mathbf{1}$ is a *terminal object* in $\mathbf{A}$ if for each object $A$ there is a unique morphism $A \to \mathbf{1}$.

($a$) Prove that the empty set is initial and every one point set is terminal in **Set**.

($b$) Prove that a zero-dimensional vector space is both initial and terminal in the category **Vec**–$F$ of vector spaces over a field $F$.

($c$) Prove that every two initial objects in a category $\mathbf{A}$ are uniquely isomorphic (there is a unique isomorphism from one to the other), and similarly for terminal objects.

($d$) If $\mathbf{A}$ contains an object $Z$ that is both initial and terminal (a *null object*), prove that for each pair of objects $A, B$ in $\mathbf{A}$ there is a unique morphism $A \to B$ that factors as $A \to Z \to B$. Also, if $W$ is any other such object, prove that this composite equals the composite $A \to W \to B$. [*Hint:* Consider the unique maps from $W$ to $Z$ and vice versa.]

**4.** Prove that a covariant functor takes retracts to retracts and retractions to retractions. State the corresponding result for contravariant functors.

**5.** If $E$ is a terminal object in the category $\mathbf{A}$ and $f : E \to X$ is a morphism in $\mathbf{A}$, prove that $f$ is a monomorphism (in fact, something stronger is true—what is it?).

**6.** Let $\mathbf{A} = (\mathbb{N}^+, \mathsf{Morph}\,, \varphi)$, where $\mathbb{N}^+$ denotes the positive integers, $\mathsf{Morph}\,(p, q)$ denotes all $p \times q$ matrices with integer coefficients, and

$$\varphi : \mathsf{Morph}\,(p, q) \times \mathsf{Morph}\,(q, r) \to m(p, r)$$

is matrix multiplication. Verify that $\mathbf{A}$ is a category.

**7.** If $f$ is a morphism in a category $\mathbf{A}$, a morphism $g$ (in the same category) is called a *quasi-inverse* for $f$ if and only if $f \circ g \circ f = f$. Prove that every morphism that has a quasi-inverse is itself the quasi-inverse of some morphism in the category.

**8.** In the category of sets, show that the Axioms of Choice implies that every mapping has a quasi-inverse. Also, in the matrix category of Exercise 6, show that every matrix has a quasi-inverse. [*Hint:* Look at the associated linear transformations, and choose bases in a suitable manner.]

**NOTE.** In fact, there are canonical choices of quasi-inverses. See the following Wikipedia articles for further information on generalizations of matrix inverses:

`http://en.wikipedia.org/wiki/Moore-Penrose_inverse`

`http://en.wikipedia.org/wiki/Group_inverse`

`http://planetmath.org/encyclopedia/DrazinInverse.html`

**9.** Suppose that $\mathbf{C}$ is a category in which every map has a quasi-inverse. Prove that every monomorphism in $\mathbf{C}$ is a retract. Using this, give examples of mappings in the category of topological spaces (and continuous mappings) which do not have quasi-inverses.

**10.** Let $\mathbf{A}$ and $\mathbf{B}$ be small categories. Prove that one can define a *product category* $\mathbf{A} \times \mathbf{B}$ whose objects are given by ordered pairs $(X, Y)$, where $X$ and $Y$ are objects of $\mathbf{A}$ and $\mathbf{B}$ respectively, whose morphisms are given by ordered pairs $(f, g)$ of morphisms $f$ in $\mathbf{A}$ and $g$ in $\mathbf{B}$, and whose domain, codomain and composition operations are given as follows:

$$\begin{aligned}
\mathrm{Domain}(f,g) &= \big(\mathrm{Domain}(f), \mathrm{Domain}(g)\big) \\
\mathrm{Codomain}(f,g) &= \big(\mathrm{Codomain}(f), \mathrm{Codomain}(g)\big) \\
(f_1, g_1) \circ (f_0, g_0) &= (f_1 \circ f_0, g_1 \circ g_0)
\end{aligned}$$

Prove that $\mathbf{A} \times \mathbf{B}$ with these definitions of objects, morphisms, domains, codomains and composition forms a category, and show that "projections onto the first and second coordinates" define covariant functors from this category into $\mathbf{A}$ and $\mathbf{B}$ respectively.

**11.** Suppose that we are in a category $\mathbf{C}$ with morphisms $f : X \to Y$ and $g : Y \to Z$. Prove that if any two of $f$, $g$ and $g \circ f$ are isomorphisms, then so is the third.

**12.** Let $\mathbf{IC_0}$ be the category whose objects are open intervals in the real line and whose morpphisms are continuous mappings, and let $\mathbf{IC_1}$ be the subcategory with the same objects, but whose morphisms are maps with continuous first derivatives. Give an example of a morphism in $\mathbf{IC_1}$ which is an isomorphism in $\mathbf{IC_0}$ but not in $\mathbf{IC_1}$ (hence subcategories are not necessarily closed under taking inverses).

**13.** Let $\{X_\alpha\}$ be an indexed family of objects in a category $\mathbf{C}$. Then a *categorical product* of the $X_\alpha$ is given by an object $P$ and morphisms $p_\alpha : P \to X_\alpha$ such that for each indexed family of maps $f_\alpha$ from a fixed object $Y$ into the objects $X_\alpha$, there is a unique $f : Y \to P$ such that $p_\alpha \circ f = f_\alpha$ for all $\alpha$. — All the standard examples of product constructions turn out to have this property.

($a$) Prove that if $(P, p_\alpha)$ and $(Q, q_\alpha)$ are categorical products, then there is a unique isomorphism $h : Q \to P$ such that $q_\alpha = p_\alpha \circ h$ for all $\alpha$. [*Hint:* The only morphism $\varphi$ from $P$ to itself satisfying $p_\alpha = p_a lpha \circ \varphi$ for all $\alpha$ is the identity.]

($b$) Formuate the dual notion of coproduct in a category (a product in the opposite category), and state the dual of the conclusion in ($a$).

($c$) Show that the (external) direct sum is both a product and coproduct in $\mathbf{VEC}_{\mathbb{F}}$ for finite families of vector spaces, and show that the coproduct can be viewed as a proper subspace of the product for infinite families.

**14.** Let $\mathbf{FLD}$ be the category of (commutative) fields with morphisms given by field homomorphisms. Show that the category $\mathbf{FLD}$ does not have products. [*Hints:* Suppose we could construct a product $\mathbb{A}$ of the complex numbers with itself in this category, and consider the morphisms from $\mathbb{C}$ to itself given by the identity and complex conjugation. Recall that every homomorphism of fields is injective.]

**15.** Let $\mathbf{TOP}$ be the category of topological spaces and continuous mappings. Show that there is a *homotopy category* $\mathbf{HTP}$ whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps from one space to another. [*Hint:* The key thing to note is that one has identities and a decent well-defined notion of composition in $\mathbf{HTP}$.]

**16.** We have mentioned that the reason for specifying codomains as part of the structure for morphisms is that functors to not necessarily preserve the injectivity of mappings. Illustrate this for the fundamental group functor $\pi_1(X, x)$ on pointed topological spaces by giving an example of a continuous map of pointed spaces $f : (X, x) \to (Y, y)$ such that $f$ is injective but $f_*$ is surjective and not injective, and also give an example of a continuous map of pointed spaces $f : (X, x) \to (Y, y)$ such that $f$ is surjective but $f_*$ is injective and not surjective.