Preface

Perhaps the simplest motivation for algebraic topology is the following basic question:

If m and n are distinct positive integers, is $\mathbf{R}^m$ ever homeomorphic to $\mathbf{R}^n$?

Results from point set topology imply the answer is **NO** if one of m and n is equal to 1. If a homeomorphism $h : \mathbf{R}^m \to \mathbf{R}$ existed then for each $\mathbf{x} \in \mathbf{R}^m$ we could conclude that $\mathbf{R}^n - \{\mathbf{x}\}$ is homeomorphic to $\mathbf{R} - \{h(\mathbf{x})\}$. Since $\mathbf{R}^m - \{\mathbf{x}\}$ is connected for all $\mathbf{x} \in \mathbf{R}$ if $m > 1$ while $\mathbf{R} - \{t\}$ is not connected for any choice of $t \in \mathbf{R}$, it follows that $\mathbf{R}^m - \{\mathbf{x}\}$ is never homeomorphic to $\mathbf{R} - \{t\}$ if $m > 1$ and hence $\mathbf{R}^m$ cannot be homeomorphic to $\mathbf{R}$. Similarly, results on fundamental groups imply that for all relevant choices of $\mathbf{x}$ the set $\mathbf{R}^m - \{\mathbf{x}\}$ is simply connected if $m > 2$ while $\mathbf{R}^2 - \{\mathbf{x}\}$ has an infinite cyclic fundamental group, so we also know that $\mathbf{R}^m$ is not homeomorphic to $\mathbf{R}^2$ if $m > 2$. One basic goal of an introductory course in algebraic topology is to show that $\mathbf{R}^m$ is never homeomorphic to $\mathbf{R}^n$ if $m \neq m$.

The idea behind proving such results is to define certain abelian groups which give an **algebraic picture** of a given topological space; in particular, if two topological spaces are homeomorphic, then their associated groups will be algebraically isomorphic. Unfortunately, the definitions for these **homology groups** are less straightforward than the definition of the fundamental group, and much of the work in this course involves the construction of such groups and the proofs that they have good formal properties.

In analogy with standard results for fundamental groups, the homology groups of two spaces will be isomorphic if the spaces satisfy a condition that is somewhat weaker than the existence of a homeomorphism between them; namely, an the groups are isomorphic if the two spaces have the same *homotopy type* as defined on page 363 of the book by Munkres cited below.

Since the constructions for the associated groups are somewhat complicated, it is natural to expect that they should be useful for more than simply answering the homeomorphism question for Euclidean spaces. In particular, one might ask if these groups (and a course in algebraic topology) can shed new light on some questions left open in undergraduate or beginning graduate courses in mathematics.

1. The material in introductory graduate level courses does not really give much insight into the popular characterization of topology as a "rubber sheet geometry." In other words, topology is generally viewed as the study of properties that do not change under various sorts of bending or stretching operations. Some aspects of this already appear in the study of fundamental groups, and one objective of this course is to develop these ideas much further.

2. As a refinement of the problem at the beginning of this preface, one can ask if there is some topological criterion which characterizes the algebraic notion of n–dimensionality, at least for spaces that are relatively well-behaved.

3. An algebraic topology course should also yield better insight into several issues that arise in undergraduate courses, including (a) the Fundamental Theorem of Algebra, (b) various facts about planar and nonplanar networks, (c) insides and outsides of plane curves and closed surfaces in 3–dimensional space, and (d) Euler's Formula for "nice" polyhedra in $\mathbf{R}^3$; namely, if P is a polyhedron bounding a convex body in $\mathbf{R}^3$, then the numbers V, E and F of vertices, edges and faces satisfy the equation $E + 2 = V + F$.

4. If time permits, another goal will be to give a unified approach to certain results in multivariable calculus involving the ∇ operator, Green's Theorem, Stokes' Theorem and the Divergence Theorem, and to formulate analogs for higher dimensions.

Throughout the course we shall use the following book as a reference for many topics and definitions:

> **J. R. Munkres.** *Topology* (Second Edition), *Prentice-Hall, Saddle River NJ*, 2000. ISBN: 0–13–181629–2.

The official text for this course is the following book:

> **A. Hatcher.** *Algebraic Topology* (Third Paperback Printing), *Cambridge University Press, New York NY*, 2002. ISBN: 0–521–79540–0.

This book can be legally downloaded from the Internet at no cost for personal use, and here is the link to the online version:

`www.math.cornell.edu/~hatcher/AT/ATpage.html`

One important feature of homology groups is that if $f : X \to Y$ is a continuous mapping of topological spaces, then there is an associated homomorphism f_* from the homology groups of X to the homology groups of Y; this is again similar to the situation for fundamental groups of pointed spaces, and it plays an important role in addressing the issues listed above. In fact, algebraic topology turns out to be an effective means for analyzing the following central problem:

> Given two "reasonably well-behaved" spaces X and Y, describe the homotopy classes of continuous mappings from X to Y.

In general, the descriptions of the homotopy classes can be3 quite complicated, and only a few cases of such problems can be handled using the methods of a first course, but we shall mention a few special cases at various points in the course.

Many of the basic properties of homology groups and homomorphisms are best stated using the formalisms of **Category Theory**, and many of the constructions and theorems in algebraic topology are best stated within the framework of **Homological Algebra**. We shall develop these subjects in the course to the extent that we need them.

Prerequisites

The name "algebraic topology" suggests that the subject uses input from both algebra and topology, and this is in fact the case; since topology began as a branch of geometry, it is also reasonable to expect that some geometric input is also required. Our purpose here is to summarize the main points from prerequisite courses that will be needed. Additional background material which is usually not covered explicitly in the prerequisites will be described in the first unit of these notes.

Set theory

Everything we shall need from set theory is contained in the following online directory:

`http://math.ucr.edu/~res/math144`

In particular, a fairly complete treatment is contained in the documents `setsnotes`n`.*`, where $1 \leq n \leq 8$ and the file type `*` is one of `doc`, `ps` or `pdf`. In most cases the **pdf** versions are the most convenient to use, but parts of the **doc** files are in color rather than black and white.

There are two features of the preceding that are somewhat nonstandard. The first is the definition of a function from a set A to another set B. Generally this is given formally by the graph, which is a subset $G \subset A \times B$ such that for each $a \in A$ there is a unique $b \in B$ such that $(a, b) \in G$. Our definition of function will be a **triple** $f = (A, G, B)$, where $G \subset A \times B$ satisfies the condition in the preceding sentence. The reason for this is that we must specify the target set or **codomain** of the function explicitly; in fact, the need to specify the codomain has already arisen at least implicitly in prerequisite graduate topology courses, specifically in the definition of the fundamental group. A second nonstandard feature is the concept of **disjoint union** or **sum** of an indexed family $\{ X_\alpha \}$ of sets. The important features of the disjoint sum, which is written $\amalg_\alpha X_\alpha$, are that it is a union of subsets Y_α which are canonically in 1–1 correspondence with the sets X_α and that $Y_\alpha \cap Y_\beta = \emptyset$ if $\alpha \neq \beta$. Another source of information on such objects is Unit V of the online notes for Mathematics 205A which are cited below.

Topology

This course assumes familiarity with the basic material in graduate level topology courses through the theory of fundamental groups and covering spaces (in other words, the material in Mathematics 205A and 205B). Everything we need from the first of these courses can be found in the following online directory:

`http://math.ucr.edu/~res/math205A`

In particular, the files `gentopnotes2005.*` contain a fairly complete set of lecture notes for the course. This material is based upon the textbook by Munkres cited in the Preface. Two major differences between the notes and Munkres appear in Unit V. The discussion of quotient topologies is somewhat different from that of Munkres, and in analogy with the previously mentioned discussion of set-theoretic disjoint sums there is a corresponding construction of disjoint sum for an indexed family of topological spaces.

The necessary material on fundamental groups and covering spaces appears in the following sections of Munkres:

51 – 56

58 – 59

61 – 64

67 – 71

79 – 82

Supplementary exercises for Chapter 13.

At many points of these notes we shall rely heavily on the contents of these sections.

Algebra

As in the later parts of Munkres, we shall assume some familiarity with certain topics in group theory. Nearly everything we need is in Sections 67 – 69 of Munkres, but we shall also need the following basic result:

STRUCTURE THEOREM FOR FINITELY GENERATED ABELIAN GROUPS. *Let G be a finitely generated abelian group (so every element can be written as a monomial in integral powers of some finite subset $S \subset G$). Then G is isomorphic to a direct sum*

$$(H_1 \oplus \cdots \oplus H_b) \oplus (K_1 \oplus \cdots \oplus K_s)$$

where each H_i is infinite cyclic and each K_j is finite of order t_j such that t_{j+1} divides t_j for all j. — For the sake of uniformity set $t_j = 1$ if $j > s$. Then two direct sums as above which are given by $(b; t_1, \cdots)$ and $(b'; t'_1, \cdots)$ are isomorphic if and only if $b = b'$ and $t_j = t'_j$ for all j.

A proof of this fundamental algebraic result may be found in Sections II.1 and II.2 of the following standard graduate algebra textbook:

Hungerford, Thomas W. Algebra. Reprint of the 1974 original. (Graduate Texts in Mathematics, 73.) *Springer-Verlag, New York–Berlin–etc.*, 1980. ISBN: 0-387-90518-9.

Material from standard undergraduate linear algebra courses will also be used as needed.

Analysis

We shall assume the basic material from an upper division undergraduate course in real variables as well as material from a lower division undergraduate course in multivariable calculus through the theorems of Green and Stokes as well as the 3–dimensional Divergence Theorem. The classic text by W. Rudin (*Principles of Mathematical Analysis*, Third Edition) is an excellent reference for real variables, and the following multivariable calculus text contains more information on the that subject than one can usually find in the usual 1500 page calculus texts.

J. E. Marsden and A. J. Tromba. *Vector Calculus* (Fifth Edition), *W. H. Freeman & Co., New York NY*, 2003. ISBN: 0–7147–4992–0.

I. Foundational and Geometric Background

Aside from the formal prerequisites, algebraic topology relies on some background material from other subjects that is generally not covered in prerequisites. In particular, two concepts from the foundations of mathematics, namely **categories** and **functors**, play a central role in formulating the basic concepts of algebraic topology. Furthermore, since algebraic topology places heavy emphasis on spaces that can be constructed from certain fundamental building blocks, some relatively elementary but fairly detailed properties of the latter are indispensable. The purpose of this unit is to develop enough of category theory so that we can use it to formulate things efficiently and to describe the topological and geometric properties of a class of well-behaved spaces called **polyhedra** that will be needed in the course.

I.1 : Categories and functors

(Hatcher, § 2.3)

If mathematics is the study of abstract systems, then category theory may be viewed as an abstract formal setting for working with such systems. In fact, the theory was originally developed by S. Eilenberg and S. MacLane in the 1940s to provide an effective conceptual framework for handling various constructions and phenomena related to algebraic topology. The formal definition may be viewed as a generalization of familiar properties of ordinary set-theoretic functions.

Definition. A **CATEGORY** is a system $\mathbf{C}$ consisting of

(a) a class $\mathrm{Obj}\,(\mathbf{C})$ of sets called the **objects** of $\mathbf{C}$,

(b) for each ordered pair of objects X and Y a set $\mathrm{Morph}_{\mathbf{C}}(X,Y)$ called the **morphisms** from X to Y,

(c) for each ordered triple of objects X, Y and Z a *composition pairing* $\mathrm{Morph}_{\mathbf{C}}(X,Y) \times \mathrm{Morph}_{\mathbf{C}}(Y,Z) \longrightarrow \mathrm{Morph}_{\mathbf{C}}(X,Z)$, whose value for (f,g) is generally written $g \circ f$, such that the following hold:

(1) The sets $\mathrm{Morph}_{\mathbf{C}}(X,Y)$ and $\mathrm{Morph}_{\mathbf{C}}(Z,W)$ are disjoint unless $X = Z$ and $Y = W$.

(2) For each object X there is an identity morphism $1_X = \mathrm{id}_X \in \mathrm{Morph}_{\mathbf{C}}(X,X)$ such that for each $f \in \mathrm{Morph}_{\mathbf{C}}(X,Y)$ and each $g \in \mathrm{Morph}_{\mathbf{C}}(Z,X)$ we have $f \circ 1_X = f$ and $1_X \circ g = g$.

(3) The composition pairings satisfy an associative law; in other words, if $f \in \mathrm{Morph}_{\mathbf{C}}(X,Y)$, $g \in \mathrm{Morph}_{\mathbf{C}}(Y,Z)$, and $h \in \mathrm{Morph}_{\mathbf{C}}(Z,W)$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

By the assumptions, for each $f \in \mathrm{Morph}_{\mathbf{C}}(X,Y)$ the objects X and Y are uniquely determined, and they are called the **domain** and **codomain** of f respectively. When working within a given category we generally use familiar notation like $f : X \to Y$ to indicate that $f \in \mathrm{Morph}_{\mathbf{C}}(X,Y)$.

As in set theory, at some points one must take care to avoid difficulties with classes that are "too large" to be sets (for example, we cannot discuss the set of all sets), but in practice it is always possible to circumvent such problems by careful choices of definitions and wordings (for example, using the theory of *Grothendieck universes*), so we shall generally not dwell on such points.

By the remarks preceding the definition of a category, it is clear that we have a category **SETS** whose objects are given by all sets, whose morphisms are set-theoretic functions from one set to another (with the conventions mentioned in the Prerequisites!), and whose composition is merely ordinary composition of mappings. Here are some further examples:

1. Given a field $\mathbf{F}$, there is the category $\mathbf{VEC_F}$ whose objects are vector spaces, whose morphisms are $\mathbf{F}$-linear transformations, and whose composition is ordinary composition. The important facts here are that the identity on a vector space is a linear transformation, and the composite of two linear transformations is a linear transformation.

2. There is also a category **GRP** whose objects are groups and whose morphisms are group homomorphisms (with the usual composition). Once again, the crucial properties needed to check the axioms for a category are that identity maps are homomorphisms and the composite of two homomorphisms is a homomorphism.

3. Within the preceding example, there is the subcategory **ABGRP** whose objects are abelian groups, with the same morphisms and compositions. In this category, the set of morphisms from one object to another has a natural abelian group structure given by pointwise addition of functions, and the resulting abelian group of homomorphisms is generally denoted by $\mathrm{Hom}(X, Y)$.

4. If P is a partially ordered set with ordering relation $\leq$, then one has an associated category whose objects are the elements of P and such that $\mathrm{Morph}\,(x, y)$ consists of a single point if $x \leq y$ and is empty otherwise. This is an example of a **small** category in which the class of objects is a set.

5. One can also use partially ordered sets to define a category **POSETS** whose objects are partially ordered sets and whose morphisms are monotonically nondecreasing functions from one partially ordered set to another; as in most other cases, composition has its usual meaning.

6. If G is a group, then G also defines a small category as follows: There is exactly one object, the morphisms of this object to itself are given by the elements of G, and composition is given by the multiplication in G.

7. There is a category **TOP** whose objects are topological spaces, whose morphisms are continuous maps between topological spaces, and whose composition is the usual notion. Again, the crucial properties needed to verify the axioms for a category are that identity maps are continuous and composites of continuous maps are also continuous.

8. There are also categories whose objects are topological spaces and whose morphisms are **open** maps or **closed** maps. The categories with various types of morphisms are distinct.

9. One also has a category **MET–UNIF** whose objects are metric spaces and whose morphisms are **uniformly continuous** mappings (with the usual composition).

10. Given an arbitrary category $\mathbf{C}$, one has the **dual** or **opposite** category $\mathbf{D} = \mathbf{C}^{\mathbf{OP}}$ with the same objects as $\mathbf{C}$, but with $\mathrm{Morph}_{\mathbf{D}}(X, Y) = \mathrm{Morph}_{\mathbf{C}}(Y, X)$ and composition $*$ defined by $g * f = f \circ g$. Note that if $\mathbf{D} = \mathbf{C}^{\mathbf{OP}}$ then $\mathbf{C} = \mathbf{D}^{\mathbf{OP}}$.

In most of the preceding examples of categories, there is a fundamental notion of **isomorphism**, and in fact one can formulate this abstractly for an arbitrary category:

Definition. Let **C** be a category, and let X and Y be objects of **C**. A morphism $f : X \to Y$ is an *isomorphism* if there is a morphism $g : Y \to X$ (an inverse) such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

This generalizes notions like an invertible linear transformation, a group isomorphism, and a homeomorphism of topological spaces.

PROPOSITION 1. *Suppose that $f : X \to Y$ is an isomorphism in a category* **C** *and g and h are inverses to f. Then $h = g$.*

Proof. Consider the threefold composite $h \circ f \circ g$. Since $h \circ f = 1_X$, this is equal to g, and since $f \circ g = 1_Y$, it is also equal to h.■

Functors

The examples of categories illustrate a basic principle in modern mathematics: Whenever one defines a type of mathematical system, there is usually a corresponding type of morphism for such systems (and in some cases there are several reasonable choices for morphisms). Since a category is an example of a mathematical system, it is natural to ask whether there is a corresponding notion of morphisms in this case too. In fact, there are two concepts of morphism that turn out to be important and useful. We shall start with the simpler one.

Definition. Let **C** and **D** be categories. A *covariant functor* assigns (i) to each object X of **C** an object $T(X)$ of **D**, (ii) to each morphism $f : X \to Y$ in **C** a morphism $T(f) : T(X) \to T(Y)$ in **D** such that the following hold:

(1) For each object X in **C** we have $T(1_X) = 1_{T(X)}$.

(2) For each pair of morphisms f and g in **C** such that $g \circ f$ is defined, we have $T(g \circ f) = T(g) \circ T(f)$.

HISTORICAL TRIVIA. Eilenberg and MacLane "borrowed" the word **category** from the philosophical writings of the 18$^{\text{th}}$ century German philosopher I. Kant and the word **functor** from the philosophical writings of the 20$^{\text{th}}$ century German-American philosopher R. Carnap, who was strongly influenced by Kant's writings on the philosophy of science.

Examples of covariant functors

Numerous constructions from undergraduate and elementary graduate courses can be interpreted as functors; in many cases this does not shed much additional light on the objects constructed, but in other cases the concept does turn out to be extremely useful.

1. Given a category **C**, there is the identity functor from **C** to itself, which takes all objects and morphisms to themselves.

2. Given a category **C** and another nonempty category **D**, for each object A of **D** there is a constant functor k_A from **C** to **D** which sends every object of **C** to A and every morphism to the identity morphism 1_A.

3. In categories where the objects are given by sets with some extra structure and the morphisms are ordinary functions with additional properties, there are **forgetful functors** which take objects to the underlying sets and morphisms to the underlying mappings of sets. For example, there are forgetful functors from $\mathbf{VEC_F}$, **GRP**, **POSETS**, and **TOP** to **SETS**. Likewise, there is an obvious forgetful functor from **MET–UNIF** to **TOP** which takes a metric space to its underlying topological space and simply views a uniformly continuous mapping as a continuous mapping.

4. There is a **power set functor** P_* on the category **SETS** defined as follows: The set $P_*(X)$ is just the set of all subsets (also known as the power set), and if $f : X \to Y$ is a set-theoretic function, then $P_*(f) : P_*(X) \to P_*(Y)$ takes an element $A \in P(X)$ — which by definition is just a subset of X — to its image $f[A] \subset Y$. A short argument is needed to verify this construction actually defines a covariant functor, but it is elementary. First, we need to check that for every set X we have $P_*(1_X) = 1_{P(X)}$; this follows because $1_X[A] = A$ for all $A \subset X$. Next, we must check that $P_*(g \circ f) = P_*(g) \circ P_*(f)$ for all composable f and g. But this is a consequence of the elementary identity $g[\,f[A]\,] = g \circ f[A]$.

5. If we are given two partially ordered sets and a mapping f from the first to the second such that $u \le v$ implies $f(u) \le f(v)$, then f may be interpreted as a covariant functor on the associated categories.

6. If we are given two groups and a homomorphism f from the first to the second, then f may be interpreted as a covariant functor on the associated categories.

7. Finally, we shall give a more substantial example that played a central role in mathematics 205B. Define a new category $\mathbf{TOP}_*$ of *pointed topological spaces* whose objects are pairs (X, y), where X is a topological space and $y \in X$; the point y is said to be the *basepoint* of the pointed space. A morphism $f : (X, y) \to (Z, w)$ in this category will be a continuous mapping from X to Z (usually also denoted by f) which maps y to w (*i.e.*, a **basepoint preserving** continuous mapping). The fundamenal group $\pi_1(X, y)$ then has a natural interpretation as a covariant functor, for if f is a morphism of pointed spaces, then then one has an associated homomorphism f_* from $\pi_1(X, y)$ to $\pi_1(Z, w)$, and these have the required properties that $1_{(X,y)*}$ is the identity and $(g \circ f)_* = g_* \circ f_*$.

Contravariant functors and examples

Experience shows there are many instances in which it is useful to work with functors that **reverse** the order of function composition; such objects are called *contravariant functors*.

Definition. Let **C** and **D** be categories. A *contravariant functor* assigns (i) to each object X of **C** an object $U(X)$ of **D**, (ii) to each morphism $f : X \to Y$ in **C** a morphism $U(f) : U(Y) \to U(X)$ in **D** (note that the domain and codomain are the opposites of those in the covariant case!) such that the following hold:

(1) For each object X in **C** we have $U(1_X) = 1_{U(X)}$.

(2) For each pair of morphisms f and g in **C** such that $g \circ f$ is defined, we have $U(g \circ f) = U(f) \circ U(g)$.

The simplest examples of contravariant functors are given by the *pseudo-identity functors*, which map the objects and morphisms in the category **C** to their obvious counterparts in the opposite category $\mathbf{C^{OP}}$. In fact, there is an obvious correspondence between contravariant functors from **C** to **D** and covariant functors from **C** to $\mathbf{D^{OP}}$, or equivalently covariant functors from $\mathbf{C^{OP}}$ to **D**. The best way to motivate the definition is to give some less trivial examples.

1. Let **C** be the category of all vector spaces over some field, and consider the construction which associates to each vector space its *dual space* V^* of linear mappings from V to the scalar field F. There is a simple way of defining a corresponding construction for morphisms; if $L : V \to W$ is a linear transformation, consider the linear transformation $L^* : W^* \to V^*$ whose value on a linear functional $h : W \to F$ is given by $L^*(h) = h \circ L$, which is a linear functional on V. Standard results in linear algebra show that L^* is a linear transformation, that L^* is an identity map if L is an identity map, and if L is a composite $L_1 \circ L_2$, then we have $L^* = L_2^* \circ L_1^*$.

2. There is a contravariant power set functor P^* on the category **SETS** defined as follows: As before, the set $P^*(X)$ is just the set of all subsets, but now if $f : X \to Y$ is a set-theoretic function, then $P^*(f) : P^*(Y) \to P^*(X)$ takes an element $B \in P(Y)$ — which by definition is just a subset of Y — to its **inverse image** $f^{-1}[B] \subset X$. As in the case of P_*, a short elementary argument is needed to verify this construction actually defines a contravariant functor. The construction preserves identity maps because $1_X^{-1}[B] = B$ for all $B \subset X$, and the identity $P_*(g \circ f) = P^*(f) \circ P^*(g)$ is essentially a restatement of the elementary identity $f^{-1}[\,g^{-1}[B]\,] \;=\; (g \circ f^{-1}[B]$.

3. The preceding example actually yields a little more. Define a *Boolean algebra* to be a set with two binary operations $\cap$ and $\cup$, a unary operation $x \to x'$, and special elements 0 and 1 such that the system satisfies the usual properties for unions, intersections, and complementation for the algebra $P(X)$ of subsets of a set X, where 0 corresponds to the empty set and 1 corresponds to X. One then has an associated category **BOOL–ALG** whose objects are Boolean algebras and whose morphisms preserve unions, intersection, complementation, and the special elements. Obviously each power set $P(X)$ is a Boolean algebra, and in fact P^* defines a contravariant functor from **SETS** to **BOOL–ALG**. — In contrast, the covariant functor P_* does NOT define such a functor because $P_*(f)$ does not preserves intersections even though it does preserve unions (for example, we can have $f[A] \cap f[B] \neq \emptyset$ when $A \cap B = \emptyset$).

4. The desirability of having both contravariant and covariant functors is illustrated by the following examples. Given a category **C**, modulo foundational questions we can informally view the set $\mathrm{Morph}_{\mathbf{C}}(X, Y)$ of morphisms from X to Y as a function of two variables on **C**. What happens if we hold one of these variables constant to get a single variable construction? — Suppose first that we hold X constant and set $A_X(Y) = \mathrm{Morph}_{\mathbf{C}}(X, Y)$. Then we can make A_X into a covariant functor aas follows: Given a morphism $g : Y \to Z$, let $A_X(g)$ take $f \in A_X(Y) = \mathrm{Morph}_{\mathbf{C}}(X, Y)$ to the composite $g \circ f$. The axioms for a category then imply that $A_X(1_Y)$ is the identity and that $A_X(h \circ g) = A_X(h) \circ A_X(g)$ if g and h are composable. — Now suppose that we hold Y constant and set $B_Y(X) = \mathrm{Morph}_{\mathbf{C}}(X, Y)$. Then we can make B_Y into a **contravariant** functor as follows: Given a morphism $k : W \to X$, let $B_Y(g)$ take $f \in B_Y(X) = \mathrm{Morph}_{\mathbf{C}}(X, Y)$ to the composite $f \circ k$. The axioms for a category then imply that $B_Y(1_X)$ is the identity and that $B_Y(k \circ h) = B_Y(h) \circ B_Y(k)$ if h and k are composable.

5. In the preceding example, suppose that **C** is the category of topological spaces and continuous mappings, and let Y be the real numbers with the usual topology. In this case the contravariant functor B_Y has the algebraic structure of a commutative ring with unit given by pointwise multiplication of continuous real valued functions, and if $f : W \to X$ is continuous then $B_Y(f)$ is in fact a homomorphism of commutative rings with unit. Therefore, if we define a category of continuous rings with unit (whose morphisms are unit preserving homomorphisms), it follows that B_Y defines a functor from topological spaces and continuous mappings to commutative rings with unit. — In contrast, there is no comparable structure for the covariant functor B_X if X is the real numbers.

Properties of functors

One of the most important properties of functors is that they send isomorphic objects in one one category to isomorphic objects in the other.

PROPOSITION 2. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories, let* $T : \mathbf{C} \to \mathbf{D}$ *be a (covariant or contravariant) functor, and let* $f : X \to Y$ *be an isomorphism in* $\mathbf{C}$*. Then* $T(f)$ *is an isomorphism in* $\mathbf{D}$*. Furthermore, if* g *is the inverse to* f*, then* $T(g)$ *is the inverse to* $T(f)$*.*

Proof. CASE 1. Suppose the functors are covariant. Then we have

$$1_{T(X)} \quad = \quad T(1_X) \quad = \quad T(g \circ f) \quad = \quad T(g) \circ T(f)$$

$$1_{T(Y)} \quad = \quad T(1_Y) \quad = \quad T(f \circ g) \quad = \quad T(f) \circ T(g)$$

and hence $T(g)$ is inverse to $T(f)$. — *CASE 2.* Suppose that the functors are contravariant. Then we have

$$1_{T(X)} \quad = \quad T(1_X) \quad = \quad T(g \circ f) \quad = \quad T(f) \circ T(g)$$

$$1_{T(Y)} \quad = \quad T(1_Y) \quad = \quad T(f \circ g) \quad = \quad T(g) \circ T(f)$$

and hence $T(g)$ is inverse to $T(f)$.■

The next result states that a composite of two functors is also a functor.

PROPOSITION 3. *Suppose that* $\mathbf{C}$*,* $\mathbf{D}$ *and* $\mathbf{E}$ *are categories and that* $F : \mathbf{C} \to \mathbf{D}$ *and* $G : \mathbf{D} \to \mathbf{E}$ *are functors (in each case, the functor may be covariant or contravariant). Then the composite* $G \circ F$ *also defines a functor; this functor is covariant if* F *and* G *are both covariant or contravariant, and it is contravariant if one of* F*,* G *is covariant and the other is contravariant.*■

This result has a curious implication:

COROLLARY 4. *There is a "category of small categories"* **SMCAT** *whose objects are small categories and whose morphisms are covariant functors from one small category to another.*■

SEMANTIC TRIVIA. (For readers who are familiar with contravariant and covariant tensors.) In the applications of linear algebra to differential geometry and topology, one often sees objects called *contravariant tensors* and *covariant tensors*, and for finite-dimensional vector spaces these are given by finitely iterated tensor products $V \otimes \cdots \otimes V$ of V with itself in the contravariant case and similar objects involving V^* in the covariant case; for our purposes it will suffice to say that if U and W are vector spaces with bases $\{ \mathbf{u}_i \}$ and $\{ \mathbf{w}_j \}$ respectively, then their tensor product $U \otimes W$ is a vector space having a basis of the form $\{ \mathbf{u}_i \otimes \mathbf{w}_j \}$ where i and j are allowed to vary independently (hence the dimension of $U \otimes W$ is $[\dim U] \cdot [\dim W]$). Since the identity functor on the category of vector spaces is covariant and the dual space functor is covariant, at first it might seem that something is the opposite of what it should be. However, the classical tensor notation refers to the manner in which the **coordinates** transform; now coordinates for a vector space may be viewed linear functionals on that space, or equivalently as elements of the dual space, which is contravariant. Therefore individual coordinates on $V \otimes \cdots \otimes V$ correspond to elements of the *dual space* of the latter, and in fact the construction which associates the space $(V \otimes \cdots \otimes V)^*$ to V defines a *contravariant* functor on the category of finite-dimensional vector spaces over the given scalars; likewise, the construction which associates the space $(V^* \otimes \cdots \otimes V^*)^*$ to V defines a **covariant** functor on the category of finite-dimensional vector spaces over the given scalars.

Natural transformations

The final concept in category theory to be considered here is the notion of **natural transformation** from one functor to another. In fact, the motivation for category theory in the work of Eilenberg and MacLane was a need to discuss "natural mappings" in a mathematically precise manner. There are actually two definitions, depending whether both functors under consideration are covariant or contravariant.

Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and let F and G be covariant functors from $\mathbf{C}$ to $\mathbf{D}$. A *natural transformation* θ from F to G associates to each object X in $\mathbf{C}$ a morphism $\theta_X : F(X) \to G(X)$ such that for each morphism $f : X \to Y$ we have $\theta_Y \circ F(f) = G(f) \circ \theta_X$.

The morphism identity is often expressed graphically by saying the the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow \theta_X & & \downarrow \theta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

is a **commutative diagram**. The idea is that all paths of arrows from one object-vertex to another yield the same function.

The definition of a natural transformation of contravariant functors is similar.

Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and let T and U be contravariant functors from $\mathbf{C}$ to $\mathbf{D}$. A *natural transformation* θ from F to G associates to each object X in $\mathbf{C}$ a morphism $\theta_X : T(X) \to U(X)$ such that for each morphism $f : X \to Y$ we have $\theta_X \circ T(f) = U(f) \circ \theta_Y$.

Here is the corresponding commutative diagram:

$$\begin{array}{ccc} T(Y) & \xrightarrow{T(f)} & T(X) \\ \downarrow \theta_Y & & \downarrow \theta_X \\ U(Y) & \xrightarrow{U(f)} & U(X) \end{array}$$

Once again we need to give some decent examples

1. Given any functor $T : \mathbf{C} \to \mathbf{D}$, there is an obvious identity transformation j^T from T to itself; specifically, j^T_X is the identity map on $T(X)$.

2. Let $\mathbf{C}$ be one of the categories as above for which a diagonal functor can be defined. Then there is a natural diagonal transformation Δ from the identity to the diagonal functor such that for each object X the mapping $\Delta_X : X \to X \times X$ is the diagonal map.

3. On the category of vector spaces over some field F, one can iterate the dual space functor to obtain a covariant *double dual space functor* $(V^*)^*$. There is a natural transformation $e_V : V \to (V^*)^*$ defined as follows: For each $\mathbf{v} \in V$, let $e_V(\mathbf{v}) : V^* \to F$ be the linear function given by evaluation at v; in other words, the value of $e_V(\mathbf{v})$ on a linear functional f is given by $f(\mathbf{v})$. If V is finite-dimensional, this map is an isomorphism.

Note that if V is finite-dimensional then V and its dual space V^* are isomorphic, but the isomorphis depends upon some additional data such as the choice of a basis or an inner product. In contrast, *the natural isomorphism e_V does not depend upon any such choices.*

4. In the category of sets or topological spaces and continuous mappings, let A be an arbitrary object and define functors L_A and R_A such that $L_A(X) = A \times X$ and $R_A(X) = X \times A$. One can make these into covariant functors by sending the morphism $f : X \to Y$ to $L_A(X) = 1_A \times f$ and $R_A(f) = f \times 1_A$. There is an obvious natural transformation $t : L + A \to R_A$ such that $t_A(X) : A \times X \to X \times A$ sends (a, x) to (x, a) for all $a \in A$ and $x \in X$, and it is an elementary exercise to verify that this is a natural transformation such that each map $t_A(X)$ is an *isomorphism*; in other words, t_A is a natural isomorphism from the functor L_A to the functor R_A.

5. For the morphism examples A_X and B_Y discussed previously, if $h : W \to X$ is a morphism in the category, then it defines a natural transformation $h^* : A_X \to A_W$ which sends $f \in A_X(Y)$ to $f \circ h \in A_W(Y)$; the naturality condition follows from associativity of composition. Similarly, if $g : Y \to Z$ is a morphism then there is a natural transformation $g_* : B_Y \to B_Z$ sending f to $g \circ f$; once again, the key naturality condition follows from the associativity of composition. Furthermore, h^* is a natural isomorphism if h is an isomorphism and g_* is a natural isomorphism if g_* is an isomorphis,

A basic exercise in category theory is to prove the following:

PROPOSITION 5. *There are $1 - 1$ correspondences between natural transformations from A_X to A_W and morphisms from W to X and between natural transformations from B_Y to B_Z and morphisms from Y to Z.*

Sketch of proof. The main point is to retrieve the function from the natural transformation. Given $\theta : A_X \to A_W$, one does this by considering the image of 1_X, and given $\varphi : B_Y \to B_Z$, one does this by considering the image of 1_Y.■

Finally, we have the following result on natural isomorphisms (*i.e.*, natural transformations θ such that each map θ_X is an isomorphism:

PROPOSITION 6. *Let $\theta : F \to G$ be a natural transformation such that for each object X the map θ_X is an isomorphism. The there is a natural transformation $\varphi_X : G \to F$ such that for each X the map φ_X is inverse to θ_X.*

Proof. The main thing to check is that the relevant diagrams are commutative; we shall only do the case where F and G are covariant, leaving the other case to the reader. Since $\theta_X \circ \varphi_X$ is the identity on $G(X)$ and $\varphi_X \circ \theta_X$ is the identity on $F(X)$, we have

$$\theta_Y \circ \varphi_Y \circ G(f) \quad = \quad G(f) \quad = \quad g(f) \circ \theta_X \circ \varphi_X \quad = \quad \theta_Y \circ F(f) \circ \varphi_X$$

and if we compose with the inverse θ_X on the left of these expressions we obtain

$$\varphi_Y \circ G(f) \quad = \quad F(f) \circ \varphi_X$$

which is the naturality condition.■

We say that two functors are *naturally isomorphic* if there is a natural isomorphism from one to the other.

Equivalences of categories

One can obviously define an isomorphism of categories to be a covariant functor $T : \mathbf{C} \to \mathbf{D}$ for which there is an inverse covariant functor $U : \mathbf{D} \to \mathbf{C}$ such that the composites $T \circ U$ and $U \circ T$ are the identities on $\mathbf{C}$ and $\mathbf{D}$ respectively. However, there is a less rigid notion of category equivalence that suffices for most purposes.

Definition. A covariant functor $T : \mathbf{C} \to \mathbf{D}$ is a *category equivalence* (or equivalence of categories) if there is a covariant functor $U : \mathbf{D} \to \mathbf{C}$ such that the composites $T \circ U$ and $U \circ T$ are *naturally isomorphic to* the identities on $\mathbf{C}$ and $\mathbf{D}$ respectively.

In particular, if T and U define an equivalence of categories, then every object in $\mathbf{D}$ is isomorphic to an object of the form $T(X)$, and conversely every object in $\mathbf{C}$ is isomorphic to an object of the form $U(A)$.

I.2 : Barycentric coordinates and polyhedra

(Hatcher, § 2.1)

Drawings to illustrate many of the concepts in this and other sections of the notes can be found in the following document(s):

`http://math.ucr.edu/∼res/math246A/algtopfigures.∗`

Here the suffix $*$ is one of `doc`, `ps` or `pdf`.

A more leisurely and detailed discussion of barycentric coordinates, and more generally the use of linear algebra to study geometric problems, is contained in Section I.4 of the following online document, in which $*$ is one of the options in the preceding paragraph:

`http://math.ucr.edu/∼res/math133/geomnotes1.∗`

The files `math133exercises1.∗`, `math133solutions1.∗` and `solvedproblems`n`.∗`, where $n =$ 1 or 2, in the directory

`http://math.ucr.edu/∼res/math133`

contain further material on these topics.

Affine independence and barycentric coordinates

The crucial algebraic information is contained in the following result.

PROPOSITION 1. *Suppose that the ordered set of vectors* $\mathbf{v}_0, \cdots, \mathbf{v}_n$ *lie in some vector space* V. *Then the vectors* $\mathbf{v}_1 - \mathbf{v_0}, \cdots, \mathbf{v}_n - \mathbf{v}_n$ *are linearly independent if and only if every vector* $\mathbf{x} \in V$ *has at most one expansion of the form* $t_0\mathbf{v}_0 + \cdots + t_n\mathbf{v}_n$ *such that* $t_0 + \cdots + t_n = 1$.

A finite ordered set of vectors satisfying either (hence both) conditions is said to be *affinely independent*. Note that since the second condition does not depend upon the choice of ordering, a set of vectors is affinely independent if and only if for some arbitrary j the vectors $\mathbf{v}_i - \mathbf{v}_j$ (where $i \neq j$) is linearly independent. A linear combination in which the coefficients add up to 1 is called an *affine combination*.

Sketch of proof. To show the first statement implies the second, use the fact that $\mathbf{x} - \mathbf{v}_0$ has at most one expansion as a linear combination of $\mathbf{v}_1 - \mathbf{v_0}, \cdots, \mathbf{v}_n - \mathbf{v}_n$. To prove the reverse implication, show that if $\mathbf{x} - \mathbf{v}_0$ has more than one expansion as a linear combination of $\mathbf{v}_1 - \mathbf{v_0}, \cdots, \mathbf{v}_n - \mathbf{v}_n$, then $\mathbf{x}$ has more than one expansion as an affine combination of $\mathbf{v}_0, \cdots, \mathbf{v}_n$.■

COROLLARY 2. *If $S = \{\mathbf{v}_0, \cdots, \mathbf{v}_n\}$ is affinely independent, then every nonempty subset of S is affinely independent.*

This follows immediately from the uniqueness of expansions of vectors as affine combinations of vectors in S.■

The coefficients t_i are called **barycentric coordinates**. If we put physical weights of t_i units at the respective vertices $\mathbf{v}_i$, then the center of gravity for the system will be at the point $t_0\mathbf{v}_0 + \cdots + t_n\mathbf{v}_n$. If, say, $n = 2$, then this center of gravity will be inside the triangle with the given three vertices if and only if each t_i is positive, and it will be on the triangle defined by these vertices if and only if each t_i is nonnegative and at least one is equal to zero.

More generally, if $\mathbf{v}_0, \cdots, \mathbf{v}_n$ are affinely independent then the **n-simplex** with vertices $\mathbf{v}_0, \cdots, \mathbf{v}_n$ is the set of all points expressible as affine combinations such that each coefficient is nonnegative (*i.e.*, **convex combinations**).

Frequently the n-simplex described above will be denoted by $\mathbf{v}_0 \cdots \mathbf{v}_n$. Note that if $n = 0$, then a 0-simplex consists of a single point, while a 1-simplex is a closed line segment, a 2-simplex is given by a triangle and the points that lie "inside" the triangle (also called a *solid triangle*), and a 3-simplex is given by a pyramid with a triangular base (*i.e.*, a *tetrahedron*) together with the points inside this pyramid (also called a *solid tetrahedron*).

The following definition will also play an important role in our discussions.

Definition. If $\mathbf{v}_0, \cdots, \mathbf{v}_n$ form the vertices of a simplex $\mathbf{v}_0 \cdots \mathbf{v}_n$, then the **faces** of this simples are the simplices whose vertices are given by proper subsets of $\{\mathbf{v}_0, \cdots, \mathbf{v}_n\}$; note that such proper subsets are affinely independent by Corollary 2. If a proper subset $T \subset S$ has $k+1$ elements, then we shall say that the simplex $\Delta(T)$ whose vertices are given by T is a **k-face** of the original n-simplex, which in this notation is equal to $\Delta(S)$.

Sets with simplicial decompositions

In calculus textbooks, the derivation of Green's Theorem is often completed only for special sorts of closed regions such as the simplex whose vertices are $(0,0)$, $(1,0)$ and $(1,1)$. One then finds discussions indicating how the general case can be retrieved from special cases by splitting a general region into pieces that are nicely homeomorphic to closed regions of the special type; in particular, there is one such discussion on page 523 of the text by Marsden and Tromba, and it is taken further in the online document with figures for these notes (see Figure I.2.8 in the document `algtopfigures.pdf`).

Here are the formal descriptions.

Definition. A subset $P \subset \mathbf{R}^m$ is a *polyhedron* if
(i) P is a finite union of simplices $A_1, \cdots, A_q$,
(ii) For each pair of indices $i \neq j$, the intersection $A_i \cap A_j$ is a common face.

The simplices $A_1, \cdots, A_q$ are said to form a *simplicial decomposition* of P, and if $\mathbf{K}$ is the collection of simplices given by the A_j and all their faces, then the ordered pair $(P, \mathbf{K})$ is called a (finite) *simplicial complex*.

If X is an arbitrary topological space, then a (finite) *triangulation* of X consists of a simplcial complex $(P, \mathbf{K})$ and a homeomorphism $t : P \to X$.

With these definitions, we can say that Green's Theorem holds for "decent" closed plane regions because Such regions have nice triangulations.

SIMPLE EXAMPLE. Consider the solid rectangle in the plane given by $[a,b] \times [c,d]$, where $a < b$ and $c < d$. Everyday geometrical experience shows this can be split into two 2-simplices along a diagonal, and in fact it is the union of two 2-simplices, one with vertices (a,c), (a,d) and (b,d), and the other with vertices (a,c), (b,c) and (b,d). A point (x,y) which lies in the solid rectangle will be in the first simplex if and only if

$$(y-c)(b-a) \quad \leq \quad (d-c)(x-a)$$

and this point will be in the second simplex if and only if

$$(y-c)(b-a) \quad \geq \quad (d-c)(x-a)$$

Generalizations of this example will play an important role in the standard approach to algebraic topology.

If $(P, \mathbf{K})$ is a simplicial complex, then a subset $\mathbf{L} \subset \mathbf{K}$ is said to be a *subcomplex* if $\sigma \in \mathbf{L}$ implies that every face of σ also lies in $\mathbf{L}$. The union of the simplices in $\mathbf{L}$ is a closed subspace of P which is denoted by $|\mathbf{L}|$. With this notation we have $P = |\mathbf{K}|$.

Decompositions of prisms

The rectangular example has the following important generalization:

PROPOSITION 3. *Suppose that $A \subset \mathbf{R}^m$ is a simplex with vertices $\mathbf{v}_0, \cdots, \mathbf{v}_n$. Then $A \times [0,1] \subset \mathbf{R}^{m+1}$ has a simplicial decomposition with exactly $n+1$ simplices of dimension $n+1$.*

Proof. For each i between 0 and n let $\mathbf{x}_i = (\mathbf{v}_i, 0)$ and $\mathbf{y}_i = (\mathbf{v}_i, 1)$. We claim that the vectors

$$\mathbf{x}_0, \cdots, \mathbf{x}_i, \mathbf{y}_i \cdots, \mathbf{y}_n$$

are affinely independent and the corresponding simplices

$$\mathbf{x}_0 \cdots \mathbf{x}_i \mathbf{y}_i \cdots \mathbf{y}_n$$

(where $0 \leq i \leq n$) form a simplicial decomposition of $A \times [0,1]$.

An illustration for the case $n = 2$ is given in Figure I.2.11 of `algtopfigures.pdf`).

To prove affine independence, take a fixed value of i and suppose we have

$$\sum_{j<i} t_j \, \mathbf{x}_j \;+\; a \, \mathbf{x}_i \;+\; b \, \mathbf{y}_i \;+\; \sum_{j>i} t_j \, \mathbf{y}_j \;=$$

$$\sum_{j<i} t'_j \, \mathbf{x}_j \;+\; a' \, \mathbf{x}_i \;+\; b' \, \mathbf{y}_i \;+\; \sum_{j>i} t'_j \, \mathbf{y}_j$$

where the coefficients in each expression add up to 1; the summation will be taken to be zero if the limits reduce to $j < 0$ or $j > n$. If we view $\mathbf{R}^{m+1}$ as $\mathbf{R}^m \times \mathbf{R}$ and project down to $\mathbf{R}^m$ we obtain the equation

$$\sum_{j<i} t_j\,\mathbf{v}_j \;+\; (a+b)\,\mathbf{x}_i \;+\; \sum_{j>i} t_j\,\mathbf{v}_j \;=\; \sum_{j<i} t'_j\,\mathbf{v}_j \;+\; (a'+b')\,\mathbf{v}_i \;+\; \sum_{j>i} t'_j\,\mathbf{v}_j$$

and by the affine independence of the vectors $\mathbf{v}_k$ it follows that $t_j = t'_j$ if $j \neq i$ and also that $a + b = a' + b'$. On the other hand, if we project down to the second coordinate (the copy of $\mathbf{R}$), then we obtain

$$b \;+\; \sum_{j>i} t_j \;=\; b' \;+\; \sum_{j>i} t'_j$$

and since $t_j = t'_j$ for all j it follows that $b = b'$. Finally, since the sum of all the coefficients is equal to 1, the preceding observations imply that $1 - a = 1 - a'$, and therefore we also have $a = a'$. Therefore the vectors

$$\mathbf{x}_0,\;\cdots\;,\mathbf{x}_i,\mathbf{y}_i\;\cdots\;,\mathbf{y}_n$$

are affinely independent.

We shall next check that every point $(\mathbf{z}, u) \in A \times [0,1]$ lies in one of the simplices

$$\mathbf{x}_0\;\cdots\;\mathbf{x}_i\mathbf{y}_i\;\cdots\;\mathbf{y}_n$$

listed above. Write $\mathbf{z} = \sum_j\, t_j\,\mathbf{v}_j$ where $t_j \geq 0$ for all j and $\sum\, t_j = 1$. It follows that $u \leq 1 = \sum_{j\geq 0}\, t_j$; let $i \leq n$ be the largest nonnegative integer such that $u \leq \sum_{j\geq i}\, t_j$. We claim that $(\mathbf{z}, u)$ lies in the simplex $\mathbf{x}_0\;\cdots\;\mathbf{x}_i\mathbf{y}_i\;\cdots\;\mathbf{y}_n$. Let $b = \sum_{j\geq i}\, t_j \;-\; u$, and let $a = u - \sum_{j>i}\, t_j \;=\; t_i - b$. Then we have $a,\; b \geq 0$, and

$$(\mathbf{z}, u) \;=\; \sum_{j<i} t_j\,\mathbf{x}_j \;+\; a\,\mathbf{x}_i \;+\; b\,\mathbf{y}_i \;+\; \sum_{j>i} t_j\,\mathbf{y}_j$$

where all the coefficients are nonnegative and add up to 1.

To conclude the proof, we need to show that the intersection of two simplices as above is a common face. Suppose that $k < i$ and

$$(\mathbf{z}, u) \;\in\; \Big(\mathbf{x}_0\;\cdots\;\mathbf{x}_i\mathbf{y}_i\;\cdots\;\mathbf{y}_n\Big) \,\cap\, \Big(\mathbf{x}_0\;\cdots\;\mathbf{x}_k\mathbf{y}_k\;\cdots\;\mathbf{y}_n\Big)\,.$$

Then we must have

$$\sum_{j\leq i} p_j\,\mathbf{x}_j \;+\; \sum_{j\geq i} q_j\,\mathbf{y}_j \;=\; \sum_{j\leq k} p'_j\,\mathbf{x}_j \;+\; \sum_{j\geq k} q'_j\,\mathbf{y}_j$$

where all the coefficients are nonnegative and the coefficients on each side of the equation add up to 1. If we project down to $\mathbf{R}^m$ we obtain $p_j + q_j = p'_j + q'_j$ for all j (by convention, we take a coefficient to be zero if it does not lie in the corresponding summation as above). It follows immediately that $p_j = p'_j$ if $j < k$, while $p_j = q'_j$ if $k < j < i$ and $q_j = q'_j$ if $j > i$. Furthermore, if we project down to the last coordinate we see that

$$u \;=\; \sum_{j\geq i} q_j \;=\; \sum_{j\geq k} q'_k\;.$$

Since $q_j = q'_j$ if $j > i$, it follows that

$$q_i \quad = \quad \sum_{k \le j \le i} q'_j$$

and since all the coefficients are nonnegative, it follows that $q_i \ge q'_i$. On the other hand, we also have $q'_i = p'_i + q'_i = p_i + q_i$, and hence we conclude that $q_i = q'_i$ and $p_i = 0$. Applying the first of these, we see that

$$0 \quad = \quad \sum_{k \le j < i} q'_j$$

and hence the nonnegativity of the coefficients implies that $q'_j = 0$ for all j such that $k \le j < i$. We also know that $p'_j = 0$ for $j > k$, and therefore it follows that $p'_j + q'_j = 0$ when $k < j < i$ The equations $p_j + q_j = p'_j + q'_j$ and the nonnegativity of all terms now imply that $p_j = q_j = 0$ when $k < j < i$.

The conclusions of the preceding paragraph imply that the point $(\mathbf{z}, u)$ actually lies on the simplex

$$\mathbf{x}_0 \ \cdots \ \mathbf{x}_k\mathbf{y}_i \ \cdots \ \mathbf{y}_n$$

and since the latter is a common face of $\mathbf{x}_0 \ \cdots \ \mathbf{x}_i\mathbf{y}_i \ \cdots \ \mathbf{y}_n$ and $\mathbf{x}_0 \ \cdots \ \mathbf{x}_k\mathbf{y}_k \ \cdots \ \mathbf{y}_n$ it follows that the $(n+1)$-simplices

$$\mathbf{x}_0 \ \cdots \ \mathbf{x}_i\mathbf{y}_i \ \cdots \ \mathbf{y}_n$$

(where $0 \le i \le n$) form a simplicial decomposition of $A \times [0,1]$.■

COROLLARY 4. *If $P \subset \mathbf{R}^m$ is a polyhedron, then $A \times [0,1] \subset \mathbf{R}^{m+1}$ is also a polyhedron.*

Before discussing the proof of this we note one important special case.

COROLLARY 5. *For each positive integer m, the hypercube $[0,1]^m \subset \mathbf{R}^m$ is a polyhedron.*

Proof of Corollary 5. If $m = 1$ this follows because the unit interval is a 1-simplex; by Corollary 4, if the result is true for $m = k$ then it is also true for $m = k + 1$. Therefore the result is true for all m by induction.■

Proof of Corollary 4. Let $\mathbf{K}$ be a simplicial decomposition for P, and let $\mathbf{K}^*$ be obtained from $\mathbf{K}$ by including all the faces of simplices in $\mathbf{K}$. Choose a linear ordering of the vertices in $\mathbf{K}^*$ (note there are finitely many). For each vertex $\mathbf{v}$ of $\mathbf{K}^*$, as before let $\mathbf{x} = (\mathbf{v}, 0)$ and $\mathbf{y} = (\mathbf{v}, 1)$. Then $P \times [0,1]$ is the union of all simplices of the form

$$\mathbf{x}_0 \ \cdots \ \mathbf{x}_i\mathbf{y}_i \ \cdots \ \mathbf{y}_n$$

where $\mathbf{v}_i < \mathbf{v}_{i+1}$ with respect to the given linear ordering of the vertices in $\mathbf{K}^*$ **and also** the vertices $\mathbf{v}_i$ are the vertices of a simplex in $\mathbf{K}^*$. The set $P \times [0,1]$ is the union of these simplices by Proposition 3 and the fact that P is the union of the simplices $\mathbf{v}_0 \ \cdots \ \mathbf{v}_n$. The fact that these simplices form a simplicial decomposition will follow from the construction and the next result.

LEMMA 6. *Suppose that we have two polyhedra P_1 and P_2 in $\mathbf{R}^\mathbf{q}$, and there exist simplicial decompositions $\mathbf{K}_1$ and $\mathbf{K}_2$ such that the following hold:*

(i) Both $\mathbf{K}_1$ and $\mathbf{K}_2$ are closed under taking faces of simplices.

(ii) The set $\mathbf{L}_1$ of all simplices in $\mathbf{K}_1$ contained in $P_1 \cap P_2$ equals the set $\mathbf{L}_2$ of all simplices in $\mathbf{K}_2$ contained in $P_1 \cap P_2$, and this collection determines a simplicial decomposition of $P_1 \cap P_2$.

Then $\mathbf{K}_1 \cup \mathbf{K}_2$ determines a simplicial decomposition of $P_1 \cup P_2$.

The hypothesis clearly applies to the construction in Proposition 3, so Corollary 4 indeed follows once we prove Lemma 6.■

Proof of Lemma 6. It follows immediately that $P_1 \cup P_2$ is the union of the points of the simplices in $\mathbf{K}_1 \cup \mathbf{K}_2$. Suppose now that we are given an intersection of two simplices in the latter. This intersection will be a common face if both simplices lie in either $\mathbf{K}_1$ or $\mathbf{K}_2$, so the only remaining cases are those where one simplex α lies in $\mathbf{K}_1$ and the other simplex β lies in $\mathbf{K}_2$.

We know that $\alpha \cap \beta$ us convex. Furthermore, by the hypotheses we know that $\alpha \cap \beta$ must be a union of simplices that are faces of both α and β. Therefore it follows that every point in $\alpha \cap \beta$ is a convex combination of the vertices which lie in $\alpha \cap \beta$, and consequently $\alpha \cap \beta$ is the common face determined by all vertices which lie in $\alpha \cap \beta$.■

I.3 : Subdivisions

(Hatcher, § 2.1)

TO BE COMPLETED